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Descriptive Complexity,
Canonisation,
and Definable Graph Structure Theory
## Contents

1. **Introduction**  

2. **Background from Graph Theory and Logic**  
   - 2.1 General Notation  
   - 2.2 Graphs and Structures  
   - 2.3 Logics  
   - 2.4 Transductions  

3. **Descriptive Complexity**  
   - 3.1 Logics Capturing Complexity Classes  
   - 3.2 Definable Orders  
   - 3.3 Definable Canonisation  
   - 3.4 Finite Variable Logics and Pebble Games  
   - 3.5 Isomorphism Testing and the Weisfeiler-Leman Algorithm  

4. **Treelike Decompositions**  
   - 4.1 Tree Decompositions  
   - 4.2 Treelike Decompositions  
   - 4.3 Normalising Treelike Decompositions  
   - 4.4 Tight Decompositions  
   - 4.5 Isomorphisms, Homomorphisms, and Bisimulations  
   - 4.6 The Relation Between Tree Decompositions and Treelike Decompositions  

5. **Definable Decompositions**  
   - 5.1 Decomposition Schemes  
   - 5.2 Normalising Definable Decompositions  

---

1 Introduction 7

I The Basic Theory 17

2 Background from Graph Theory and Logic 19
   - 2.1 General Notation 19
   - 2.2 Graphs and Structures 19
   - 2.3 Logics 25
   - 2.4 Transductions 34

3 Descriptive Complexity 41
   - 3.1 Logics Capturing Complexity Classes 42
   - 3.2 Definable Orders 50
   - 3.3 Definable Canonisation 53
   - 3.4 Finite Variable Logics and Pebble Games 68
   - 3.5 Isomorphism Testing and the Weisfeiler-Leman Algorithm 76

4 Treelike Decompositions 89
   - 4.1 Tree Decompositions 89
   - 4.2 Treelike Decompositions 92
   - 4.3 Normalising Treelike Decompositions 98
   - 4.4 Tight Decompositions 103
   - 4.5 Isomorphisms, Homomorphisms, and Bisimulations 109
   - 4.6 The Relation Between Tree Decompositions and Treelike Decompositions 110

5 Definable Decompositions 115
   - 5.1 Decomposition Schemes 115
   - 5.2 Normalising Definable Decompositions 117
## CONTENTS

5.3 Definable Tight Decompositions ........................................ 119  
5.4 Lifting Definability ......................................................... 120  
5.5 Parametrised Decomposition Schemes .................................. 121  
5.6 The Transitivity Lemma ..................................................... 124  

6 Graphs of Bounded Tree Width ......................................... 139  
6.1 Defining Bounded-Width Decompositions ................................. 139  
6.2 Defining Bounded-Width Decompositions Top-Down ..................... 142  

7 Ordered Treelike Decompositions ...................................... 147  
7.1 Definitions and Basic Results ............................................. 147  
7.2 Parametrised O-Decompositions Schemes ................................. 151  
7.3 Extension Lemmas ............................................................. 152  
7.4 Canonisation via Definable Ordered Treelike Decompositions .......... 157  

8 3-Connected Components .................................................. 165  
8.1 Decomposition into 2-Connected Components ............................. 165  
8.2 2-Separators of 2-Connected Graphs ..................................... 168  
8.3 Decomposition into 3-Connected Components ............................. 170  

9 Graphs Embeddable in a Surface ........................................ 177  
9.1 Surfaces and Embeddings of Graphs ..................................... 177  
9.2 Angles ................................................................. 190  
9.3 Planar Graphs ............................................................... 196  
9.4 Graphs on Arbitrary Surfaces ............................................ 202  

II Definable Decompositions of Graphs with Excluded Minors .......... 215  
10 Quasi-4-Connected Components .......................................... 217  
10.1 Hinges ................................................................. 217  
10.2 Decomposition into Quasi-4-Connected Components ..................... 237  
10.3 The Q4C Lifting Lemma .................................................. 245  

11 $K_5$-Minor Free Graphs .................................................... 253  
11.1 Decompositions ............................................................ 253  
11.2 Definability ............................................................... 256  

12 Completions of Pre-Decompositions .................................... 257  
12.1 Pre-Decompositions and Completions ................................... 257  
12.2 Ordered Completions ...................................................... 260  
12.3 Bounded Width Completions .............................................. 260  
12.4 Derivations of Pre-Decompositions ..................................... 264  
12.5 The Finite Extension Lemma for Ordered Completions .................. 265  
12.6 The Q4C Completion Lemma ............................................. 267  

M. Grohe, *Definable Graph Structure Theory*
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>13 Almost Planar Graphs</td>
<td>279</td>
</tr>
<tr>
<td>13.1 Relaxations of Planarity</td>
<td>279</td>
</tr>
<tr>
<td>13.2 Central Vertices</td>
<td>285</td>
</tr>
<tr>
<td>13.3 Defining the Central Faces</td>
<td>289</td>
</tr>
<tr>
<td>13.4 Centres and Skeletons</td>
<td>317</td>
</tr>
<tr>
<td>13.5 Decomposing Almost Planar Graphs and Their Minors</td>
<td>321</td>
</tr>
<tr>
<td>14 Almost Planar Completions</td>
<td>331</td>
</tr>
<tr>
<td>14.1 From Almost Planar to Ordered Completions</td>
<td>331</td>
</tr>
<tr>
<td>14.2 Grids</td>
<td>332</td>
</tr>
<tr>
<td>14.3 Supercentre and Superskeleton</td>
<td>346</td>
</tr>
<tr>
<td>14.4 The Completion Theorem for Quasi-4-Connected Graphs</td>
<td>347</td>
</tr>
<tr>
<td>14.5 MAP-(p)-Star Completions</td>
<td>352</td>
</tr>
<tr>
<td>14.6 Proof of the Almost Planar Completion Theorem 14.1.3</td>
<td>355</td>
</tr>
<tr>
<td>15 Almost Embeddable Graphs</td>
<td>359</td>
</tr>
<tr>
<td>15.1 Arrangements in a Surface</td>
<td>359</td>
</tr>
<tr>
<td>15.2 Shortest Path Systems</td>
<td>371</td>
</tr>
<tr>
<td>15.3 Simplifying and Safe Subgraphs</td>
<td>375</td>
</tr>
<tr>
<td>15.4 Patches</td>
<td>377</td>
</tr>
<tr>
<td>15.5 Belts</td>
<td>388</td>
</tr>
<tr>
<td>16 Decompositions of Almost Embeddable Graphs</td>
<td>399</td>
</tr>
<tr>
<td>16.1 The Combination Lemma</td>
<td>399</td>
</tr>
<tr>
<td>16.2 The Last Extension Lemma</td>
<td>405</td>
</tr>
<tr>
<td>16.3 Decomposing Almost Embeddable Graphs and Their Minors</td>
<td>426</td>
</tr>
<tr>
<td>16.4 Almost Embeddable Completions</td>
<td>432</td>
</tr>
<tr>
<td>17 Graphs with Excluded Minors</td>
<td>441</td>
</tr>
<tr>
<td>17.1 The Structure of Graphs with Excluded Minors</td>
<td>441</td>
</tr>
<tr>
<td>17.2 The Main Theorem</td>
<td>447</td>
</tr>
<tr>
<td>18 Bits and Pieces</td>
<td>455</td>
</tr>
<tr>
<td>18.1 From Graphs to Relational Structures</td>
<td>455</td>
</tr>
<tr>
<td>18.2 Lifting Canonisations</td>
<td>457</td>
</tr>
<tr>
<td>18.3 Invariant Treelike Decompositions and Polynomial Time Canonisation</td>
<td>463</td>
</tr>
<tr>
<td>18.4 Directions for Further Research</td>
<td>464</td>
</tr>
<tr>
<td>A Robertson and Seymour’s Version of the Local Structure Theorem</td>
<td>469</td>
</tr>
</tbody>
</table>
Chapter 1
Introduction

In this book, we develop a definable structure theory for finite graphs. The goal is to decompose graphs into pieces that are “simpler” than the original graphs, and to do this in a way that the decomposition is definable in some logic. A simple example of a decomposition theorem we prove here is that every graph has a “definable treelike decomposition” into 3-connected graphs. (A graph is 3-connected if it stays connected even after two arbitrary vertices are removed.) A more complicated example states that every graph that excludes $K_5$, the complete graph on five vertices, as a minor has a definable treelike decomposition into pieces that are either 3-connected planar graphs or isomorphic to the graph $L$ shown in Figure 1.1. (A minor of a graph $G$ is a graph $H$ that is obtained from a subgraph of $G$ by contracting edges. We will give more background on graph minors in the next section.)

The main applications of our definable structure theory are in descriptive complexity, and for these applications we need our decompositions to be definable in least fixed-point logic LFP, or equivalently, in inflationary fixed-point logic IFP. (For technical reasons, it will be more convenient for us to work with IFP.) These fixed-point logics are extensions of first-order predicate logic by fixed-point operators that allow it to formalise inductive definitions. In this book, we will exclusively study decompositions definable in IFP, but much of the general theory we develop here applies to other logics as well, for example, to monadic second-order logic.

There is a standard graph theoretic notion of tree decomposition, playing a central role in modern graph structure theory. Ideally, we would like our definable decompositions to be tree decompositions, but it turns out that in general tree decompositions are not logically definable, because they are not invariant under automorphisms of the underlying graph. Instead, we introduce a new notion of treelike decompositions. Treelike decompositions inherit many of the desirable properties of tree decompositions, yet they can be made automorphism-invariant and, as it turns out, are often definable in logics like IFP.

Our main theorem, the Definable Structure Theorem 17.2.1, says that all classes of graphs that exclude some fixed graph as a minor admit IFP-definable treelike decompositions into pieces that admit an IFP-definable linear order. Linearly ordered finite graphs are easy to deal with in many ways. For example, they have trivial automorphism groups. More importantly, many results in descriptive complexity require structures to be linearly ordered. It is a longstanding open question whether there is a logical characterisation of the polynomial-time properties of graphs. As an application of our definable structure theorem, we obtain such
a characterisation for all properties of graphs with excluded minors. As a second important application of our structure theorem, we show that for every class of graphs that exclude some fixed graph as a minor there is a $k$ such that a simple combinatorial algorithm, namely “the $k$-dimensional Weisfeiler-Leman algorithm”, decides isomorphism of graphs in $\mathcal{C}$ in polynomial time.

The rest of this introductory chapter is structured as follows. In the next section, we describe the graph theoretic context of our results. After that, we briefly (and informally) explain the central concept of treelike decompositions of graphs. Then we say more about the applications in descriptive complexity theory and to the graph isomorphism problem. Finally, we give an outline of the rest of this book and of the proof of our main theorem and close the chapter with a few bibliographical remarks.

**Graph Minor Theory**

Recall that a graph $H$ is a minor of graph $G$ if $H$ is obtained from a subgraph of $G$ by contracting edges. (Formally, contracting an edge means deleting the edge and identifying its endvertices.) Figure 1.2 shows an example. If $\mathcal{C}$ is a class of graphs such that $H$ is not a minor of any $G \in \mathcal{C}$, then we say that $\mathcal{C}$ excludes $H$ as a minor. Graph minor theory is concerned with graph classes excluding some fixed graph as a minor. Maybe the starting point of the theory is a variant of Kuratowski’s well known characterisation of the planar graphs due to Wagner, stating that a graph is planar if and only if it excludes $K_5$ (the complete graph with 5 vertices) and $K_{3,3}$ (the complete bipartite graph with 3 vertices in both parts) as minors. It was a long standing open question whether a similar characterisation by excluded minors exists for graphs embeddable in other surfaces than the plane or equivalently, the 2-sphere. Archdeacon gave a list of 35 excluded minors characterising the class of graphs embeddable in the projective plane. No explicit excluded-minor characterisations are known for any surface except the sphere and the projective plane.

However, Robertson and Seymour proved that for every surface such a characterisation exists. Indeed, they proved a much more powerful result known as the Graph Minor...
Figure 1.2. The graph in (c) is a minor of the graph in (a) obtained by deleting the dotted edges and the white node and contracting the dashed edges in (b).

Theorem [112]. Let us call a class $C$ of graphs that is closed under taking minors a minor ideal. It is easy to see that for every surface $S$ the class of all graphs embeddable in $S$ is a minor ideal (Figure 9.4 on page 188 illustrates why). There are many other natural graph classes that are minor ideals, for example classes of bounded tree width (see Section 4.1 and Chapter 3), the class of all graphs linklessly embeddable in 3-space (a linkless embedding of graph $G$ is an embedding where no two cycles of $G$ are linked in the sense of knot theory; see [113] for an explicit excluded minor characterisation of this class), the class of all graphs knotlessly embeddable in 3-space, the class of all graphs that have a vertex cover of size at most $k$ (a vertex cover of a graph is a set of vertices that contains at least one endvertex of each edge), and the class of all graphs that have a feedback vertex set of size at most $k$ (a feedback vertex set of a graph is a set of vertices that contains at least one vertex of each cycle). Trivially, each minor ideal $M$ has a characterisation by (possibly infinitely many) excluded minors. The Graph Minor Theorem states that every minor ideal can be characterised by finitely many excluded minors, that is, for every minor ideal $M$ there is a finite list $H_1, \ldots, H_n$ of graphs such that $M$ is the class of all graphs that do not contain any $H_i$ as a minor.

To prove the Graph Minor Theorem, in a long series of articles [105] Robertson and Seymour developed a structure theory for graphs with excluded minors. In [111], they proved a structure theorem which says that graphs with excluded minors have a tree decomposition into pieces that are “almost embeddable” into some surface. Intuitively, almost embedding a graph into a surface means first removing a bounded number of vertices from the graph (these vertices are called apices) and then drawing the rest of the graph in the surface with no edges crossing except in a bounded number of regions (called vortices) in which the surface structure may be violated. The high-level structure is illustrated by Figure 1.3. Each vortex is attached to the boundary of a “hole” in the surface. The vortices may be far from being embeddable in the underlying surface, but they have a fairly simple structure that is controlled by a parameter called the width of a vortex. Thus overall there are four parameters in the definition of almost embeddability: the surface, the number of apices, the number of vortices, and the width of the vortices. These parameters are bounded in terms of the excluded minor. (We will give a precise definition of almost embeddability in Chapter 15 and the exact statement of Robertson and Seymour’s structure theorem in Chapter 17.)

Besides the Graph Minor Theorem, the structure theorem has found numerous other applications, many of them algorithmic [23, 24, 26, 27, 28, 44, 110]. The structure theorem
Chapter 1. Introduction

Figure 1.3. A graph almost embedded in a triple torus with three vortices and four apices also plays an important role in this book.

Treelike Decompositions

Tree decompositions and the related notion of tree width have been introduced by Robertson and Seymour in [106]. (Interestingly, several equivalent notions have been introduced independently by other researchers [1, 4, 62, 117].) By now, they have developed into a standard tool in structural graph theory and graph algorithms ([104] is a survey). A tree decomposition of a graph $G$ consists of a tree $T$ and a mapping $\beta$ that associates with every node $t$ of $T$ a set $\beta(t)$ of vertices of $G$ subject to certain technical conditions making sure that the structure of the tree $T$ approximates the connectivity structure of $G$. The set $\beta(t)$ is called the bag of the decomposition at $t$.

Now suppose that we want to define a tree decomposition in some logic. We could try to interpret the tree $T$ in the underlying graph $G$, that is, define a set of $\ell$-tuples of vertices of $G$ representing the nodes of $T$ and define a $2\ell$-ary relation representing the edges of $T$. Then we could define an $(\ell + 1)$-ary relation to represent the bags. Unfortunately, most interesting tree decompositions are not definable in this way, no matter what logic we use, because the decompositions are not invariant under automorphisms of the graph. What this means is that there may be an automorphism $f$ of $G$ for which we cannot find an automorphism $g$ of $T$ such that for all nodes $t$ we have $\beta(g(t)) = f(\beta(t))$. As an example, consider the decomposition of the cycle $C_5$ displayed in Figure 1.4. However, only invariant objects are logically definable in the graph.

We resolve this problem by introducing a more general notion of decomposition that we call treelike. In a treelike decomposition, we replace the tree $T$ underlying a tree decomposition by a directed acyclic graph $D$. The idea is that certain restrictions of $D$ to subtrees yield tree decompositions of $G$, and by including many such decompositions we can close the treelike decompositions under automorphisms of a graph. To get an impression how treelike decompositions look, consider Figure 1.5 which shows a treelike decomposition of the cycle $C_5$. The sets displayed in the nodes of the decomposition are the bags. Observe that the four grey nodes form exactly the tree decomposition of $C_5$ displayed in Figure 1.4. There are many

M. Grohe, Definable Graph Structure Theory
Figure 1.4. The cycle $C_5$ together with a tree decomposition of the cycle

tree decompositions of $C_5$ contained in the treelike decomposition in a similar way. Note the cyclic structure of the whole decomposition, which reflects the structure of the underlying cycle and is the reason for the invariance of the decomposition under automorphisms of the cycle. This cyclic structure is lost in a tree decomposition like the one in Figure 1.4.

We extend treelike decompositions to ordered treelike decompositions, which one may think of as treelike decompositions together with linear orders of all bags. Our main goal is to prove that certain classes of graphs admit IFP-definable ordered treelike decompositions. The Definable Structure Theorem for Graphs with Excluded Minors says that this is the case for all classes of graphs excluding some fixed graph as a minor.

Descriptive Complexity Theory

Descriptive complexity theory characterises the complexity of computational problems in terms of logical definability. The starting point of the theory was Fagin’s Theorem [34] from 1974, stating that existential second-order logic captures the complexity class $NP$. This means that a property of finite structures is decidable in nondeterministic polynomial time if and only if it is definable in existential second-order logic. Similar logical characterisations were later found for most other complexity classes. For example, Immerman [72] and independently Vardi [125] characterised the class $PTIME$ (polynomial time) in terms of least fixed-point logic, and Immerman [74] characterised the classes $NL$ (nondeterministic logarithmic space) and $L$ (logarithmic space) in terms of transitive closure logic and its deterministic variant. However, these logical characterisations of the classes $PTIME$, $NL$, and $L$, and all other known logical characterisations of complexity classes contained in $PTIME$, have a serious drawback: they only apply to properties of ordered structures, that is, structures with one distinguished relation that is a linear order of the elements of the structure. It is still an open question whether there are logics that characterise these complexity classes on arbitrary, not necessarily ordered, finite structures.

The question of whether there is a logic that captures $PTIME$ was first raised by Chandra and Harel [19] in a fundamental paper on query languages for relational databases. Chandra and Harel asked for a query language expressing precisely those queries that can be evaluated in polynomial time. Gurevich [59] rephrased the question in terms of logic. His precise definition of a “logic capturing $PTIME$” is subtle; we will discuss it in Chapter 8. Gurevich [60] conjectured that there is no logic capturing $PTIME$. Note that this conjecture implies $PTIME \neq NP$, because by Fagin’s Theorem there is a logic that captures $NP$. The
question of whether there is a logic capturing PTIME is still open today, and it is viewed as one of the main open problems in finite model theory and database theory. Only partial positive answers are known. To start with, remember the Immerman-Vardi Theorem stating that least fixed-point logic, or equivalently inflationary fixed-point logic IFP captures PTIME on ordered structures. It is easy to prove that IFP does not capture PTIME on the class of all finite structures. IFP cannot even define the property of a graph having an even number of vertices, but clearly this property is decidable in polynomial time. More generally, IFP “lacks the ability to count”. Immerman [73] proposed the extension IFP+C of IFP by counting operators as a candidate for a logic capturing PTIME. It was shown by Cai, F¨ urer, and Immerman in 1992 [16] that IFP+C does not capture PTIME, but it comes surprisingly close. Indeed, Hella, Kolaitis, and Luosto [64] proved that IFP+C captures PTIME on almost all structures (in a precise probabilistic sense).

We shall prove that IFP+C captures PTIME on all classes C of graphs that admit IFP-definable ordered treelike decompositions. Hence it follows from our Definable Structure Theorem that IFP+C captures PTIME on all classes of graphs excluding some fixed graph as a minor.

The Graph Isomorphism Problem

It is a long standing open problem whether there is a polynomial-time algorithm deciding if two graphs are isomorphic. Polynomial-time isomorphism tests are known for many natural classes of graphs including the class of planar graphs [68], classes of graphs embeddable in a fixed surface [35, 89] and more generally classes of graphs with excluded minors [101], and classes of graphs of bounded degree [87]. The isomorphism test for graphs of bounded
degree due to Luks [87] involves some nontrivial group theory, and many later isomorphism algorithms build on the group theoretic techniques developed by Babai, Luks, and others in the early 1980s. In particular, Ponomarenko’s [101] isomorphism algorithm for graphs with excluded minors heavily builds on these techniques. It follows from our Definable Structure Theorem that a simple combinatorial algorithm known as the \( k \)-dimensional Weisfeiler-Leman algorithm decides isomorphism on all classes of graphs excluding some fixed graph as a minor in polynomial time. Here the parameter \( k \) of the algorithm depends on the excluded minor.

**The Structure of this Book**

The book has two parts. The first is devoted to the general theory of definable treelike decomposition and its connections with descriptive complexity theory. After two chapters giving the necessary background in graph theory, logic, and descriptive complexity, in Chapters 4, 5, and 7 we introduce (definable, ordered) treelike decompositions and study their basic properties. In Chapter 6 we turn to graphs of bounded tree width and prove that they admit definable treelike decompositions of bounded width. (The *width* of tree decomposition or treelike decomposition is the maximum bag size minus 1, and the *tree width* of a graph is minimum of the width of all its tree decompositions.) In Chapter 8 we show that every graph has a definable treelike decomposition into its 3-connected components. The first part culminates, in Chapter 9, in a Definable Structure Theorem for Graphs Embeddable in Surface, stating that for each surface \( S \) the class of all graphs embeddable in \( S \) admits IFP-definable ordered treelike decompositions.

The first part only uses elementary graph theory. Maybe expanded by additional background material on descriptive complexity theory (for example, [38]), it could be used as the basis of course on this direction of finite model theory.

The second part is largely devoted to a proof of the Definable Decomposition Theorem for Graphs with Excluded Minors. Instead of going through the chapters one by one, we give an outline of the proof. Actually, we step back to the first part and start with an outline of the proof of the definable decomposition theorems for planar graphs and graphs embeddable in a surface.

**Step 1: Planar graphs.** We first prove that 3-connected planar graphs admit IFP-definable linear orders. The key step in the proof is to define the facial cycles of a 3-connected planar graph in IFP. We use the fact, going back to Whitney, that the facial cycles (boundaries of the faces) of a 3-connected graph embedded in the plane are precisely the chordless and nonseparating cycles. In particular, this means that the facial cycles are the same for every embedding of the graph. Once we have defined the facial cycles, we can use three parameters to fix one facial cycle and then define in IFP a linear order by “walking around this cycle in spirals”.

To show that arbitrary planar graphs admit IFP-definable ordered treelike decompositions, we use the result (proved in Chapter 8) that every graph has an IFP-definable treelike decomposition into its 3-connected components.

**Step 2: Graphs embeddable in a surface.** We exploit the fact that every surface of positive Euler genus has a noncontractible cycle. Cutting the surface open along such a cycle and glueing disks on the hole(s) yields one or two surfaces of strictly smaller Euler genus.

To define ordered treelike decompositions on graphs embeddable in a surface, we proceed by induction on the Euler genus of the surface. Planar graphs are the base case. In the inductive step, we try to define the facial cycles of a graph embedded in a surface. Either we
Chapter 1. Introduction

Figure 1.6. An almost planar graph (the outer edges connect vertices of distance two on the cycle)

succeed, then we can use the facial cycles to define a linear order in a similar way as for planar graphs. Or we find a noncontractible cycle along the way. Then we can delete this cycle, apply the induction hypothesis to the resulting graph embeddable in one or two surfaces of strictly smaller Euler genus, and extend the decomposition to the original graph.

Step 3: Almost planar graphs. Remember our informal description of almost embeddable graphs. Let $A(p,q,r,s)$ be the class of all graphs almost embeddable in a surface of Euler genus at most $r$ with at most $s$ apices and at most $q$ vortices, each of width at most $p$. In this and the following step, we want to prove that for all $p,q,r,s$ the class $A(p,q,r,s)$ admits IFP-definable ordered treelike decompositions. We proceed by induction on $q + r$. We already know how to deal with the class $A(0,0,0)$ of planar graphs and more generally the class $A(0,0,r,0)$ of all graphs embeddable in a surface of Euler genus at most $r$.

In this step, we consider the classes $A(p,1,0,0)$ of almost planar graphs. We think of almost planar graphs as graphs being embedded into a disk with a vortex glued on the boundary of the disk. Figure 1.6 shows an example. In the key lemma of this step, and actually one of the most difficult lemmas of the whole book, we show that the facial cycles of an almost planar graph that are sufficiently far from the vortex do not depend on the specific embedding. That is, no matter how we divide the graph into a vortex and a part embedded in the disk, these cycles will end up in the disk, and they will be facial cycles. Moreover, these cycles are IFP-definable. We call the subgraph of the graph induced by these cycles the centre of the graph. Using the facial cycles, we can define a linear order on each connected component of the centre. If we contract each connected component of the centre to a single vertex, we obtain a graph of tree width bounded by $O(p^2)$, which we call the skeleton of our original graph. We define a treelike decomposition of bounded width of the skeleton, and we can extend the decomposition to an ordered treelike decomposition of the original graph using the linear orders of the components of the centre.

Step 4: Almost embeddable graphs. We prove that the classes $A(p,q,r,s)$ admit IFP-definable ordered treelike decompositions by an inductive construction similar to, but far more complicated than the one in Step 2.

Step 5: Graphs with excluded minors. Let $C$ be a class of graphs excluding some fixed graph as a minor. By Robertson and Seymour’s structure theorem, there are $p,q,r,s$ such that all graphs in $C$ have a tree decomposition into pieces from $A(p,q,r,s)$. If we could define such a decomposition in IFP, then we could use the definable decomposition of the graphs in $A(p,q,r,s)$ obtained in the previous step to define ordered treelike decompositions of the
graphs in \( C \). But unfortunately I do not know how to define such a decomposition in \( \text{IFP} \).

Instead, we repeatedly apply the construction of the previous steps to inductively build up an ordered treelike decomposition of a graph in \( C \) from partial decompositions in a bottom up fashion. To be able to do this, we have to prove generalisations of the results from the previous steps, so-called completion lemmas, which roughly say that if we already have ordered treelike decompositions of parts of a graph, and if the part of the graph that is not covered by these partial decompositions has a nice structure, such as being almost embeddable in some surface, then we can complete the partial decompositions to an ordered treelike decomposition of the whole graph. The formal framework of pre-decompositions and completions will be introduced in Chapter 12.

Another difficulty of the proof is that we need the graphs we decompose to be not only 3-connected, but quasi 4-connected. Quasi 4-connectedness is a new notion introduced in Chapter 10. We prove that all graphs have definable decompositions into their "quasi-4-connected components." Unfortunately, these decompositions turn out to be quite complicated. As an immediate reward, once we have obtained the decompositions into quasi-4-connected, we easily get a Definable Decomposition Theorem for the class of graphs excluding \( K_5 \) as a minor (see Chapter 11).

It is mainly the last two issues, completion lemmas and the decomposition into quasi-4-connected components, that cause a lot of technical problems and make the proof of the Definable Decomposition Theorem so long. The proof will be completed in Chapter 17.

**Bibliographical Remarks**

The main results of this book, the Definable Structure Theorem for Graphs with Excluded Minors and its applications, are new. They have been announced in the conference paper [47]. A short and nontechnical introduction into the results has also appeared in [48]. Partial results have appeared in earlier conference papers [42, 43, 45]. The main results of Chapter 9 also appeared in the journal paper [49].

Although in the second part we use some heavy graph theory, the book is self-contained, and the only prerequisites are a little bit of elementary logic and graph theory. Complexity theory will only play a role in Chapter 3 where the connections of our structure theory with descriptive complexity theory are prepared. All known results that we use without proof are labelled as “Facts”; of course references for these facts are given.

**Acknowledgements**

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Part I

The Basic Theory
Chapter 2

Background from Graph Theory and Logic

2.1 General Notation

\( \mathbb{Z}, \mathbb{N}, \) and \( \mathbb{N}^+ \) denote the sets of integers, nonnegative integers and positive integers, respectively. For \( m, n \in \mathbb{Z} \), we let \( [m, n] := \{ \ell \in \mathbb{Z} \mid m \leq \ell \leq n \} \) and \( [n] := [1, n] \). By \( \leq_{\text{lex}} \) we denote the lexicographical order of tuples and sets of integers. That is, for tuples \( \overline{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k, \overline{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell \) we let \( \overline{m} \leq_{\text{lex}} \overline{n} \) if and only if either there exists an \( i \leq \min \{k, \ell\} \) such that \( m_i < n_i \) and \( m_j = n_j \) for \( 1 \leq j < i \) or \( k \leq \ell \) and \( m_i = n_i \) for all \( i \leq k \). For sets \( M, N \subseteq \mathbb{N} \) we let \( M \leq_{\text{lex}} N \) if and only if either \( M = N \) or there exists an \( i \in N \setminus M \) such that for all \( j < i \) it holds that \( j \in M \iff j \in N \).

We often denote tuples \( (v_1, \ldots, v_k) \) by \( \overline{v} \). For \( \overline{v} = (v_1, \ldots, v_k) \), by \( \overline{v} \) we denote the set \( \{v_1, \ldots, v_k\} \). Furthermore, for a mapping \( f \) whose domain contains \( \overline{v} \) as a subset we let \( f(\overline{v}) := (f(v_1), \ldots, f(v_k)) \). For an element \( w \) we let \( \overline{v}w := (v_1, \ldots, v_k, w) \), and for a tuple \( \overline{w} = (w_1, \ldots, w_\ell) \) we let \( \overline{vw} := (v_1, \ldots, v_k, w_1, \ldots, w_\ell) \). By \( |\overline{v}| \) we denote the length of \( \overline{v} \), that is, \( |(v_1, \ldots, v_k)| = k \). An \( \ell \)-tuple \( (w_1, \ldots, w_\ell) \) is a subtuple of a \( k \)-tuple \( (v_1, \ldots, v_k) \) if there are \( i_1, \ldots, i_\ell \in [k] \) such that \( i_1 < i_2 < \ldots < i_\ell \) and \( w_j = v_{i_j} \) for all \( j \in [\ell] \). If \( \overline{v} \in V^k \) and \( W \) is an arbitrary set, then by \( \overline{v} \cap W \) we denote the subtuple of \( \overline{v} \) consisting of all entries in \( W \), and by \( \overline{v} \setminus W \) we denote the subtuple \( \overline{v} \cap (V \setminus W) \).

The power set of a set \( S \) is denoted by \( 2^S \), and the set of all \( k \)-element subsets of \( S \) by \( \binom{S}{k} \). For a mapping \( f \) defined on \( S \), we let \( f(S) := \{ f(s) \mid s \in S \} \). If \( \equiv \) is an equivalence relation on a set \( S \), then \( s/\equiv \) denotes the equivalence class of an element \( s \in S \), and \( S/\equiv \) denotes the set of all equivalence classes. For a tuple \( \overline{s} = (s_1, \ldots, s_k) \in S^k \) we let \( \overline{s}/\equiv := (s_1/\equiv, \ldots, s_k/\equiv) \), and for a \( k \)-ary relation \( R \subseteq S^k \) we let \( R/\equiv := \{ \overline{s}/\equiv \mid \overline{s} \in R \} \). If \( S \) is a set of sets, then \( \bigcup S \) denotes the union of all sets in \( S \). The cardinality of a set \( S \) is denoted by \( |S| \).

2.2 Graphs and Structures

In this section, we briefly review the graph theoretic concepts used in this book, and at the same time introduce our notation. For background in graph theory, I refer the reader to Diestel’s textbook \[29\].

A directed graph is a pair \( G = (V(G), E(G)) \), where \( V(G) \) is a finite set, the vertex set, and \( E(G) \) is a binary relation on \( V(G) \), the edge relation. A graph is a directed graph with an irreflexive and symmetric edge relation. Thus graphs are always finite and simple, where simple means that they have no loops or parallel edges, and unless explicitly called “directed”,
graphs are undirected. We view the edges of a graph $G$ as 2-element subsets of $V(G)$ and use notations like $e = \{v, w\}$ and $v \in e$. Actually, for brevity we denote edges by $vw$ instead of $\{v, w\}$. We usually do not distinguish between the edge relation (a subset of $V(G)^2$) and the set of all edges (a subset of $V(G)$) and denote both by $E(G)$. If $e = vw \in E(G)$, then we say that $v, w$ are the endvertices of $e$, and that $v$ and $w$ are neighbours. We also say that $v$ and $w$ are incident to $e$, and that $v$ and $w$ are adjacent. We always assume that the vertex set and edge set of a graph are disjoint. The empty graph with empty vertex and edge set is denoted by $\emptyset$. The class of all graphs is denoted by $\mathcal{G}$.

### 2.2.1 Basic Notions

In the following, let $G, H$ be graphs. The union $G \cup H$ of $G$ and $H$ is defined by $V(G \cup H) := V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$, and the intersection $G \cap H$ is defined analogously. If $I = G \cup H$ and $V(G) \cap V(H) = \emptyset$ (and thus $E(G) \cap E(H) = \emptyset$), then we say that $I$ is the disjoint union of $G$ and $H$ and write $I = G \uplus H$. For every class $\mathcal{C}$ of graphs, $\cup(\mathcal{C})$ denotes the closure of $\mathcal{C}$ under disjoint unions, that is, the class of all graphs $G = G_1 \uplus \ldots \uplus G_m$, where $m \in \mathbb{N}$ and $G_1, \ldots, G_m \in \mathcal{C}$. (For $m = 0$, we interpret $G_1 \uplus \ldots \uplus G_m$ as the empty graph.)

$H$ is a subgraph of $G$ (denoted by $H \subseteq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The graph $H$ is an induced subgraph of $G$ if $H \subseteq G$ and $E(H) = E(G) \cap \left(\binom{V(H)}{2}\right)$. We write $G[W]$ to denote the induced subgraph of $G$ with vertex set $W \subseteq V(G)$. For an arbitrary set $S$ we let $G \setminus S := G[V(G) \setminus S]$, and for a graph $H$ we let $G \setminus H := G \setminus V(H)$. For a vertex $w \in V(G)$ we let $G \setminus w := G \setminus \{w\}$. For a set $F \subseteq \left(\binom{V(G)}{2}\right)$ we let $G - F := (V(G), E(G) \setminus F)$, and $G + F := (V(G), E(G) \cup F)$. Sometimes, it is convenient to use these notations even for sets $F$ that are not subsets of $\left(\binom{V(G)}{2}\right)$, then they are meant to refer to the set $F' := F \cap \left(\binom{V(G)}{2}\right)$. Furthermore, if $F = \{f\}$ is a singleton set, we may write $G - f$ and $G + f$ instead of $G - \{f\}$ and $G + \{f\}$.

The order of $G$ is the number $|G| := |V(G)|$ of vertices of $G$. We let $\|G\| := |E(G)|$. For every $k \in \mathbb{N}$, the class of all graphs of order at most $k$ is denoted by $\mathcal{G}_k$. A graph $G$ is a $k$-enlargement of a graph $H$ if $H \subseteq G$ and $|G \setminus H| \leq k$. For class $\mathcal{C}$ of graphs, $\mathcal{N}_k(\mathcal{C})$ denotes the class of all $k$-enlargements of graphs in $\mathcal{C}$.

For a mapping $f$ defined on $V(G)$, we let

$$f(G) := \left( f(V(G)), \{f(v)f(w) \mid vw \in E(G)\}\right).$$

Note that $f(G)$ is not necessarily a simple graph; it may have loops. A homomorphism from $G$ to $H$ is a mapping $h : V(G) \to V(H)$ such that $h(G) \subseteq H$. An embedding is an injective homomorphism. A homomorphism or embedding $h$ is strong if $h(G)$ is an induced subgraph of $H$. An isomorphism is a bijective strong homomorphism. The graphs $G$ and $H$ are isomorphic (denoted by $G \cong H$) if there is an isomorphism from $G$ to $H$. An automorphism of a graph $G$ is an isomorphism from $G$ to $G$. We always assume classes of graphs to be closed under isomorphism. Sometimes, in particular in an algorithmic or complexity theoretic context, we refer to (isomorphism closed) classes of graphs as properties of graphs.

The set of all edges incident to a vertex $v \in V(G)$ is denoted by $E^G(v)$, and the set of all neighbours of $v$ is denoted by $N^G(v)$. For a set $W \subseteq V(G)$, we let $N^G(W) := \bigcup_{w \in W} N^G(w) \setminus W$, and for a subgraph $H \subseteq G$ we let $N^G(H) := N^G(V(H))$. We omit the index $G$ if $G$ is clear from the context, and we do the same for similar notations introduced later. The boundary of a subset $W \subseteq V(G)$ is the set $\partial^G(W) := N(G \setminus W) = \{w \in W \mid \exists v \in V(G) \setminus W : vw \in E(G)\}$. 

M. Grohe, Definable Graph Structure Theory
The boundary of a subgraph $H \subseteq G$ is the set $\partial^G(H) := \partial(V(H))$. The degree of a vertex $v \in V(G)$ is the number $\deg^G(v) := |N(v)|$. A branch vertex of $G$ is a vertex of degree at least 3. The minimum degree of $G$ is $\min\{\deg(v) \mid v \in V(G)\}$, and the maximum degree is defined analogously. $G$ is $k$-regular, for some $k \in \mathbb{N}$, if $\deg^G(v) = k$ for all $v \in V(G)$.

For every finite set $V \neq \emptyset$, we let $K[V]$ be the complete graph with vertex set $V$ and edge set $\binom{V}{2}$. For every $n \in \mathbb{N}^+$, we let $K_n := K[\{n\}]$. For all $m,n \in \mathbb{N}^+$, we let $K_{m,n}$ be the complete bipartite graph with vertex set $\{m+n\}$ and edge set $\{ij \mid i \in [m], j \in [m+1,n]\}$. A clique in a graph $G$ is a set $W \subseteq V(G)$ such that $G[W]$ is a complete graph. A $k$-clique in $G$ is a clique $W$ in $G$ of order $|W| = k$. An independent set in $G$ is a set $W \subseteq V(G)$ such that $E(G[W]) = \emptyset$. A matching of $G$ is a set of mutually disjoint edges of $G$.

### 2.2.2 Paths, Cycles, and Connectivity

A path is a graph $P$ with vertex set $V(P) = \{v_0, \ldots, v_n\}$, for some $n \in \mathbb{N}$, and edge set $E(P) := \{v_{i-1}v_i \mid i \in [n]\}$. We call $v_0$ and $v_n$ the endvertices of $P$ we also say that $P$ is a path from $v_0$ to $v_n$. We call $v_1, \ldots, v_{n-1}$ the internal vertices of $P$. The length of a path $P$ is the number $|P|$ of edges of $P$. For every $m \in \mathbb{N}$, the natural path on $[m]$ is the path $P$ of length $(m-1)$ with $V(P) := \{m\}$ and $E(P) := \{i \mid i \in [m-1]\}$. A path in $G$ is a subgraph of $G$ that is a path. An isolated path in $G$ is a path $P \subseteq G$ such that $\deg^G(v) = 2$ for all internal vertices $v$ of $P$. Two paths $P, P'$ are internally disjoint if no internal vertex of $P$ is a vertex of $P'$ and vice versa. The distance $\dist^G(w_1, w_2)$ between two vertices $w_1, w_2 \in V(G)$ is the length of a shortest path from $w_1$ to $w_2$, or $\infty$ if no such path exists. The distance between two vertex sets $W_1, W_2 \subseteq V(G)$ is $\dist^G(W_1, W_2) := \min(|\dist^G(w_1, w_2) \mid w_1 \in W_1, w_2 \in W_2|).

A cycle of length $n \in \mathbb{N}^+, n \geq 3$ is a graph $C$ with vertex set $V(C) = \{v_1, \ldots, v_n\}$ and edge set $E(C) := \{v_{i-1}v_i \mid i \in [2, n]\} \cup \{v_nv_1\}$. A cycle in $G$ is a subgraph of $G$ that is a cycle. A cycle in $G$ is chordless if it is an induced subgraph of $G$. A segment of a path or cycle $Q$ is a path $P$ such that $P \subseteq Q$. For any two vertices $v, w$ of a path $P$, we denote the unique segment of $P$ with endvertices $v, w$ by $vPw$. Note that for a cycle $C$ and vertices $v, w$ there are precisely two, internally disjoint segments of $C$ with endvertices $v, w$.

A walk in $G$ is a sequence $W := v_0e_1v_1e_2\ldots e_nv_n$, where $n \in \mathbb{N}$, $v_0, \ldots, v_n \in V(G)$, $e_1, \ldots, e_n \in E(G)$ such that $v_{i-1}, v_i \in e_i$ for all $i \in [n]$. The length of $W$, denoted by $|W|$, is $n$, and the endvertices of $W$ are $v_0$ and $v_n$. We also say that $W$ is a walk from $v_0$ to $v_n$. We let $V(W) = \{v_0, \ldots, v_n\}$ and $E(W) = \{e_1, \ldots, e_n\}$ and call $G(W) := (V(W), E(W))$ the subgraph of $G$ corresponding to $W$. The walk $W$ is a simple walk if the vertices $v_0, \ldots, v_n$ are pairwise distinct. $W$ is a closed walk if $n \geq 1$ and $v_0 = v_n$, and a simple closed walk if it is a closed walk and $v_1, \ldots, v_n$ are pairwise distinct. Note that if $W$ is a simple walk then $G(W)$ is a path in $G$. If $W$ is a simple closed walk of length at least 3 then $G(W)$ is a cycle in $G$. Conversely, every walk in $G$ corresponds to two simple walks, one in each direction, and every cycle of length $n$ to $2n$ simple closed walks.

A nonempty graph $G$ is connected if for all $v, w \in V(G)$ there is a path from $v$ to $w$ in $G$. A graph is disconnected if it is not connected. The empty graph is disconnected by definition. The class of all connected graphs is denoted by $\mathcal{Z}$. A connected component of a nonempty graph $G$ is a maximal (with respect to inclusion) connected subgraph. A set $W \subseteq V(G)$ is connected if $G[W]$ is connected. A graph $G$ is $k$-connected, for some $k \in \mathbb{N}^+$, if $|G| > k$ and for every $S \subseteq V(G)$ with $|S| < k$ the graph $G \setminus S$ is connected. The class of all $k$-connected graphs is denoted by $\mathcal{Z}_k$. Furthermore, $\mathcal{Z}_k^*$ denotes the union of $\mathcal{Z}_k$ with all complete graphs $K[V]$ with $1 \leq |V| \leq k$. 

Preliminary Version
A set $S \subseteq V(G)$ is a separator of $G$, or separates $G$, if there are vertices $v, w \in V(G) \setminus S$ such that there is a path from $v$ to $w$ in $G$, but not in $G \setminus S$. Usually, we are only interested in separators of connected graphs. A set $W \subseteq V(G)$ that is not a separator is nonseparating, and a subgraph $H \subseteq G$ is nonseparating in $G$ if its vertex set is. The order of a separator $S$ is its cardinality $|S|$. A $k$-separator of $G$ is a separator of $G$ of order $k$. For sets $W_1, W_2 \subseteq V(G)$, a set $S \subseteq V(G)$ separates $W_1$ from $W_2$, or is a $(W_1, W_2)$-separator, if there is no path from a vertex in $W_1 \setminus S$ to a vertex in $W_2 \setminus S$ in the graph $G \setminus S$. The set $S$ is a minimal $(W_1, W_2)$-separator if $S$, but no proper subset of $S$, is a $(W_1, W_2)$-separator. For a proof (actually, three different proofs) of the following basic theorem, I refer the reader to [29].

**Fact 2.2.1 (Menger’s Theorem).** Let $G$ be a graph, $W_1, W_2 \subseteq V(G)$, and $k \in \mathbb{N}$. Then there is a family of $k$ disjoint paths from $W_1$ to $W_2$ if and only if there is no $(W_1, W_2)$-separator of order less than $k$.

We will occasionally use the following corollary to Menger’s Theorem. Let $G$ be a $k$-connected graph and $v \in V(G)$, $W \subseteq V(G)$ with $|W| \geq k$. Then there are distinct vertices $w_1, \ldots, w_k \in W$ and paths $P_i$ from $v$ to $w_i$, for $i \in [k]$, that are mutually internally disjoint.

Let $S \subseteq V(G)$. For a connected component $A$ of $G \setminus S$, the vertices in $N(A) \subseteq S$ are called vertices of attachment of $A$. An $S$-bridge is a subgraph of $G$ consisting of a connected component of $G \setminus S$ together with all its vertices of attachment and all edges with at least one endvertex in the component. More formally, an $S$-bridge is a subgraph $B \subseteq G$ such that $B \setminus S$ is a connected component of $G \setminus S$, and $V(B) \cap S = N(B \setminus S)$, and $E(B) = \{e \in E(G) \mid e \cap (V(B) \setminus S) \neq \emptyset \}$. For a subgraph $H \subseteq G$, an $H$-bridge is either a $V(H)$-bridge or a subgraph consisting of a single edge $e \in E(G) \setminus E(H)$ with both endvertices in $V(H)$. Hence if $H$ is an induced subgraph, an $H$-bridge is just a $V(H)$-bridge. The vertices of attachment of an $S$-bridge or an $H$-bridge $B$ are the vertices in $V(B) \cap S$ or $V(B) \cap V(H)$, respectively.

### 2.2.3 Minors

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. Let us make this formal. For a connected subgraph $A \subseteq G$, we define $G/A$, the graph obtained from $G$ by contracting $A$, as follows: We choose an arbitrary vertex $v_A \notin V(G) \setminus V(A)$, and let $V(G/A) := (V(G) \setminus V(A)) \cup \{v_A\}$. We let $E(G/A) := E(G \setminus A) \cup \{wv_A \mid w \in N^G(A)\}$.

The graph $G/A$ is only unique up to the choice of the vertex $v_A$, but of course for all choices of $v_A$ the resulting graphs are isomorphic, and they have the common induced subgraph $G \setminus A$. If $v_A \in V(A)$, we say that we contract $A$ to $v_A$. For an arbitrary subgraph $B \subseteq G$ with connected components $A_1, \ldots, A_m$, we let

$$G/B := G/A_1/\cdots/A_m.$$  

Note that $G/B$ does not depend on the order in which we contract the components. For a set $F \subseteq E(G)$ we let $G(F) := (\bigcup F, F)$ and $G/F := G(G(F)$. For a single edge $e \in E(G)$ we let $G/e := G/\{e\}$. We say that a graph $H$ is a contraction of a graph $G$ if there is a set $F \subseteq E(G)$ such that $H \cong G/F$. We say that $H$ is a minor of $G$ (we write $G \preceq H$) if $H$ is a contraction of a subgraph of $G$, that is, there is a subgraph $G' \subseteq G$ and a set $F \subseteq E(G')$ such that $H \cong G'/F$.

It is easy to see that $H \preceq G$ if and only if there is a family $(Y_w)_{w \in V(H)}$ of disjoint connected subsets of $V(G)$ such that for all edges $wv' \in E(H)$ there is an edge $vv' \in E(G)$ with $v \in Y_v$. 

M. Grohe, *Definable Graph Structure Theory*
and \( v' \in Y_w \). We call \((Y_w)_{w \in V(H)}\) an image of \( H \) in \( G \) and the sets \( Y_w \) the branch sets of the image. Often, we consider minors \( H \preceq G \) with \( V(H) \subseteq V(G) \). For such minors \( H \), an image \((Y_w)_{w \in V(H)}\) of \( H \) in \( G \) is faithful if for all \( w \in V(H) \) it holds that \( w \in Y_w \).

For every graph \( G \) we let \( \mathcal{M}(G) := \{ H \in \mathcal{G} \mid H \preceq G \} \), and for a class \( \mathcal{C} \) of graphs we let \( \mathcal{M}(\mathcal{C}) := \bigcup_{G \in \mathcal{G}} \mathcal{M}(G) \). A class \( \mathcal{C} \) is a minor ideal if it is closed under taking minors, that is, if \( \mathcal{C} = \mathcal{M}(\mathcal{C}) \). Furthermore, we let \( \mathcal{X}(G) := \{ H \in \mathcal{G} \mid G \not\preceq H \} \) and \( \mathcal{X}(\mathcal{C}) := \bigcap_{G \in \mathcal{G}} \mathcal{X}(G) \). We call \( \mathcal{X}(G) \) the class of graphs excluding \( G \), and for each \( H \in \mathcal{X}(G) \) we say that \( H \) excludes \( G \), or that \( H \) is \( G \)-minor free. Note that all classes \( \mathcal{X}(\mathcal{C}) \) are minor ideals. Conversely, it is easy to see that all minor ideals \( \mathcal{C} \) are of the form \( \mathcal{X}(D) \) for some class \( D \); simply let \( D := \mathcal{G} \setminus \mathcal{C} \).

The following deep theorem due to Robertson and Seymour states that finitely many excluded minors are enough to characterise a minor ideal.

**Fact 2.2.2 (Graph Minor Theorem [112]).** Let \( \mathcal{C} \) be a minor ideal. Then there are graphs \( G_1, \ldots , G_n \) such that

\[
\mathcal{C} = \mathcal{X}(\{G_1, \ldots , G_n\}).
\]

The proof is based on a structure theory for graphs with excluded minors, which is also the basis of our main theorem. We will state the main structure theorem in Chapter 17 (Theorem 17.1.1) and give a brief outline of the main ideas of its proof there. Another consequence of the structure theory for graphs with excluded minors is the following algorithmic result, also due to Robertson and Seymour.

**Fact 2.2.3 ([110]).**

1. For every \( k \in \mathbb{N} \) there is a cubic time algorithm that, given a graph \( G \) and vertices \( s_1, \ldots , s_k, t_1, \ldots , t_k \in V(G) \), decides if there are mutually disjoint paths \( P_1, \ldots , P_k \subseteq G \) such that for every \( i \in [k] \) the endvertices of \( P_i \) are \( s_i \) and \( t_i \).

2. For every graph \( H \) there is a cubic time algorithm that decides whether a given graph contains \( H \) as a minor.

Combined with the Graph Minor Theorem, this yields the following far reaching algorithmic result.

**Corollary 2.2.4.** Every minor ideal is decidable in cubic time.

A graph \( G \) is a subdivision of a graph \( H \) if \( G \) is obtained from \( H \) by replacing each edge \( e = vw \in E(H) \) by a path \( P_e \) of positive length from \( v \) to \( w \) such that for all edges \( e \in E(H) \) the path \( P_e \) is internally disjoint from \( V(H) \) and for distinct edges \( e, e' \in E(H) \) the paths \( P_e \) and \( P_{e'} \) are internally disjoint. \( H \) is a topological subgraph (or topological minor) of \( G \) if there is a subgraph \( G' \subseteq G \) that is isomorphic to a subdivision of \( H \). It is easy to see that if \( H \) is a topological subgraph of \( G \) then it is a minor of \( G \). If \( H \) is a graph of maximum degree at most 3, then the converse holds as well, that is, \( H \) is a minor of \( G \) if and only if it is a topological subgraph. A topological subgraph ideal is a class of graphs that is closed under taking topological subgraphs.

### 2.2.4 Trees and Forests

A forest is an undirected acyclic graph, and a tree is a connected forest. Recall that the empty graph is disconnected by definition, so trees are always nonempty. The class of all trees is denoted by \( T \). It will be a useful convention to call the vertices of trees and forests nodes. For any two nodes \( t, u \in V(T) \) of a tree \( T \) there is a unique path from \( t \) to \( u \) in \( T \); we denote...
this path by $tTu$. A rooted tree is a triple $T = (V(T), E(T), r(T))$, where $(V(T), E(T))$ is a tree and $r(T) \in V(T)$ is a distinguished node called the root.

A spanning tree of a graph $G$ is a tree $T \subseteq G$ with $V(T) = V(G)$.

### 2.2.5 Directed Graphs

Recall that a directed graph, or digraph, is a pair $D = (V(D), E(D))$, where $V(D)$ is the finite vertex set and $E(D) \subseteq V(D)^2$. We do not require $E(D)$ to be irreflexive, that is, we allow digraphs to have loops. Edges of a digraph are ordered pairs of vertices, which we usually denote by $vw$ instead of $(v, w)$. For an edge $e = vw$ we call $w$ the head and $v$ the tail of $e$. We use standard graph theoretic terminology for digraphs, without going through it in detail. To avoid the awkward term “subdigraph”, subgraphs of a digraph are understood to be digraphs as well. Paths and cycles in a digraph are always meant to be directed; otherwise we call them “paths or cycles of the underlying undirected graph”. The head and tail of a directed path are defined in the obvious way. Note that cycles in digraphs may have length 1 or 2. For a digraph $D$ and a vertex $v \in V(D)$, we let $N_D^+(v) := \{w \in V(D) \mid vw \in E(D)\}$ and $N_D^-(v) := \{w \in V(D) \mid wv \in E(D)\}$. The out-degree of $v$ is the number $|N_D^+(v)|$, and the in-degree of $v$ is $|N_D^-(v)|$. For vertices $v, w \in V(D)$ we let $\text{dist}^D(v, w)$ be the length of the shortest (directed) path from $v$ to $w$ or $\infty$ if no such path exists. (Thus in general, $\text{dist}^D$ is not symmetric.) Homomorphisms, strong homomorphisms, embeddings, and isomorphisms between digraphs are required to preserve the direction of the edges.

Directed acyclic graphs will be of particular importance in this paper, and we introduce some additional terminology for them. Let $D$ be a directed acyclic graph. A node $w$ is a child of a node $v$, and $v$ is a parent of $w$, if $vw \in E(D)$. We let $\leq^D$ be the reflexive transitive closure of the edge relation $E(D)$ and $\prec^D$ its irreflexive version. Then $\leq^D$ is a partial order on $V(D)$. If $v \leq^D w$ then we say that $v$ is an ancestor of $w$ and $w$ is a descendant of $v$. We call $\leq^D$-minimal elements roots of $D$ and $\leq^D$-maximal elements leaves. Thus roots are precisely the vertices of in-degree 0 and leaves are the vertices of out-degree 0. The height $\text{ht}^D(v)$ of a vertex $v$ in a directed acyclic graph $D$ is the maximum length of a (directed) path $Q \subseteq D$ from a root to $v$, and the depth $\text{dep}^D(v)$ of $v$ is the maximum length of a path $Q \subseteq D$ from $v$ to a leaf.

A directed tree is a directed acyclic graph $T$ such that $T$ has exactly one root $r$, and all vertices $t \in V(T) \setminus \{r\}$ have in-degree 1. There is an obvious one-to-one correspondence between rooted trees and directed trees: for a rooted tree $T = (V(T), E(T), r(T))$ we define the corresponding directed tree $T'$ by $V(T') := V(T)$ and $E(T') := \{(t, u) \mid \{t, u\} \in E(T) \text{ and } t \text{ occurs on the path } rT'u\}$. We freely jump back and forth between rooted trees and directed trees, depending on which will be more convenient. In particular, we use the terminology introduced for directed acyclic graphs (roots, leaves, parents, children, ancestors, et cetera) for rooted trees. Note that the height of a node in a rooted or directed tree is its distance from the root, and its depth is the maximum distance to a leaf in the subtree rooted at this vertex. Let $T$ be a directed tree. A subtree of a directed tree $T$ is subgraph $T' \subseteq T$ that is a directed tree. A set $U \subseteq V(T)$ induces a subtree of $T$ if $T[U]$ is a subtree of $T$. Observe that $U \subseteq V(T)$ induces a subtree if and only if it is connected in the undirected tree underlying $T$. Furthermore, as (directed and undirected) trees are always nonempty, a set $U \subseteq V(T)$ that induces a subtree is nonempty.

M. Grohe, Definable Graph Structure Theory
2.2.6 Relational Structures

A vocabulary is a finite set of relation symbols, each of which has an arity \( k \in \mathbb{N} \). In the following, let \( \tau \) be a vocabulary. A structure \( A \) of vocabulary \( \tau \) (for short: \( \tau \)-structure) consists of a finite vertex set \( V(A) \) and a relation \( R(A) \subseteq V(A)^k \) for every \( k \)-ary relation symbol \( R \in \tau \). We call a structure \( A \) numerical if \( V(A) \subseteq \mathbb{N} \). To improve readability, for binary relation symbols such as \( \leq \) we write \( \leq^A \) instead of \( \leq(A) \) and use infix notation.

Graphs and digraphs are structures of vocabulary \( \{ E \} \), where \( E \) is a binary relation symbol. Actually, graphs and digraphs are almost the only structures we consider in this book.

We can easily generalise graph theoretic notions such as subgraphs (yielding substructures) and induced subgraphs (yielding induced substructures), union and intersection of structures, the order of a structure, (strong) homomorphisms and embeddings and isomorphisms from graphs to arbitrary structures. For vocabularies \( \tau \subseteq \nu \), the \( \tau \)-restriction of a \( \nu \)-structure \( A \) is the \( \tau \)-structure \( A|\tau \) with \( V(A|\tau) := V(A) \) and \( R(A|\tau) := R(A) \) for all \( R \in \tau \). A \( \nu \)-structure \( A \) is a \( \nu \)-expansion of a \( \tau \)-structure \( B \) if \( A|\tau = B \).

Throughout this book, we let \( \leq \) be a distinguished binary relation symbol. An ordered structure is a structure \( A \) whose vocabulary contains \( \leq \) such that \( \leq^A \) is a linear order on \( V(A) \). In particular, an ordered graph is an \( \{ E, \leq \} \)-structure \( G = (V(G), E(G), \leq^G) \) where \( (V(G), E(G)) \) is a graph and \( \leq^G \) is a linear order of \( V(G) \). Note the difference between the symbols \( \leq \) and \( \leq \); the latter is used to denote the natural order of the integers. (No confusion should arise by mixing up the two symbols anywhere in this book, though, as it will always be clear from the context which symbol is required.)

Classes of structures may contain structures of different vocabularies. If \( \mathcal{C} \) is a class of structures and \( \tau \) a vocabulary, then \( \mathcal{C}|\tau \) denotes the class of all \( \tau \)-structures in \( \mathcal{C} \). The class of all structures is denoted by \( \mathcal{S} \), and the class of all ordered structures is denoted by \( \mathcal{O} \). As classes of graphs, we always assume classes of structures to be closed under isomorphism. We sometimes call classes of \( \tau \)-structures properties of \( \tau \)-structures.

2.2.7 Notational Conventions

Let us close this section by fixing some notational conventions that will serve as a guideline, though will not always be followed strictly. Vocabularies will always be denoted by \( \tau \) or \( \nu \) and variants such as \( \tau' \) or \( \nu_1 \). Integers will be usually denoted by \( i, j, k, \ell, m, n \). Vertices of structures will usually be denoted by \( s, t, u, v, w \). We use the letters \( p, q, r \) to refer to either vertices of a structure or integers. Structures are usually denoted by \( A, B, C, D \), or \( G, H \) if they are graphs, or \( T \) if they are trees. Classes of graphs or structures will be denoted by calligraphic letters such as \( \mathcal{C} \).

2.3 Logics

In this section, we introduce the main logics used in this book, inflationary fixed-point logic and inflationary fixed-point logic with counting. A thorough introduction into the logics and their expressive power can be found in the textbooks [31, 39, 86] and the monograph [95].

Throughout this section, let \( \tau \) be a vocabulary.

We start by recalling first-order logic \( \text{FO} \). \( \text{FO} \) has individual variables ranging over the vertices of the structure at hand. Atomic \( \text{FO} \)-formulae of vocabulary \( \tau \) are of the form \( x = y \), where \( x, y \) are individual variables, and \( R\bar{x} \), where \( R \in \tau \) and \( \bar{x} \) is a tuple of individual
variables whose length matches the arity of $R$. Arbitrary FO-formulae of vocabulary $\tau$ are formed from atomic formulae by applying the standard Boolean connectives $\neg$ (negation), $\land$ (conjunction), $\lor$ (disjunction), $\rightarrow$ (implication) and quantifiers $\exists x$ (existential) and $\forall x$ (universal). We interpret FO-formulae of vocabulary $\tau$ in $\tau$-interpretations $(A, \alpha)$, where $A$ is a nonempty $\tau$-structure and $\alpha$ an assignment that associates a vertex $\alpha(x) \in V(A)$ with each individual variable $x$. The meaning of the formulae is the obvious one; we just mention that an interpretation $(A, \alpha)$ satisfies an atomic formula $R \bar{x}$ if $\alpha(\bar{x}) \in R(A)$. It may not be entirely obvious how to extend the semantics to the empty structure, but this can be done in a rather straightforward way.\footnote{The meaning of formulae in the empty structure is not relevant anywhere in this book, and we ignore this issue in the following.}

We write $(A, \alpha) \models \varphi$ to denote that an interpretation $(A, \alpha)$ satisfies a formula $\varphi$.

We denote the class of all FO-formulae of vocabulary $\tau$ by $\text{FO} [\tau]$. More generally, for any logic $L$ we denote the set of $L$-formulae of vocabulary $\tau$ by $L [\tau]$. Before we continue, we introduce a few more notational conventions: We write $\text{true}$ to abbreviate the formula $\forall x \, x = x$ and $\text{false}$ to abbreviate $\neg \text{true}$. We write $x \neq y$ instead of $\neg (x = y)$. For tuples $\bar{x} = (x_1, \ldots, x_k)$ and $\bar{y} = (y_1, \ldots, y_k)$ we write $\bar{x} = \bar{y}$ to abbreviate $x_1 = y_1 \land \ldots \land x_k = y_k$.

**Example 2.3.1.** The following FO-sentence in the vocabulary $\{E\}$ of graphs defines the class of (simple undirected) graphs.

$$\text{graph} := \forall x \neg E(x, x) \land \forall x \forall y (E(x, y) \rightarrow E(y, x)).$$

That is, for every $\{E\}$-structure $G$ we have $G \models \text{graph}$ if and only if $G$ is a graph.

**Example 2.3.2.** The following FO-sentence in the vocabulary $\{\leq\}$ defines the class of all linearly ordered structures.

$$\text{linear-order} := \forall x \, x \leq x$$
$$\land \forall x \forall y (x \leq y \land y \leq x) \rightarrow x = y$$
$$\land \forall x \forall y \forall z (x \leq y \land y \leq z) \rightarrow x \leq z$$
$$\land \forall x \forall y (x \leq y \lor y \leq x).$$

### 2.3.1 Inflationary Fixed-Point Logic

**Inflationary fixed-point logic** IFP is an extension of first-order logic by a fixed-point operator that allows it to formalise inductive definitions.

Let us first define the syntax. Besides individual variables, IFP also has relation variables of prescribed arities. A $k$-ary relation variable ranges over $k$-ary relations over the vertex set of the structure at hand. In addition to the atomic formulae of FO, IFP has atomic formulae of the form $X \bar{x}$, where $X$ is a relation variable and $\bar{x}$ is a tuple of individual variables whose length matches the arity of $X$. Arbitrary IFP-formulae are formed from atomic formulae by applying the standard Boolean connectives $\neg, \land, \lor, \rightarrow$, first-order quantifiers $\exists x$, $\forall x$ (but no second-order quantifiers applying to relation variables), and an **inflationary fixed-point operator** ifp with the following syntax. If $\varphi$ is an IFP-formula, $X$ is a $k$-ary relation variable, and $\bar{x}, \bar{x}'$ are $k$-tuples of individual variables, then

$$\text{ifp} \left( X \bar{x} \leftarrow \varphi \right) \bar{x}'$$

(2.3.1)
is a new IFP-formula.

As FO-formulae, we interpret IFP-formulae in interpretations \((A, \alpha)\), but now the assignment \(\alpha\) not only assigns vertices to individual variables, but also a \(k\)-ary relation \(\alpha(X) \subseteq V(A)^k\) to each \(k\)-ary relation variable \(X\). We need an additional piece of notation. If \(\alpha\) is an assignment in a structure \(A\) and \(x\) an individual variable and \(v \in V(A)\), we let \(\alpha(v/x)\) be the assignment with \(\alpha(x) = v\) and \(\alpha(v/y)(Y) = \alpha(Y)\) for all (individual or relation) variables \(Y \neq x\). We use a similar notation \(\alpha(R/X)\) for relation variables and also combine several modifications of \(\alpha\), as in \(\alpha(R/X, \pi/\pi')\).

The semantics of IFP extends the semantics of FO as follows. An interpretation \((A, \alpha)\) satisfies an atomic formula \(X\varphi\) if \(\alpha(\varphi) \in \alpha(X)\). To define the semantics of the ifp-operator, consider the formula displayed in (2.3.1), and let \((A, \alpha)\) be an interpretation. We define a mapping \(F : 2^{V(A)} \rightarrow 2^{V(A)^k}\) on the \(k\)-ary relations \(R \subseteq V(A)^k\) by

\[
F(R) := \{ \pi \in V(A)^k \mid (A, \alpha(R/X, \pi/\pi')) \models \varphi \}. \quad (2.3.2)
\]

We inductively define a sequence \((X^i)_{i \in \mathbb{N}}\) of \(k\)-ary relations on \(V(A)\) as follows:

\[
X^0 := \emptyset, \quad (2.3.3)
\]

\[
X^{i+1} := X^i \cup F(X^i) \quad \text{(for all } i \in \mathbb{N}). \quad (2.3.4)
\]

Note that we have \(X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq V(A)^k\), and as \(V(A)\) is a finite set, the sequence reaches a fixed point, which we denote by \(X^\infty\). Then

\[
(A, \alpha) \models \text{ifp}(X\varphi \leftarrow \varphi) \iff \alpha(\varphi) \in X^\infty.
\]

This completes the definition of IFP.

The free variables of an FO-formula are defined in the usual way (variables are free unless they are in the range of a quantifier binding them), and to define the free variables of an IFP-formula we let the free variables of a formula \(\text{ifp}(X\varphi \leftarrow \varphi)\varphi'\) be all variables in \(\varphi'\) and all free variables of \(\varphi\) except \(X\) and the variables in \(\overline{\varphi}\). A formula without free variables is a sentence. Obviously, it only depends on the values \(\alpha\) assigns to the free variables of a formula \(\varphi\) whether \((A, \alpha) \models \varphi\). In particular, if \(\varphi\) is a sentence it only depends on \(A\) if \((A, \alpha) \models \varphi\). This allows us to write \(A \models \varphi\) instead of \((A, \alpha) \models \varphi\) for sentences \(\varphi\).

We often annotate formulae \(\varphi\) by tuples of variables, as in \(\varphi(X_1, \ldots, X_k)\). Here \(X_1, \ldots, X_k\) can be individual or relation variables. If we write \(\varphi(X_1, \ldots, X_k)\), this always implies that all free variables of \(\varphi\) are in \(\{X_1, \ldots, X_k\}\). Instead of \((A, \alpha) \models \varphi\) we may now write \(A \models \varphi[\alpha(X_1), \ldots, \alpha(X_k)]\). Another use of the notation \(\varphi(X_1, \ldots, X_k)\) is to denote substitutions of individual variables by other individual variables and of relation variables by formulae. For example, let \(X\) be a binary relation variable and \(x_1, x_2, y, z\) individual variables. Let \(\psi(X, y)\) and \(\varphi(x_1, x_2)\) be IFP-formulae. Then \(\psi(\varphi, z)\) denotes the formula obtained from \(\psi\) by substituting \(X\) by \(\varphi\) and \(y\) by \(z\), possibly renaming bound variables of \(\psi\) if they clash with \(z\) or the free variables of \(\varphi\).

**Example 2.3.3.** Consider the IFP-formula

\[
\text{path}(x, y) := \text{ifp} \left( Y(y) \leftarrow x = y \lor \exists z(Y(z) \land E(z, y)) \right)(y)
\]
in the vocabulary \{E\} of graphs. Let \(G\) be a graph and \(\alpha\) an assignment with \(\alpha(x) =: v \in V(G)\). Then the mapping \(F\) defined as in (2.3.2) (with \(\varphi := (x = y \lor \exists z (Y(z) \land E(z, y)))\)) maps each \(W \subseteq V(G)\) to

\[
F(W) = \{v\} \cup \bigcup_{w \in W} N(w) \subseteq \{v\} \cup W \cup N(W).
\]

Thus for the sequence \((X^i)_{i \in \mathbb{N}}\) defined as in (2.3.3) and (2.3.4) we obtain \(X^1 = \{v\}, X^2 = \{v\} \cup N(v)\), and for all \(i \in \mathbb{N}^+\) the set \(X^i\) consists of all vertices \(w \in V(G)\) with \(\text{dist}(v, w) \leq i - 1\). Hence

\[
G \models \text{path}[v, w] \iff \text{there is a path from } v \text{ to } w \text{ in } G. \tag{2.3.5}
\]

Furthermore, the sentence

\[
\text{conn} := \exists x \ x = x \land \forall x \forall y \text{ path}(x, y)
\]

is satisfied by a graph \(G\) if and only if \(G\) is connected. Remember that the empty graph is disconnected by definition.

**Example 2.3.4.** Note that the formula \(\text{path}(x, y)\) of the previous example keeps its meaning in digraphs. That is, (2.3.5) also holds if \(G\) is a digraph. Hence the sentence

\[
\text{dag} := \forall x \neg \exists y (\text{path}(x, y) \land E(y, x))
\]

is satisfied by a digraph \(D\) if and only if \(D\) is acyclic.

Consider the sentence

\[
\text{dtree} := \text{dag} \land \exists x \left( \forall y \neg E(y, x) \land \forall x' (x' \neq x \to \exists y E(y, x')) \right) \\
\land \forall x \neg \exists y_1 \exists y_2 (y_1 \neq y_2 \land E(y_1, x) \land E(y_2, x)).
\]

In a digraph \(D\), the first line of the formula says that \(D\) is acyclic. The second line says that \(D\) has exactly one root. The third line says that every node has in-degree at most 1. Thus \(D \models \text{dtree}\) if and only if \(D\) is a directed tree.

I leave it as an exercise for the reader to write an \(\mathcal{IFP}\)-sentence \(\text{tree}\) saying that an undirected graph is a tree.

### 2.3.2 Simultaneous and Deflationary Fixed Points

The inflationary fixed-point operator has a dual *deflationary fixed-point operator*, which we denote by \(\text{dfp}\). The syntax is analogous to that for \(\text{ifp}\). If \(\varphi\) is an \(\mathcal{IFP}\)-formula, \(X\) is a \(k\)-ary relation variable, and \(\overline{x}, \overline{x}'\) are \(k\)-tuples of individual variables then with the \(\text{dfp}\)-operator we may form a new formula

\[
\text{dfp} \left( X \overline{x} \leftarrow \varphi \right) \overline{x}'.
\]

To define the semantics of this new formula, let \((A, \alpha)\) be an interpretation. We define the mapping \(F : 2^{V(A)^k} \to 2^{V(A)^k}\) as in (2.3.2) and a sequence \((X_i)_{i \in \mathbb{N}}\) of \(k\)-ary relations on \(V(A)\) by:

\[
X_0 := V(A)^k,
\]

\[
X_i := \left[\bigcup_{w \in X_{i-1}} N(w) \subseteq \bigcup_{w \in X_{i-1}} \left( X_{i-1} \cup w \cup N(w) \right)\right].
\]

\[
G \models \varphi \iff \text{there is an assignment } \alpha \text{ with } \alpha(x) =: v \in V(G) \text{ such that } (A, \alpha) \models \varphi.
\]
\[ X_{i+1} := X_i \cap F(X_i) \quad \text{(for all } i \in \mathbb{N}). \]

Note that we have \( V(A)^k = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \), and thus the sequence reaches a fixed point, which we denote by \( X_\infty \). Then

\[(A, \alpha) \models \text{dfp} \left( X \xleftarrow{\varphi} \bar{x}' \right) : \iff \alpha(\bar{x}') \in X_\infty. \]

Adding a deflationary fixed-point operator to the logic does not increase the expressive power. Let \( \tilde{\varphi} \) be the formula obtained from \( \neg \varphi \) by replacing each atom of the form \( X \bar{z} \) by \( \neg X \bar{z} \). Then it is not hard to see that the formula \( \text{dfp} \left( X \xleftarrow{\varphi} \bar{x}' \right) \) is equivalent to

\[-\text{ifp} \left( X \xleftarrow{\tilde{\varphi}} \bar{x}' \right) \quad (2.3.7)\]

Hence there is no need to add a deflationary fixed-point operator to our language, but we may use formulae of the form \( (2.3.6) \) as abbreviations of the corresponding formulae \( (2.3.7) \).

Another version of the inflationary fixed-point operator that does not increase the expressive power of the logic, but is often very convenient, is a \textit{simultaneous fixed-point operator}. For some \( m \in \mathbb{N}^+ \), let \( \varphi_1, \ldots, \varphi_m \) be \text{IFP}-formulae. For each \( i \in [m] \), let \( X_i \) be a \( k_i \)-ary relation variable and \( \bar{x}_i \) a \( k_i \)-tuple of individual variables. Furthermore, let \( \bar{x}' \) be another \( k_1 \)-tuple of individual variables. By means of a simultaneous ifp-operator, we may form the following new formula:

\[ \text{ifp} \left( \begin{array}{c} X_1 \bar{x}_1 \leftarrow \varphi_1 \\ \vdots \\ X_m \bar{x}_m \leftarrow \varphi_m \end{array} \right) \bar{x}'. \quad (2.3.8) \]

To define the semantics of this new formula, let \((A, \alpha)\) be an interpretation. For all \( j \in [m] \) we define an operator \( F_j : 2^{V(A)^{k_1}} \times \cdots \times 2^{V(A)^{k_m}} \to 2^{V(A)^{k_j}} \) by

\[
F_j(R_1, \ldots, R_m) := \left\{ \bar{v} \in V(A)^{k_j} \mid (A, \alpha(R_1/X_1, \ldots, R_m/X_m, \bar{v}/\bar{x}_j)) \models \varphi_j \right\},
\]

for all \( R_1 \subseteq V(A)^{k_1}, \ldots, R_m \subseteq V(A)^{k_m} \). Then by induction on \( i \in \mathbb{N} \) we define relations \( X_i^j \subseteq V(A)^{k_j} \) for all \( j \in [m] \) as follows:

\[
X_j^0 := \emptyset, \\
X_j^{i+1} := X_j^i \cup F_j(X_j^0, \ldots, X_j^i) \quad \text{for all } i \in \mathbb{N}.
\]

Again the sequence \((X_1^i, \ldots, X_m^i)_{i \in \mathbb{N}}\) reaches a fixed point, which we denote by \((X_1^\infty, \ldots, X_m^\infty)\). Then

\[ (A, \alpha) \models \text{ifp} \left( \begin{array}{c} X_1 \bar{x}_1 \leftarrow \varphi_1 \\ \vdots \\ X_m \bar{x}_m \leftarrow \varphi_m \end{array} \right) \bar{x}' : \iff \alpha(\bar{x}') \in X_1^\infty. \]

By a result known as the \textit{Lemma on Simultaneous Induction} (see [31] for a proof), every simultaneous inflationary fixed-point formula of the form \( (2.3.8) \) is equivalent to a proper \text{IFP}-formula (without simultaneous fixed-point operators). We shall freely use simultaneous fixed-point operators throughout this book and appeal to the Lemma on Simultaneous Induction without explicitly mentioning it.
2.3.3 Least Fixed-Point Logic

Least fixed-point logic LFP is another fixed-point extension of first-order logic, which is actually much more common than IFP. We may view LFP as the fragment of IFP consisting of all formulae ψ such that for all subformulae of ψ of the form ifp (Xϕ ← ϕ)ψ′ the formula ϕ is positive in X, that is, X only appears in the scope of an even number of negation symbols. Suppose that X is k-ary. It is not hard to show that if ϕ is positive in X then for every interpretation (A, α) the operator F defined as in (2.3.2) is monotone. This implies that X∞ is the least fixed point of the operator F.

If ϕ is positive in X, we may write ifp (Xϕ ← ϕ)ψ instead of ifp (Xϕ ← ϕ)ψ′ and speak of a least fixed-point operator. There is also a dual least fixed-point operator gfp corresponding to the deflationary fixed-point operator dfp.

Surprisingly, LFP has the same expressive power as IFP. On finite structures, this was proved by Gurevich and Shelah [61], and on arbitrary structures by Kreutzer [81]. This expressive equivalence justifies our exclusive use of IFP. The reason we prefer IFP over LFP is that it has a simpler counting extension (to be defined next).

2.3.4 Inflationary Fixed-Point Logic with Counting

Inflationary fixed-point logic with counting IFP+C is an extension of IFP by a counting mechanism that enables the logic to determine the cardinalities of definable sets of vertices and do calculations with these cardinalities. For every structure A, we let Num(A) := [0, |A|]. Note that Num(A) is the set of cardinalities of subsets of V(A).

IFP+C has two types of individual variables: Individual variables of type v, which we usually call vertex variables, range over the vertex set V(A) of the structure A at hand. Vertex variables correspond to the individual variables of FO and IFP. Individual variables of type n, which we usually call number variables, range over Num(A). We make no notational distinction between vertex variables and number variables and denote them by the lowercase letters x, y, z and variants like x′, y1. The type of a tuple (x1, . . . , xk) of individual variables is the tuple (t1, . . . , tk), where ti ∈ {v, n} is the type of xi for all i ∈ [k]. Relation variables are also typed; each k-ary relation variable has a type τ ∈ {v, n}k. A relation variable of type (t1, . . . , tk) ranges over relations R ⊆ U1 × U2 × . . . × Uk, where Ui := V(A) if ti = v and Ui := Num(A) if ti = n. Relation variables of type (v, . . . , v) are called plain. We denote relation variables by the uppercase letters X, Y, Z and variants. Sometimes we use uppercase letters for variables that may be individual or relation variables.

Atomic IFP+C-formulae of vocabulary τ are of the form x = y, where x, y are individual variables of the same type, x ≤ y, where x, y are number variables, Rτ, where R ∈ τ and τ is a tuple of vertex variables whose length matches the arity of R, and Xτ, where X is a relation variable and τ a tuple of individual variables of the same type as X. Arbitrary IFP+C-formulae are formed from atomic formulae by applying the standard Boolean connectives ¬, ∧, ∨, →, first-order quantifiers ∃x, ∀x, where x is an individual variable, an inflationary fixed-point operator that allows us to form new formulae ifp (Xϕ ← ϕ)ψ′, where ϕ is an IFP+C-formula, X is a relation variable, and ψ′ a tuple of individual variables of the same type as X, and a counting operator that allows us to form new formulae, #x ϕ = y, where ϕ is an IFP+C-formula, x is a vertex variable, and y is a number variable.

To define the semantics, let (A, α) be an interpretation. We consider the atomic formulae first. For formulae of the form x = y, Rτ, and Xτ, the semantics is defined as for the
2.3. Logics

corresponding IFP-formulae, except that equality may also be applied to two number variables, ranging over the numbers in \(\text{Num}(A)\), and relation variables \(X\) may by of mixed type. The formula \(x \leq y\) also has the obvious meaning: \((A, \alpha) \models x \leq y \iff \alpha(x) \leq \alpha(y)\), where the inequality sign on the right hand side of this equivalence refers to the natural order of the integers. The Boolean connectives and quantifiers are interpreted as usual, just noting that if \(x\) is a number variable the quantifiers \(\exists x\) and \(\forall x\) range over the elements of \(\text{Num}(A)\). The ifp-operator has the same semantics as in the logic IFP, except that now the relations may be of mixed type. It remains to explain the meaning of the counting operator. Intuitively, \(#x\varphi = y\) says that the number of \(x\) satisfying \(\varphi\) is \(y\). Formally,

\[
(A, \alpha) \models #x\varphi = y \iff |\{v \in V(A) \mid (A, \alpha(v/x)) \models \varphi\}| = \alpha(y).
\]

This completes the definition of IFP+C.

The Lemma on Simultaneous Induction also holds for IFP+C (see [37] for a proof). Hence we may also use simultaneous fixed-point operators in IFP+C-formulae. It is easy to see that we may also use deflationary fixed-point operators without increasing the expressive power.

To illustrate the expressiveness of IFP+C, we give a few examples.

**Example 2.3.5.** In this example, we show how to define basic arithmetic on the numbers. All individual variables occurring in the formulae are number variables. Let \(A\) be a structure of an arbitrary (possibly empty) vocabulary.

1. The IFP+C-formula \(\text{zero}(x) := \forall y \ x \leq y\) defines the number 0. That is, for all \(i \in \text{Num}(A)\) it holds that \(A \models \text{zero}[i] \iff i = 0\). In the following, we write \(x = 0\) instead of \(\text{zero}(x)\) in IFP+C-formulae.

Similarly, the formula \(\text{one}(x) := \neg \text{zero}(x) \land \forall y (\text{zero}(y) \lor x \leq y)\) defines 1 in all nonempty structures \(A\), the formula \(\text{two}(x) := \neg \text{zero}(x) \land \neg \text{one}(x) \land \forall y (\text{zero}(y) \lor \text{one}(y) \lor x \leq y)\) defines 2 in all structures \(A\) with at least two elements, et cetera. For all \(i \in \mathbb{N}\), we write \(x = i\) to denote the formula defining \(i\), and we also use notations such as \(x \neq i\), \(x \leq i\), \(x < i\), \(x \geq i\), \(x > i\) as abbreviations for corresponding formulae. For example, \(x > 1\) abbreviates \(\exists y(y = 1 \land y < x)\). Here we write \(y < x\) instead of \(y \leq x \land \neg y = x\), and we use similar abbreviations elsewhere.

Finally, the IFP+C-formula \(\text{largest}(x) := \forall y \ y \leq x\) defines the number \(|A|\).

2. The IFP+C-formula \(\text{succ}(x, y) := x < y \land \forall z (z \leq x \lor y \leq z)\) defines the successor relation on \(\text{Num}(A)\). That is, for all \(i, j \in \text{Num}(A)\) it holds that \(A \models \text{succ}[i, j] \iff i + 1 = j\). In the following, we write \(x + 1 = y\) instead of \(\text{succ}(x, y)\).

3. The IFP+C-formula

\[
\text{plus}(x_1, x_2, x_3) := \text{ifp}\left(Y(y_1, y_2) \leftarrow \left(\begin{array}{c}
y_1 = 0 \land y_2 = x_2 \\
\exists z_1 \exists z_2 (Y(z_1, z_2) \\
\land z_1 + 1 = y_1 \land z_2 + 1 = y_2)
\end{array}\right)\right) (x_1, x_3)
\]

defines the graph of the addition function. That is, for all \(i_1, i_2, i_3 \in \text{Num}(A)\) it holds that

\[
A \models \text{plus}[i_1, i_2, i_3] \iff i_1 + i_2 = i_3.
\]

In the following, we write \(x_1 + x_2 = x_3\) instead of \(\text{plus}(x_1, x_2, x_3)\).
Example 2.3.6. Consider the IFP+C-formula

$$\text{times}(x_1, x_2, x_3) := \text{ifp} \left( Y(y_1, y_2) \leftarrow \left( (y_1 = 0 \land y_2 = 0) \right) \lor \exists z_1 \exists z_2 (Y(z_1, z_2) \land z_1 + 1 = y_1 \land z_2 + x_2 = y_2) \right) (x_1, x_3)$$

defines the graph of the multiplication function. In the following, we write $x_1 \cdot x_2 = x_3$ instead of $\text{times}(x_1, x_2, x_3)$.

(5) The IFP+C-formula $\text{even}(x) := \exists y \ y + y = x$ defines the set of even numbers in $\text{Num}(A)$. \hfill \diamond

Example 2.3.7. A Eulerian cycle in a graph is a closed walk on which every edge occurs exactly once. It is a well-known fact that a graph has a Eulerian cycle if and only if it is connected and every vertex has even degree.

The formula $\text{even-deg}(x) := \exists y \left( \#x' \ E(x, x') = y \land \text{even}(y) \right)$, where $x, x'$ are vertex variables, $y$ is a number variable, and $\text{even}$ is the formula defined in Example 2.3.5(5), says that vertex $x$ has even degree. Hence the IFP+C-sentence

$$\text{conn} \land \forall x \ \text{even-deg}(x),$$

where conn is the IFP-sentence defined in Example 2.3.3, says that a graph has a Eulerian cycle.

Remark 2.3.8. For the sake of completeness, let us also define first-order logic with counting FO+C. Essentially, FO+C is the fragment of IFP+C consisting of all formulae without fixed-point operators. However, it has turned out to be useful to add arithmetic on the numbers to FO+C. For this reason, we add atomic formulae $x + y = z$ and $x \cdot y = z$ for number variables $x, y, z$, with the obvious meaning.

We saw in Example 2.3.5 that addition and multiplication on the numbers is definable in IFP+C. Hence every FO+C-formula is equivalent to an IFP+C-formula.

By standard techniques from finite model theory, it can be proved that there is neither an IFP-sentence nor an FO+C-sentence defining the class of graphs with a Eulerian cycle. Hence IFP+C is strictly more expressive than both IFP and FO+C. The two logics IFP and FO+C are incomparable in expressive power: it can be proved that there is no IFP-sentence
saying that the vertex set of a structure has even cardinality (cf. Example 3.4.6), but this property is expressible by the FO+C-sentence $\exists x \exists y (x + x = y \land \forall z \ z \leq y)$. Conversely, it can be proved that there is no FO+C-sentence defining connectedness, but we have already seen that connectedness is expressible in IFP. Both IFP an FO+C are strictly more expressive than FO. Indeed, it is not hard to prove that neither the class of connected graphs nor the class of graphs with an even number of vertices are definable in FO.

2.3.5 Signatures, Assignments, and Interpretations

We introduce some additional terminology and notation for dealing with free variables in formulae. A signature is a set of variables. An assignment for a signature $\xi$ in a structure $A$ is a mapping $\alpha$ that associates an element $\alpha(x) \in V(A)$ with each vertex variable $x \in \xi$, an integer $\alpha(y) \in \text{Num}(A)$ with each number variable $y \in \xi$, and a relation $\alpha(X)$ of appropriate type with each relation variable $X \in \xi$. The set of all assignments for $\xi$ in $A$ is denoted by $A^{\xi}$. For a vocabulary $\tau$ and a signature $\xi$, a $(\tau, \xi)$-interpretation is a pair $(A, \alpha)$ consisting of a $\tau$-structure $A$ and an assignment $\alpha \in V(A)^\xi$. A $\tau$-interpretation is a $(\tau, \xi)$-interpretation for the set $\xi$ of all variables. Most of the time we are only concerned with graphs. A graph interpretation for a signature $\xi$ is a pair $(G, \alpha)$, where $G$ is a graph and $\alpha \in G^\xi$.

It is often more convenient to view signatures as tuples rather than sets. We do so without further mentioning it. That is, if $X_1, \ldots, X_k$ are variables, then we often denote the signature $\{X_1, \ldots, X_k\}$ by $(X_1, \ldots, X_k)$. We denote assignments for this signature as tuples $(P_1, \ldots, P_k)$, interpretations in the form $(A, P_1, \ldots, P_k)$, and the set of all assignments by $A^{(X_1, \ldots, X_k)}$. Note that if $(x_1, \ldots, x_k)$ is a tuple of distinct vertex variables, then $A^{(x_1, \ldots, x_k)} = V(A)^k$.

We may view formulae with free variables as defining sets of assignments. Let $A$ be a structure. If $\varphi$ is a formula and $\xi$ a signature that contains all free variables of $\varphi$, then we may view $\varphi$ as defining the set

$$\{ \alpha \in A^\xi \mid (A, \alpha) \models \varphi \}.$$

To denote the relations defined by formulae, it is again useful to annotate formulae with variables. For a formula $\varphi(X_1, \ldots, X_k)$ (with free variables among $X_1, \ldots, X_k$), we let

$$\varphi[A, X_1, \ldots, X_k] := \{(P_1, \ldots, P_k) \in A^{(X_1, \ldots, X_k)} \mid (A, P_1, \ldots, P_k) \models \varphi\}.$$

Often, we also use the following variant of this notation. For $j < k$ and $(P_1, \ldots, P_j) \in A^{(X_1, \ldots, X_j)}$ we let

$$\varphi[A, P_1, \ldots, P_j, X_{j+1}, \ldots, X_k] := \{(P_{j+1}, \ldots, P_k) \in A^{(X_{j+1}, \ldots, X_k)} \mid (A, P_1, \ldots, P_k) \models \varphi\}.$$

We often abbreviate this by $\varphi[A, P, \bar{X}]$ if the indices are clear from the context. Most often, we use this notation if $X_{j+1}, \ldots, X_k$ are vertex variables. Then $\varphi[A, P_1, \ldots, P_j, X_{j+1}, \ldots, X_k]$ is a $(k - j)$-ary relation on $V(A)$.

**Example 2.3.9.** Let $\text{path}(x, y)$ be the formula defined in Example 2.3.3. Then for every graph $G$ and vertex $v \in V(G)$, the set $\text{path}[G, v, y]$ is the vertex set of the connected component of $G$ that contains $v$. 

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Preliminary Version
**Example 2.3.10.** Let \( \text{dist}(x_1, x_2, y) \) be the formula defined in Example [2.3.6](#). Then for every graph \( G \) and every \( i \in \text{Num}(G) \), the binary relation

\[
\text{dist}[G, x_1, x_2, i]
\]

consists of all pairs \((v_1, v_2)\) of vertices with \( \text{dist}(v_1, v_2) = i \).

While locally, that is, in a fixed structure, formulae with free variables define sets of assignments, globally they define classes of interpretations. We say that a class \( \mathcal{I} \) of \( (\tau, X) \)-interpretations is \( L \)-definable if there is a formula \( \varphi(X) \in L[\tau] \) such that for all \( (\tau, X) \)-interpretations \((A, \mathcal{P})\) it holds that

\[
(A, \mathcal{P}) \in \mathcal{I} \iff A \models \varphi[\mathcal{P}].
\]

Consider \((\tau, \xi)\)-interpretations \((A, \alpha)\) and \((B, \beta)\). An isomorphism from \((A, \alpha)\) to \((B, \beta)\) is an isomorphism \( f \) from \( A \) to \( B \) such that for every \( X \in \xi \) it holds that \( f(\alpha(X)) = \beta(X) \). The interpretations \((A, \alpha)\) and \((B, \beta)\) are isomorphic (we write \((A, \alpha) \equiv (B, \beta)\)) if there is an isomorphism from \((A, \alpha)\) to \((B, \beta)\). For every reasonable logic\(^2\) \( L \), an \( L \)-definable class \( \mathcal{Q} \) of \((\tau, \xi)\)-interpretations is closed under isomorphism. We call isomorphism closed classes \( \mathcal{Q} \) of \((\tau, \xi)\)-interpretations \((\tau, \xi)\)-queries. Note that \((\tau, \emptyset)\)-queries are just isomorphism closed classes of \( \tau \)-structures.

### 2.4 Transductions

**Transductions** are definitions of structures within other structures. They will be used to define **canonisations** in the next chapter, but will also appear on many other occasions throughout this book. Traditionally, transductions are known as **syntactical interpretations**, but we prefer “transduction” here to avoid confusion with the (semantic) interpretations introduced in Section [2.3.5](#). The term “transduction” was introduced by Courcelle in his studies of monadic second-order logic on graphs.

Throughout this section, we make the following assumptions:

**Assumption 2.4.1.** \( L \) is one of the logics \( \text{IFP} \) or \( \text{IFP+C} \), and \( \tau, \tau', \tau'', \nu \) are vocabularies.

It will be useful to start with an example on definable equivalence relations. Recall our notation for equivalence relations from p.[19](#).

**Example 2.4.2.** Let \( \varphi(y_1, y_2) \in \text{IFP}[\tau] \), and let \( A \) be a \( \tau \)-structure. Let \( \equiv_\varphi \) be the reflexive, symmetric, transitive closure of the binary relation \( \varphi[A, y_1, y_2] \); we call \( \equiv_\varphi \) the equivalence relation generated by \( \varphi[A, y_1, y_2] \). The equivalence relation \( \equiv_\varphi \) is definable in \( \text{IFP} \) as follows: Let

\[
\text{eqrel}_\varphi(y_1, y_2) := \text{ifp}\left(Y(y_1, y_2) \leftarrow y_1 = y_2 \lor \varphi(y_1, y_2) \lor \varphi(y_2, y_1) \lor \exists z (Y(y_1, z) \land Y(z, y_2)) \right)(y_1, y_2).
\]

Then \( \equiv_\varphi = \text{eqrel}_\varphi[A, y_1, y_2] \).

More generally, for every formula \( \varphi(X, y_1, y_2) \in \text{IFP}[\tau] \) we can define a formula \( \text{eqrel}_{\varphi}(X, y_1, y_2) \) such that for all \((\tau, X)\)-interpretations \((A, \mathcal{P})\) the binary relation \( \text{eqrel}_{\varphi}[A, \mathcal{P}, y_1, y_2] \) is the equivalence relation on \( V(A) \) generated by \( \varphi[A, \mathcal{P}, y_1, y_2] \).\(^2\)

\(^2\)All logics considered in this book have this invariance property, and if we were to define the notion of a logic, we would require this invariance property as one of the basic axioms.
2.4. Transductions

**Definition 2.4.3.** (1) An $L[\tau, v]$-transduction is a tuple

$$\Theta(\overline{X}) = \left( \theta_{\text{dom}}(\overline{X}), \theta_{V}(\overline{X}, \overline{y}), \theta_{=}((\overline{X}, \overline{y}_1, \overline{y}_2), (\theta_R(\overline{X}, \overline{y}_R))_{R \in v} \right),$$

of formulae in $L[\tau]$, where $\overline{X}$ is a tuple of variables, $\overline{y}$, $\overline{y}_1$, $\overline{y}_2$ are tuples of individual variables of the same type, and for each $k$-ary $R \in v$, there are tuples $\overline{y}_{R,1}, \ldots, \overline{y}_{R,k}$ of individual variables that all have the same type as $\overline{y}$ such that $\overline{y}_R = \overline{y}_{R,1} \cdots \overline{y}_{R,k}$.

In the following, let $\Theta(\overline{X}) = \left( \theta_{\text{dom}}(\overline{X}), \theta_{V}(\overline{X}, \overline{y}), \theta_{=}((\overline{X}, \overline{y}_1, \overline{y}_2), (\theta_R(\overline{X}, \overline{y}_R))_{R \in v} \right)$ be an $L[\tau, v]$-transduction.

(2) The domain of $\Theta(\overline{X})$ is the class $D_{\Theta(\overline{X})}$ of all $(\tau, \overline{X})$-interpretations $(A, \overline{P})$ such that $A \models \theta_{\text{dom}}(\overline{P})$.

(3) For all $(A, \overline{P}) \in D_{\Theta(\overline{X})}$ we let $\Theta[A, \overline{P}]$ be the $v$-structure with vertex set $V(\Theta[A, \overline{P}]) := \theta_{V}[A, \overline{P}, \overline{y}] / \equiv$,

where $\equiv$ is the equivalence relation generated by $\theta_{=}([A, \overline{P}, \overline{y}_1, \overline{y}_2]$ viewed as a binary relation on $A^\overline{y}$. Furthermore, for each $k$-ary $R \in v$ we let

$$R(\Theta[A, \overline{P}]) := \left( \theta_R[A, \overline{P}, \overline{y}_R] \cap \theta_{V}[A, \overline{P}, \overline{y}]^k \right) / \equiv.$$

Here we view the relation $\theta_R[A, \overline{P}, \overline{y}_R]$ as a $k$-ary relation on $A^\overline{y}$.

**Remark 2.4.4.** Let me point the reader’s attention to one subtlety in the definition of $L[\tau, v]$-transductions: we do not require the formula $\theta_{=}((\overline{X}, y_1, y_2)$ in a transduction $\Theta(\overline{X})$ to define the equivalence relation $\equiv$, but only to generate it. This is not a problem, because in IFP and stronger logics, the equivalence relation generated by a definable binary relation is definable as well (cf. Example 2.4.2). Remember that we assumed $L$ is one of the logics IFP or IFP+C. If we were to define transductions for first-order logic FO, we would have to impose the restriction that $\theta_{=}((\overline{X}, \overline{y}_1, \overline{y}_2)$ actually defines an equivalence relation, because the transitive closure of a binary relation is not definable in FO.

It will be convenient to have some additional terminology. Let

$$\Theta(\overline{X}) = \left( \theta_{\text{dom}}(\overline{X}), \theta_{V}(\overline{X}, \overline{y}), \theta_{=}((\overline{X}, \overline{y}_1, \overline{y}_2), (\theta_R(\overline{X}, \overline{y}_R))_{R \in v} \right)$$

be an $L[\tau, v]$-transduction. The variables appearing in the tuple $\overline{X}$ are called the parameters of the transduction. The variables appearing in the tuple $\overline{y}$ are called the domain variables. Usually, all parameters will be vertex variables, and either all domain variables will be vertex variables or all domain variables will be number variables. If all domain variables are number variables, then we call $\Theta(\overline{X})$ a numerical $L[\tau, v]$-transduction. The length of the tuple $\overline{y}$ is the dimension of the transduction. We call $\Theta(\overline{X})$ simple if $\theta_{=}((\overline{X}, \overline{y}_1, \overline{y}_2) = (\overline{y}_1 = \overline{y}_2)$. If a transduction $\Theta(\overline{X})$ with domain variables $\overline{y}$ is simple, then for all $(A, \overline{P}) \in D_{\Theta(\overline{X})}$, we view $V(\Theta[A, \overline{P}])$ as a subset of $A^\overline{y}$ rather than a set of 1-element subsets of $A^\overline{y}$ (the trivial equivalence classes). That is, for $\overline{q} \in A^\overline{y}$ we identify the equivalence class $\{\overline{q}\}$ with $\overline{q}$.

**Example 2.4.5.** Let $\subseteq$ be a binary relation symbol. Consider the IFP$[\{E\}, \{\subseteq\}]$-transduction $\Theta$ defined as follows:

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Preliminary Version


- \( \theta_{\text{dom}} := \text{true} \);
- \( \theta_V(y) := \text{true} \);
- \( \theta_{\leq}(y_1, y_2) := \text{path}(y_1, y_2) \land \text{path}(y_2, y_1) \), where \( \text{path}(x, y) \) is the formula from Example 2.3.3 saying that there is a path from \( x \) to \( y \). We apply the formula to digraphs here. Hence the paths are directed.
- \( \theta_{\leq}(y_1, y_2) := \text{path}(y_1, y_2) \).

For every digraph \( D \) we let \( D' \) be the digraph obtained from \( D \) by “contracting” every strongly connected component to a single vertex. \( D' \) is sometimes called the \textit{component graph} of \( D \). Then \( D' \) is a directed acyclic graph, and thus the reflexive transitive closure \( \leq D' \) is a partial order on \( V(D') \). It is easy to see that

\[
\Theta[D] = (V(D'), \leq D').
\]

In particular, if \( D \) is a directed acyclic graph then \( \Theta[D] = (V(D), \leq D) \).

\( \text{L}[\tau, v] \)-transductions \( \Theta(\overline{X}) \) define a mapping from the class \( D_{\Theta(\overline{X})} \) of \( (\tau, \overline{X}) \)-interpretations to the class \( S[v] \) of \( v \)-structures. The crucial observation is that they induce a reverse mapping from \( \text{L}[v] \) to \( \text{L}[\tau] \). A proof of this observation for first-order logic (taking Remark 2.4.4 into account), which is also known as the\textit{Lemma on Syntactical Interpretations}, can be found in [32]. The proof for \text{IFP} and \text{IFP+C} is an easy adaptation of the one for first-order logic.

\textbf{Fact 2.4.6 (Transduction Lemma).} Let \( \Theta(\overline{X}) \) be an \( \text{L}[\tau, v] \)-transduction, and let \( \overline{y} \) be its tuple of domain variables. Then for every formula \( \varphi(z_1, \ldots, z_k) \in \text{L}[v] \), where \( z_1, \ldots, z_k \) are vertex variables, there is a formula \( \varphi^{-\Theta}(\overline{x}, \overline{y}_1, \ldots, \overline{y}_k) \in \text{L}[\tau] \), where \( \overline{y}_1, \ldots, \overline{y}_k \) are tuples of individual variables of the same type as \( \overline{y} \), such that for all \( (A, \overline{P}) \in D_{\Theta(\overline{X})} \) and all tuples \( \overline{q}_1, \ldots, \overline{q}_k \in A^{\overline{y}} \) we have

\[
A \models \varphi^{-\Theta}[\overline{P}, \overline{q}_1, \ldots, \overline{q}_k] \iff \overline{q}_1 / =, \ldots, \overline{q}_k / = \in V(\Theta[A, \overline{P}]) \text{ and } \Theta[A, \overline{P}] \models \varphi[\overline{q}_1 / =, \ldots, \overline{q}_k / =],
\]

where \( = \) is the equivalence relation generated by \( \theta_\equiv[A, \overline{P}, \overline{q}_1, \overline{q}_2] \) viewed as a binary relation on \( V(A)^{\overline{y}} \).

As a corollary, we state the restriction of the Transduction Lemma to sentences, which is much easier to digest.

\textbf{Corollary 2.4.7.} Let \( \Theta(\overline{X}) \) be an \( \text{L}[\tau, v] \)-transduction. Then for every sentence \( \varphi \in \text{L}[v] \) there is a formula \( \varphi^{-\Theta}(\overline{X}) \in \text{L}[\tau] \) such that for all \( (A, \overline{P}) \in D_{\Theta(\overline{X})} \) we have

\[
A \models \varphi^{-\Theta}[\overline{P}] \iff \Theta[A, \overline{P}] \models \varphi.
\]

We also need a version of the Transduction Lemma where the formula \( \varphi \) has free number variables. To avoid unnecessary complications, we restrict this version to simple 1-dimensional transductions.

\textbf{Fact 2.4.8.} Let \( \Theta(\overline{X}) \) be a simple 1-dimensional \( \text{IFP+C}[\tau, v] \)-transduction, and let \( y \) be its domain variable. Let \( z_1, \ldots, z_k, y_1, \ldots, y_k \) be individual variables such that for all \( i \in [k] \), if \( z_i \) is a vertex variable then \( y_i \) is of the same type as \( y \) and if \( z_i \) is a number variable...
then $y_i$ is a number variable. Then for every formula $\varphi(z_1, \ldots, z_k) \in \text{IFP} + \text{C}[v]$ there is a formula $\varphi^-(\overline{x}, y_1, \ldots, y_k) \in \text{IFP} + \text{C}[\tau]$ such that for all $(A, \overline{P}) \in D_{\Theta(\overline{x})}$ and all tuples $(q_1, \ldots, q_k) \in A^{(y_1, \ldots, y_k)}$ we have

$$A \models \varphi^-(\overline{P}, q_1, \ldots, q_k) \iff (q_1, \ldots, q_k) \in \Theta(A, \overline{P})^{(y_1, \ldots, y_k)} \text{ and } \Theta(A, \overline{P}) \models \varphi[q_1, \ldots, q_k].$$

Note that in the previous fact we use our convention that for simple transductions we identify the 1-element equivalence classes with their elements.

### 2.4.1 Graph Transductions

Most transductions considered in this book will map graphs to graphs. Hence it is worthwhile to introduce some terminology for such transductions and consider a few examples.

**Definition 2.4.9.** An L-graph transduction is an L$\left[\{E\},\{E\}\right]$-transduction $\Theta(\overline{X})$ such that for all $(G, \overline{P}) \in D_{\Theta(\overline{x})}$ the $\{E\}$-structures $G$ and $\Theta[G, \overline{P}]$ are both graphs.

There is a simple syntactical transformation turning an arbitrary L$\left[\{E\},\{E\}\right]$-transduction $\Theta(\overline{X})$ into an L-graph transduction $\Theta^{\text{graph}}(\overline{X})$ that coincides with $\Theta(\overline{X})$ whenever it behaves like graph transduction. We let

$$\text{irrefl}(\overline{X}) := (\forall x \lnot E(x, x))^\Theta, \quad \text{symm}(\overline{X}) := \left(\forall x \forall y (E(x, y) \rightarrow E(y, x))\right)^\Theta.$$  

These formulae say that the edge relation of the $\{E\}$-structure defined by $\Theta(\overline{X})$ is irreflexive and symmetric, respectively. Let $\text{graph}$ be an FO-sentence defining the class of graphs. We let

$$\theta^{\text{graph}}_{\text{dom}}(\overline{X}) := \text{graph} \land \theta_{\text{dom}}(\overline{X}) \land \text{irrefl}(\overline{X}) \land \text{symm}(\overline{X}),$$

and we let $\Theta^{\text{graph}}(\overline{X})$ be the transduction obtained from $\Theta(\overline{X})$ by replacing $\theta_{\text{dom}}(\overline{X})$ by $\theta^{\text{graph}}_{\text{dom}}(\overline{X})$. Then for all $\{E\}, \overline{X}$-interpretations $(G, \overline{P})$ we have $(G, \overline{P}) \in D_{\Theta^{\text{graph}}(\overline{X})}$ if and only if $(G, \overline{P}) \in D_{\Theta(\overline{X})}$ and both $G$ and $\Theta[G, \overline{P}]$ are graphs. Hence $\Theta^{\text{graph}}(\overline{X})$ is a graph transduction. Furthermore, if $(G, \overline{P}) \in D_{\Theta^{\text{graph}}(\overline{X})}$, then $\Theta^{\text{graph}}[G, \overline{P}] = \Theta[G, \overline{P}]$.

**Example 2.4.10.** Let $\Theta(\overline{x})$, where $\overline{x} = (x_1, \ldots, x_k)$, be the simple LFP-graph transduction with $\theta_{\text{dom}}(\overline{x}) := \text{graph}$, $\theta_E(\overline{x}, y) = \bigwedge_{i=1}^k y \neq x_i$, and $\theta_E(\overline{x}, y_1, y_2) = E(y_1, y_2)$. (As the transduction is simple, we have $\theta_{=}(\overline{x}, y_1, y_2) = (y_1 = y_2)$.) Then for every graph $G$ and every tuple $\overline{v} \in V(G)^k$ it holds that

$$\Theta[G, \overline{v}] = G \setminus \overline{v}. \quad (2.4.1)$$

Let path$(z_1, z_2)$ be the LFP-sentence from Example 2.3.3 saying that there is a path from $z_1$ to $z_2$. Let $\text{path}^{-\Theta}(\overline{x}, y_1, y_2)$ be the formula obtained by applying the Transduction Lemma (Fact 2.4.6) to $\Theta$. Then for every graph $G$, every tuple $\overline{v} \in V(G)^k$, and all $w_1, w_2 \in V(G)$ it holds that

$$G \models \text{path}^{-\Theta}[\overline{v}, w_1, w_2] \iff \text{there is a path from } w_1 \text{ to } w_2 \text{ in } G \setminus \overline{v}.$$  

Remember that a graph is $(k + 1)$-connected if and only if $|G| \geq k + 2$ and for every set $W \subseteq V(G)$ with $|W| \leq k$ the graph $G \setminus W$ is connected. Thus the sentence

$$\text{conn}_{k+1} := \exists x_1 \ldots \exists x_{k+2} \bigwedge_{i,j \in [k+2], i \neq j} x_i \neq x_j$$

is connected.
\[ \land \forall x_1 \ldots \forall x_k \forall y_1 \forall y_2 \left( \bigvee_{i=1}^{k} x_i = y_j \lor \text{path}^\Theta(x_1, \ldots, x_k, y_1, y_2) \right) \]

says that a graph is \((k+1)\)-connected. \(\square\)

**Example 2.4.11.** Let \(\Theta(\mathcal{X})\) be a simple 1-dimensional \(\mathcal{IFP}\)-graph transduction, and let \((G, \mathcal{P}) \in D_{\Theta(\mathcal{X})}\). Suppose that

\[ \theta_E[G, \mathcal{P}, y_1, y_2] \subseteq E(G). \]

Then \(H := \Theta[G, \mathcal{P}]\) is a subgraph of \(G\).

By the Transduction Lemma, for all formulae \(\varphi(z_1, \ldots, z_k)\) and all \(w_1, \ldots, w_k \in V(G)\) it holds that

\[ G \models \varphi^\Theta[\mathcal{P}, w_1, \ldots, w_k] \iff w_1, \ldots, w_k \in V(H) \text{ and } H \models \varphi[w_1, \ldots, w_k]. \]

Equivalently, \(\varphi^\Theta[G, \mathcal{P}, z_1, \ldots, z_k] = \varphi[H, z_1, \ldots, z_k]\).

We say that a graph transduction \(\Theta(\mathcal{X})\) is a **subgraph transduction** if the condition \(\theta_E[G, \mathcal{P}, y_1, y_2] \subseteq E(G)\) is satisfied for all \((G, \mathcal{P}) \in D_{\Theta(\mathcal{X})}\). \(\square\)

As the previous example shows, it is straightforward to define subgraphs by transductions. For minors, the situation is much more complicated. The following examples show a few, increasingly complicated types of transductions that we can use to define minors.

**Example 2.4.12.** To define a contraction of a graph \(G\), we only have to specify the vertex sets of the connected subgraphs that are contracted. That is, we have to specify an equivalence relation \(\equiv\) on \(V(G)\) whose classes \(v/\equiv\) are connected in \(G\). A **contraction transduction** is a 1-dimensional graph transduction \(\Theta(\mathcal{X})\) such that

- \(\theta_V(\mathcal{X}, y) = \text{true}\);
- for all \((G, \mathcal{P}) \in D_{\Theta(\mathcal{X})}\), all classes of the equivalence relation \(\equiv\) generated by \(\theta_{\equiv}[G, \mathcal{P}, y_1, y_2]\) are connected in \(G\);
- \(\theta_E(\mathcal{X}, y_1, y_2) = E(y_1, y_2) \land \neg\text{eqrel}_{\theta_{\equiv}}(\mathcal{X}, y_1, y_2)\).

Now let \(\Theta(\mathcal{X})\) be a contraction transduction and \((G, \mathcal{P}) \in D_{\Theta(\mathcal{X})}\). Let \(\equiv\) be the equivalence relation on \(V(G)\) generated by \(\theta_{\equiv}[G, \mathcal{P}, y_1, y_2]\), and let \(V_1, \ldots, V_m\) be its equivalence classes. Then for all \(i \in [m]\) the subgraph \(A_i := G[V_i]\) is connected, and we have

\[ \Theta[G, \mathcal{P}] \cong G/A_1/\ldots/A_m. \]

Note that we have a canonical isomorphism from \(\Theta[G, \mathcal{P}]\) to \(G/A_1/\ldots/A_m\) mapping \(V_i \in V(\Theta[G, \mathcal{P}])\) to the vertex \(a_i\) of \(G/A_1/\ldots/A_m\) that corresponds to the contracted subgraph \(A_i\). We may even identify the element \(a_i\), which we left unspecified in the definition of contractions, with the set \(V_i\) for all \(i \in [m]\) and regard the two graphs as equal.

By the Transduction Lemma, for all formulae \(\varphi(z_1, \ldots, z_k)\) and all \(w_1, \ldots, w_k \in V(G)\) such that \(w_j \in V_{i_j}\) for all \(j \in [k]\), it holds that

\[ G \models \varphi^\Theta[\mathcal{P}, w_1, \ldots, w_k] \iff G/A_1/\ldots/A_m \models \varphi[a_{i_1}, \ldots, a_{i_k}]. \]
2.4. Transductions

Example 2.4.13. In this example, we consider a special case of the contraction transductions introduced in the previous example. Let $\psi(\overline{X}, x)$ be an IFP-formula. For every graph $G$ and every assignment $\overline{P} \in G^\overline{X}$, we define a minor $H$ of $G$ as follows. Let $W := \varphi[G, \overline{P}, x]$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus W$. Then we let

$$H := G/A_1/\cdots/A_m.$$  

For each $i \in [m]$, let $a_i$ be the vertex of $H$ corresponding to $A_i$. Then $V(H) = W \cup \{a_1, \ldots, a_m\}$.

Let $\Theta(\overline{X})$ be the 1-dimensional graph transduction defined by

- $\theta_{\text{dom}}(\overline{X}) := \text{graph};$
- $\theta_{\text{V}}(\overline{X}, y) := \text{true};$
- $\theta_{=}(\overline{X}, y_1, y_2) := (\varphi(\overline{X}, y_1) \land y_2 = y_1) \lor (\neg \varphi(\overline{X}, y_1) \land \neg \varphi(\overline{X}, y_2) \land E(y_1, y_2));$
- $\theta_{\text{E}}(\overline{X}, y_1, y_2) := E(y_1, y_2) \land \neg \text{eqrel}_{\theta_{=}}(\overline{X}, y_1, y_2).$

Let $\equiv$ be the equivalence relation on $V(G)$ generated by $\theta_{=}[G, \overline{P}, y_1, y_2]$. Then $v/\equiv = \{v\}$ for all $v \in W$ and $v/\equiv = V(A_i)$ for all $i \in [m], v \in V(A_i)$. Thus $\Theta(\overline{X})$ is a contraction transduction, and we have $\Theta(G, \overline{P}) \cong H$.

By the Transduction Lemma, for all formulae $\varphi(z_1, \ldots, z_k)$ and all $w_1, \ldots, w_k \in V(G)$ it holds that

$$G \models \varphi^{-\Theta}[\overline{P}, w_1, \ldots, w_k] \iff H \models \varphi[w_1', \ldots, w_k'],$$

where for all $i \in [k]$ we let $w_i' := w_i$ if $w_i \in W$ and $w_i' := a_j$ if $w_i \in V(A_j)$ for some $j \in [m]$. In particular, for all $w_1, \ldots, w_k \in W$ it holds that

$$G \models \varphi^{-\Theta}[\overline{P}, w_1, \ldots, w_k] \iff H \models \varphi[w_1, \ldots, w_k].$$

Example 2.4.14. There is a natural way of generalising contraction transductions to arbitrary minors. A naive minor transduction is a 1-dimensional graph transduction satisfying the following conditions for all $(G, \overline{P}) \in D_{\Theta(\overline{X})}$:

- Let $\equiv$ be the equivalence relation generated by $\theta_{=}[G, \overline{P}, y_1, y_2]$ on $V(G)$. Then for all $v \in \theta_{\text{V}}[G, \overline{P}, y]$ the subgraph $G[v/\equiv]$ is connected.
- For all $v_1, v_2 \in \theta_{\text{V}}[G, \overline{P}, y]$, if $G \models \theta_{\text{E}}[\overline{P}, v_1, v_2]$ then $v_1v_2 \in E(G)$ and $v_1 \neq v_2$.

Now let $\Theta(\overline{X})$ be a naive minor transduction and $(G, \overline{P}) \in D_{\Theta(\overline{X})}$. Then obviously $H := \Theta[G, \overline{P}]$ is a minor of $G$. By the Transduction Lemma, for all formulae $\varphi(z_1, \ldots, z_k)$ and all $w_1, \ldots, w_k \in V(G)$ we have

$$G \models \varphi^{-\Theta}[\overline{P}, w_1, \ldots, w_k] \iff w_1/\equiv, \ldots, w_k/\equiv \in V(H) \text{ and } H \models \varphi[w_1/\equiv, \ldots, w_k/\equiv].$$

While naive minor transductions seem to be a fairly generic for defining minors, it turns out that most transductions defining minors which appear later in this book are not naive minor transductions. The following example is typical in this respect.
Example 2.4.15. Consider the graph $G$ shown in Figure 2.1. Obviously, we have $K_4 \cong G$. For example, $K_4 \cong G/\{tu,v_1w_1,v_2w_2,v_3w_3\}$. However, there is no (parameter free) naive minor transduction $\Theta_n$ such that $\Theta_n[G] \cong K_4$. This can be seen by analysing the definable equivalence relations on definable subsets of $V(G)$ with exactly four equivalence classes and noting that none of these equivalence relations has edges between all of its classes. Hence we cannot define the edge relation of the $K_4$-minor in such a way that it only contains edges of $G$, as this is required for a naive minor transduction.

However, we can define a graph transduction $\Theta$ such that $\Theta[G] \cong K_4$ as follows.

- $\theta_{\text{dom}} := \text{graph}$.
- $\theta_V(y)$ defines the set of all vertices of degree at least 4, which is the set $\{u,v_1,v_2,v_3\}$ in the graph $G$. That is, $\theta_V(y) := \exists x_1 \ldots \exists x_4 \left( \bigwedge_{i=1}^{4} E(y,x_i) \land \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j \right)$.
- $\theta_{\equiv}(y_1,y_2) := (y_1 = y_2)$.
- $\theta_E(y_1,y_2) := (y_1 \neq y_2)$. 

Figure 2.1. The graph from Example 2.4.15
Chapter 3

Descriptive Complexity

This chapter is devoted to the main applications of our definable structure theory in descriptive complexity theory and to the graph isomorphism problem. Sections 3.1 is a short, but self-contained introduction to descriptive complexity theory. It is geared towards the “quest for a logic capturing PTIME”, the main motivation for this whole book. Section 3.5 is about the graph isomorphism problem and, mainly, a specific isomorphism algorithm known as the Weisfeiler-Leman algorithm, and Section 3.4 about finite variable logics, which connect descriptive complexity theory with the Weisfeiler-Leman algorithm. The two intermediate Sections 3.2 and 3.3 are concerned with the notion of definable canonisation.

The material of this chapter is more or less independent of the rest of the book. Only Section 7.4 really depends on it. Apart from this, we will occasionally refer to the results of this chapter and Section 7.4 when we give corollaries of our main structural results in descriptive complexity, but we will treat these results as “black boxes”. To give the reader who decides to defer this chapter to later (or skip it entirely) an idea of the line of reasoning, let me give a brief outline. Our goal is to give a logical characterisation of the polynomial time decidable properties of graphs in classes with excluded minors. In the terminology of this chapter, we look for a logic capturing PTIME on these classes, and we shall prove that IFP+C is such a logic. Our starting point is the Immerman-Vardi Theorem, stating that IFP captures PTIME on the class of all ordered structures. In Section 3.3, we will see that if a class of structures admits IFP+C-definable canonisation, then IFP+C captures PTIME on this class. IFP+C-definable canonisation means that there is an IFP+C-transduction that defines for each structure in the class an ordered copy of this structure. In Section 7.4 we will prove that if a class of graphs admits IFP-definable ordered treelike decompositions, then it admits IFP+C-definable canonisation. From then on, our main results state that certain classes of graphs admit IFP-definable ordered treelike decompositions. In this chapter, we will also see that if a class of graphs admits IFP+C-definable canonisation then there is a k such that the k-dimensional Weisfeiler-Leman algorithm correctly decides isomorphism of all graphs in this class.

None of the results presented in this chapter are new, though it is not always easy to give exact references. The question of whether IFP+C captures PTIME was first raised by Immerman [73]. Immerman and Lander [76] connect definability in finite variable counting logic and canonisation. Definable canonisation was introduced and studied to great depth by Otto in the 1990s, and his monograph [94] may be the best reference for much of the material.
Chapter 3. Descriptive Complexity

3.1 Logics Capturing Complexity Classes

This section is a brief introduction to descriptive complexity theory. For more background and detailed proofs of the theorems presented here, I refer the reader to the textbooks [31, 75, 86, 39].

3.1.1 Background from Complexity Theory

Actually, very little complexity theory is needed to understand our main results. We use multitape Turing machines as our underlying machine model. We will never get to the level of detail where we need explicit descriptions of Turing machines, their configurations, or their computations (viewed as sequences of configurations), and thus there is no need to introduce any notation for them. Algorithmic arguments will usually be presented on an informal level. However, formally we may think of a polynomial time algorithm as a pair \((M, p)\) consisting of a deterministic Turing machine \(M\) and a polynomial \(p = p(X)\) with nonnegative integer coefficients. Such a polynomial time algorithm \((M, p)\) accepts an input string \(x \in \{0, 1\}^*\) if the computation of \(M\) on input \(x\) reaches an accepting state in at most \(p(|x|)\) steps. We can define a nondeterministic polynomial time algorithms similarly with nondeterministic Turing machines. We always use the binary alphabet \(\{0, 1\}\) as input and output alphabet. Hence we encode instances of computational problems as strings \(x \in \{0, 1\}^*\), decision problems as languages \(L \subseteq \{0, 1\}^*\), and complexity classes as sets \(K \subseteq 2^{\{0,1\}^*}\) of languages. In particular, \(\text{PTIME}\) is the set of all decision problems \(L \subseteq \{0, 1\}^*\) that can be decided by a polynomial time algorithm. \(\text{NP}\) is the set of all \(L \subseteq \{0, 1\}^*\) accepted by a nondeterministic polynomial time algorithm.

3.1.2 Representations of Structures

In descriptive complexity, instances of algorithmic problems are modelled by finite structures. To relate this to the standard complexity theoretic view of problem instances as binary strings (laid out in the previous section), we need to represent structures by binary strings.

The standard representation scheme used in descriptive complexity theory generalises the adjacency matrix representation of graphs. Note that if we represent a graph by its adjacency matrix, we implicitly fix an ordering of the vertices of the graph. The same is true for other standard representations, for example, the representation of graphs by adjacency lists, or just as lists of vertices and edges. Indeed, it is hard to imagine a canonical representation scheme for graphs that does not depend on an ordering of the vertices (see Remark 3.1.1 below, however). We call a representation scheme canonical if two structures are represented by the same string if and only if they are isomorphic. The problem of finding “good” canonical representations of certain structures (graphs with excluded minors) is at the technical core of this book.

However, if a linear order of the vertices of a structure is explicitly given, the problem disappears. For example, we can represent an ordered graph by the adjacency matrix of the graph where the rows and columns are ordered according to the linear order of the vertices of the graph. To describe the general encoding scheme, we fix a vocabulary \(\tau\) that does not contain the distinguished binary relation symbol \(\leq\). We denote ordered \(\tau \cup \{\leq\}\)-structures in the form \((A, \leq_A)\), where \(A\) is a \(\tau\)-structure and \(\leq_A\) is a linear order of \(V(A)\). Recall that

M. Grohe, Definable Graph Structure Theory
\( \mathcal{O}[\tau \cup \{\leq\}] \) denotes the class of all ordered \( \tau \cup \{\leq\} \)-structure. We define a mapping

\[
\langle \cdot \rangle : \mathcal{O}[\tau \cup \{\leq\}] \to \{0,1\}^*
\]

that associates with each ordered \( \tau \cup \{\leq\} \)-structure its representation as a binary string. We fix some enumeration \( R_1, \ldots, R_\ell \) of \( \tau \). Now let \( (A, \leq^A) \in \mathcal{O}[\tau \cup \{\leq\}] \). Suppose that \( V(A) = \{v_0, \ldots, v_{n-1}\} \), where \( v_0 \leq^A v_1 \leq^A \cdots \leq^A v_{n-1} \), and \( v_0, \ldots, v_{n-1} \) are pairwise distinct. For each \( R \in \tau \), say of arity \( k \), we define a string

\[
\langle R(A) \rangle := b_0 \ldots b_{m-1},
\]

where \( m := n^k \) and for each \( i = \sum_{j=1}^{k} a_j n^{k-j} \) with \( a_1, \ldots, a_k \in [0, n - 1] \), we let \( b_i := 1 \) if \( (v_{a_1}, \ldots, v_{a_k}) \in R(A) \) and \( b_i := 0 \) otherwise. Now we let

\[
\langle A, \leq^A \rangle := 1^n0 \langle R_1(A) \rangle \langle R_2(A) \rangle \cdots \langle R_\ell(A) \rangle.
\]

Observe that the representation scheme \( \langle \cdot \rangle \) for ordered structures is canonical, that is, for all ordered structures \( (A, \leq^A), (B, \leq^B) \in \mathcal{O}[\tau \cup \{\leq\}] \) it holds that

\[
(A, \leq^A) \cong (B, \leq^B) \iff \langle A, \leq^A \rangle = \langle B, \leq^B \rangle.
\]

For a property \( \mathcal{P} \subseteq \mathcal{O}[\tau \cup \{\leq\}] \) of ordered \( \tau \cup \{\leq\} \)-structures, we let

\[
\mathcal{L}(\mathcal{P}) := \{ \langle A, \leq^A \rangle \mid (A, \leq^A) \in \mathcal{P} \}.
\]

We do not define a separate representation scheme for \( \tau \)-structures; instead we represent a \( \tau \)-structure \( A \) by the language

\[
\mathcal{L}(A) := \{ \langle A, \leq^A \rangle \mid \leq^A \text{ linear order of } V(A) \}
\]

consisting of the representations of all ordered expansions of \( A \). An algorithm deciding a property of a structure \( A \) is expected to give the correct answer on all inputs from \( \mathcal{L}(A) \). Note that at least the language \( \mathcal{L}(A) \) is canonical in the sense that for all \( \tau \)-structures \( A, B \) it holds that \( A \cong B \iff \mathcal{L}(A) = \mathcal{L}(B) \). Furthermore, if \( A \not\cong B \) then \( \mathcal{L}(A) \cap \mathcal{L}(B) = \emptyset \). For a property \( \mathcal{P} \) of \( \tau \)-structures, we let

\[
\mathcal{L}(\mathcal{P}) := \bigcup_{A \in \mathcal{P}} \mathcal{L}(A).
\]

We say that an algorithm decides the property \( \mathcal{P} \) of \( \tau \)-structures or of ordered \( \tau \cup \{\leq\} \)-structures if it decides the language \( \mathcal{L}(\mathcal{P}) \). 

Remark 3.1.1. Of course there are canonical representation schemes for (unordered) \( \tau \)-structures. A simple one maps each \( \tau \)-structure \( A \) to the lexicographically smallest string in \( \mathcal{L}(A) \). The problem with this and other known canonical representation schemes is that they are computationally intractable, that is, there is no (known) polynomial time algorithm that, given an arbitrary representation \( x \in \mathcal{L}(A) \), computes the canonical representation. Being efficiently computable is even more important for a representation scheme than being canonical. \( \blacksquare \)
Remark 3.1.2. The special treatment of the symbol \( \leq \) has the slightly awkward consequence that our representation scheme is not invariant under renaming the symbols. For example, let \( \leq' \not\equiv \tau \cup \{\leq\} \) be another binary relation symbol. Let \( \mathcal{P} \subseteq \mathcal{O}[\tau \cup \{\leq\}] \) be a property of ordered \( \tau \cup \{\leq\} \)-structures, and let \( \mathcal{P}' \) be the class of all \( \tau \cup \{\leq'\} \)-structures \( A' \) such that there is an \( A \in \mathcal{P} \) with \( A|_\tau = A'|_\tau \) and \( \leq A = \leq' (A') \). Then essentially, \( \mathcal{P} \) and \( \mathcal{P}' \) describe the same property, but their representations \( \mathcal{L}(\mathcal{P}) \) and \( \mathcal{L}(\mathcal{P}') \) differ considerably.

Fortunately, this is not really a problem, because there are polynomial time algorithms reducing \( \mathcal{L}(\mathcal{P}) \) to \( \mathcal{L}(\mathcal{P}') \) and vice versa.

We can easily extend our representation scheme from structures to interpretations. Let \( \tau \) be a vocabulary that does not contain \( \leq \), and let \( \xi \) be a signature. With each \( (\tau, \xi) \)-interpretation \( (A, \alpha) \) and each linear order \( \leq A \) of \( V(A) \) we associate a binary string

\[
\langle A, \alpha, \leq A \rangle \in \{0, 1\}^*,
\]

and then we let

\[
\mathcal{L}(A, \alpha) := \{ \langle A, \alpha, \leq A \rangle \mid \leq A \text{ linear order of } V(A) \}.
\]

It is not important how exactly we define \( \langle A, \alpha, \leq A \rangle \), as long as the representation is canonical in the sense that for any two \( (\tau, \xi) \)-interpretations \( (A, \alpha), (B, \beta) \) and linear orders \( \leq A \) of \( V(A) \), \( \leq B \) of \( V(B) \) we have \( \langle A, \alpha, \leq A \rangle = \langle B, \beta, \leq B \rangle \) if and only if the \( (\tau \cup \{\leq\}, \xi) \)-interpretations \( (A, \leq A, \alpha) \) and \( (B, \leq B, \beta) \) are isomorphic. This implies that \( \mathcal{L}(A, \alpha) = \mathcal{L}(B, \beta) \) if and only if \( (A, \alpha) \cong (B, \beta) \), and \( \mathcal{L}(A, \alpha) \cap \mathcal{L}(B, \beta) = \emptyset \) otherwise. Recall (from page 34) that a \( (\tau, \xi) \)-query is an isomorphism-closed class of \( (\tau, \xi) \)-interpretations. For every \( (\tau, \xi) \)-query \( Q \) we let \( \mathcal{L}(Q) \) be the union of all \( \mathcal{L}(A, \alpha) \) for \( (A, \alpha) \in Q \). We say that an algorithm decides the query \( Q \) if it decides \( \mathcal{L}(Q) \).

3.1.3 Fagin’s Theorem and the Immerman-Vardi Theorem

Descriptive complexity provides logical characterisations of most standard complexity classes. As a working definition, we may say that a logic captures a complexity class \( K \) if for every vocabulary \( \tau \) and every property \( \mathcal{P} \) of \( \tau \)-structures, \( \mathcal{P} \) is \( \mathcal{L} \)-definable if and only if \( \mathcal{L}(\mathcal{P}) \in K \). Actually, to exclude pathological examples the definition needs to be refined: the main additional requirement is an effective link between an \( L \)-definition of a property and a “K-algorithm” deciding the property. We will further discuss this in the next section (for the complexity class \( \text{PTIME} \)), but for now we can use the “working definition.” Actually, we also need the following generalisation of the definition. For every class \( C \) of structures, we say that \( L \) captures \( K \) on \( C \) if for every vocabulary \( \tau \) and every property \( \mathcal{P} \) of \( \tau \)-structures the following two statements are equivalent.

(i) There is an \( \mathcal{L} \)-definable property \( \mathcal{P}' \) of \( \tau \)-structures such that \( \mathcal{P} \cap C = \mathcal{P}' \cap C \).

(ii) There is a language \( \mathcal{L}' \in K \) such that \( \mathcal{L}(\mathcal{P}) \cap \mathcal{L}(C[\tau]) = \mathcal{L}' \cap \mathcal{L}(C[\tau]) \).

Observe that (i) is equivalent to the existence of a sentence \( \varphi \in L[\tau] \) such that for all structures \( A \in \mathcal{C}[\tau] \) we have \( A \in \mathcal{P} \iff A \models \varphi \). Similarly, (ii) is equivalent to the existence of a “K-algorithm” \( \mathfrak{A} \) such that for all structures \( A \in \mathcal{C}[\tau] \), if \( A \in \mathcal{P} \) then \( \mathfrak{A} \) accepts all \( x \in \mathcal{L}(A) \), and if \( A \notin \mathcal{P} \) then \( \mathfrak{A} \) accepts no \( x \in \mathcal{L}(A) \). This equivalent formulation of (ii) only applies to complexity classes \( K \) for which we have a notion of “K-algorithm” characterising it. Typically,
these are classes that can be defined by restricting resources like time or space on some machine model. Most standard complexity classes are of this form.

We are now ready to state Fagin’s Theorem and the Immerman-Vardi Theorem, arguably the two most fundamental results in descriptive complexity theory.

**Theorem 3.1.3 (Fagin’s Theorem [34]).** Existential second-order logic $\exists \text{SO}$ captures NP.

Existential second-order logic $\text{SO}$ is the extension of $\text{FO}$ where, in addition to the “first-order” quantification over vertex variables, we also allow existential and universal quantification over relation variables. Existential second-order logic $\exists \text{SO}$ is the fragment of $\text{SO}$ consisting of all formulae of the form $\exists X_1 \ldots \exists X_k \varphi$, where $X_1, \ldots, X_k$ are relation variables and $\varphi$ is a first-order formula.

**Example 3.1.4.** Recall that a graph is 3-colourable if its vertices can be coloured with three colours in such a way that no two adjacent vertices get the same colour. The following $\exists \text{SO}$-sentence defines the class of 3-colourable graphs:

$$\exists X \exists Y \exists Z \left( \forall x (X(x) \lor Y(x) \lor Z(x)) \land \forall x \forall y (E(x, y) \to (\neg (X(x) \land X(y)) \land \neg (Y(x) \land Y(y)) \land \neg (Z(x) \land Z(y)))) \right).$$

**Theorem 3.1.5 (Immerman-Vardi Theorem [70, 125]).** IFP captures $\text{PTIME}$ on the class $\mathcal{O}$ of ordered structures.

The Immerman-Vardi Theorem is usually (and was originally) stated for least fixed-point logic LFP instead of IFP, but since LFP and IFP have the same expressive power and since we always work with IFP, we prefer the IFP-version here.

In the following, we will sketch a proof of the Immerman-Vardi Theorem, and after that we will explain how it can be modified to prove Fagin’s Theorem. We start with a lemma that implies one direction of the proof of the Immerman-Vardi Theorem, but (for later reference) is stronger than needed here.

**Lemma 3.1.6.** Let $\tau$ be a vocabulary. Then there is an algorithm that, given a formula $\varphi(\bar{X}) \in \text{IFP} + \text{C}[\tau]$ and a $(\tau, \bar{X})$-interpretation $(A, \bar{P})$, decides if $A \models \varphi[\bar{P}]$ in time $|A|^{O(|\varphi|)}$.

More precisely, the algorithm of the lemma gets as its input a pair of binary strings $x_\varphi$ and $x_A$, and it accepts if and only if $x_\varphi$ represents a formula $\varphi(\bar{X}) \in \text{IFP} + \text{C}[\tau]$ (under some fixed representation scheme of IFP+C[τ]-formulae by binary strings) and $x_A \in \mathcal{L}(A, \bar{P})$ for some $(\tau, \bar{X})$-interpretation $(A, \bar{P})$ such that $A \models \varphi[\bar{P}]$. By $|\varphi|$ we denote the length of the string $|x_\varphi|$ representing $\varphi$. In the following, we disregard such coding issues.

**Proof of Lemma 3.1.6 (sketch).** We evaluate formulae recursively. Let $(A, \alpha)$ be the input interpretation and $\varphi$ the input formula. If $\varphi$ is an atomic formula of the form $R(\bar{X})$ or $X(\bar{X})$, then it is evaluated directly by inspecting the appropriate bit of the representation of the input interpretation. If $\varphi = \neg \psi$, the algorithm recursively evaluates $\psi$ and returns the opposite truth value. If $\varphi = \psi_1 \ast \psi_2$, where $\ast \in \{\land, \lor, \rightarrow\}$, the algorithm recursively evaluates $\psi_1$ and $\psi_2$ and combines the truth values appropriately. If $\varphi = \exists x \psi$ for a vertex variable $x$, then for all $v \in V(A)$ the algorithm recursively evaluates $\psi$ in $(A, \alpha(v/x))$ and returns “true” if one of the recursive calls returns “true”. If $x$ is a number variable, the algorithm goes through all $i \in \text{Num}(A)$ and recursively evaluates $\psi$ in $(A, \alpha(i/x))$. Formulae $\varphi = \forall x \psi$ can be dealt
with similarly. If \( \varphi = \#x \psi = y \), then for all \( v \in V(A) \) the algorithm recursively evaluates \( \psi \) in \( (A, \alpha(v/x)) \) and counts the number \( i \) of calls that return “true”. If \( i = \alpha(y) \), the algorithm returns “true”.

Finally, if \( \varphi = \text{ifp}(X \pi \leftarrow \psi) \pi' \), the algorithm iteratively computes the stages \( X^i \) of the fixed-point iteration (defined in (2.3.3) and (2.3.4) on page 27). To compute \( X^{i+1} \) from \( X^i \), it goes through all \( \pi \in A^\pi \) and recursively evaluates \( \psi \) in \( (A, \alpha(X^i/X, \pi/\pi)) \). Then \( X^{i+1} \) is union of \( X^i \) with the set of all \( \pi \) for which the recursive call returns “true”. The iteration stops if \( X^i = X^{i+1} \); this happens after at most polynomially many steps. Then the algorithm return “true” if \( \alpha(\pi') \in X^i \).

The depth of the recursion is precisely the “operator depth” of the formula, and in each node of the recursion tree the processing time is polynomial in the size of the input structure. Thus in each node, at most polynomially many recursive calls are made, and overall the number of nodes in the recursion tree and thus also the overall running time of the algorithm is \( n^{O(\varphi)} \).

Proof of the Immerman-Vardi Theorem 3.1.5 (sketch). We need to prove that for every vocabulary \( \tau \) with \( \leq \in \tau \) and every property \( P \) of \( \tau \)-structures the following two statements are equivalent:

(i) There is a sentence \( \varphi \in \text{IFP}[\tau] \) such that for all ordered \( \tau \)-structures \( A \) we have \( A \models \varphi \iff A \in P \).

(ii) There is a polynomial time algorithm \( (\mathcal{M}, \rho) \) such that for all ordered \( \tau \)-structures \( A \) the Turing machine \( \mathcal{M} \) accepts \( \langle A \rangle \) in at most \( \rho(|\langle A \rangle|) \) steps if and only if \( A \in P \).

Note that for vocabularies \( \tau \) with \( \leq \notin \tau \) there is nothing to prove, because \( \mathcal{O}[\tau] = \emptyset \). The direction (i)\( \Rightarrow \) (ii) follows from Lemma 3.1.6. Thus we only need to prove (ii)\( \Rightarrow \) (i).

Let \( \tau \) be a vocabulary, \( P \) a property of \( \tau \)-structures, and \( (\mathcal{M}, \rho) \) a polynomial time algorithm as in (ii). Our goal is to define a sentence \( \varphi \in \text{IFP}[\tau] \) such that for all \( A \in \mathcal{O}[\tau] \) we have \( A \models \varphi \) if and only if \( \mathcal{M} \) accepts \( \langle A \rangle \) in at most \( \rho(|\langle A \rangle|) \) steps.

Without loss of generality we restrict our attention to input structures with at least 2 elements; for each of the finitely many \( \tau \)-structures with at most 1 element we can hardwire the behaviour of \( \mathcal{M} \) in \( \varphi \). Let \( m = m(n) \) be the length of the string \( \langle A \rangle \) for \( \tau \)-structures \( A \) of order \( |A| = n \), and let \( k \in \mathbb{N} \) such that \( 2\rho(m) + 1 < n^k \) for all \( n \geq 2 \) and such that \( \mathcal{M} \) has at most \( 2^k \) states and at most \( 2^k \) symbols in its alphabet. Let \( A \in \mathcal{O}[\tau] \) with \( n := |A| \geq 2 \), and suppose that \( V(A) = \{v_0, \ldots, v_{n-1}\} \) with \( v_0 \leq^A v_1 \leq^A \ldots \leq^A v_{n-1} \). Each \( k \)-tuple \( \pi = (v_{i_1}, \ldots, v_{i_k}) \) of elements of \( V(A) \) represents a number

\[
\langle \pi \rangle := \sum_{j=1}^{k} i_j n^{j-1} \in [0, n^k - 1]. \tag{3.1.2}
\]

Suppose that \( \mathcal{M} \) has \( \ell \) tapes. It suffices to consider the first \( \rho(m) \) steps of the computation of \( \mathcal{M} \) on input \( \langle A \rangle \). In these steps, on each tape only the \( \rho(m) \) tape cells left and right of the initial position can be used. Overall, we can index the steps of the computation and the tape cells that can potentially be used by numbers in \([2\rho(m)+1] \subseteq [0, n^k-1] \). Hence we can describe a configuration of \( \mathcal{M} \) in the computation on input \( \langle A \rangle \) by a tuple \( (q, h_1, \ldots, h_{\ell}, T_1, \ldots, T_\ell) \), where \( q \in [0, n^k-1] \) represents the state, \( h_i \in [0, n^k-1] \) represent the head position on tape \( i \), and \( T_i \subseteq [0, n^k-1]^2 \) is a binary relation that contains a pair \( (b,c) \) if and only if the \( b \)th
cell on the $i$th tape is labeled by the symbol represented by $c$. We can describe the whole computation by a tuple $(Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell)$, where $Q \subseteq [n^k]^2$ is a binary relation that contains a pair $(a, b)$ if and only if the state in the $a$th step of the computation is $b$, $H_i \subseteq [n^k]^2$ is a binary relation that contains a pair $(a, b)$ if and only if the head position on the $i$th tape in the $a$th step of the computation is $b$, and $S_i \subseteq [0, n^k - 1]^3$ is a ternary relation that contains a pair $(a, b, c)$ if and only if in the $a$th step of the computation the $b$th cell on the $i$th tape is labeled by the symbol represented by $c$. Via the encoding $\langle \cdot \rangle$ of (3.1.2), we may view $(Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell)$ as a tuple of $2k$ and $3k$-ary relations over $V(A)$ as well.

We shall define an ifp-formula

$$
\psi(\overline{t}, \overline{x}) := \text{ifp}

\begin{pmatrix}
X\overline{t}\overline{x} & \leftarrow \chi(\overline{t}, \overline{x}, X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell) \\
Y_1\overline{t}\overline{y} & \leftarrow \xi_1(\overline{t}, \overline{y}, X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell) \\
\vdots & \vdots \\
Z_1\overline{t}\overline{z}\overline{z}' & \leftarrow \zeta_1(\overline{t}, \overline{z}, \overline{z}', X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell) \\
\vdots & \vdots \\
Z_\ell\overline{t}\overline{z}\overline{z}' & \leftarrow \zeta_\ell(\overline{t}, \overline{z}, \overline{z}', X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell)
\end{pmatrix}

\overline{t}\overline{x},
$$

where $X$ and the $Y_i$ are $2k$-ary relation variables, the $Z_i$ are $3k$-ary relation variables, and $\overline{t}, \overline{x}, \overline{y}, \overline{z}, \overline{z}'$ are $k$-tuples of vertex variables, such that the simultaneous fixed-point

$$(X^\infty, Y_1^\infty, \ldots, Y_\ell^\infty, Z_1^\infty, \ldots, Z_\ell^\infty)$$

in $A$ is precisely our description $(Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell)$ of the computation of $\mathfrak{M}$. Then we can let

$$\varphi := \exists \overline{x} \exists \overline{z} \left( \langle \overline{x} \rangle \leq p(m) \land \bigvee_q \langle \overline{z} \rangle = q \land \psi(\overline{t}, \overline{x}) \right),$$

where $\langle \overline{x} \rangle \leq p(m)$ is an IFP-formula in the vocabulary $\{\leq\}$ expressing that the number encoded by the tuple $\overline{x}$ is at most $p(m)$, the union is over all accepting states $q$ of $\mathfrak{M}$, and $\langle \overline{z} \rangle = q$ is an IFP-formula in the vocabulary $\{\leq\}$ expressing that the number encoded by the tuple $\overline{z}$ is $q$ (or more precisely, the number representing the state $q$).

It remains to define $\chi$ and the $\xi_i, \zeta_i$. We define formulae $\chi^0(\overline{x}), \xi^0(\overline{y})$, and $\zeta^0(\overline{z}, \overline{z}')$ describing the initial configuration of $\mathfrak{M}$ on input $A$. In particular, the formula $\zeta^0$ describes the encoding $\langle A \rangle$, which will be written on the first tape. Moreover, we define formulae $\chi^+(\overline{t}, \overline{x}, X, \overline{Y}, \overline{Z})$ and $\xi^+(\overline{t}, \overline{y}, X, \overline{Y}, \overline{Z})$ and $\zeta^+(\overline{t}, \overline{z}, \overline{z}', X, \overline{Y}, \overline{Z})$ describing the transition between configurations. From these, we can easily define the desired formulae $\chi, \xi_i, \zeta_i$. We omit the details.

**Proof of Fagin’s Theorem 3.1.3 (sketch).** Let $\tau$ be a vocabulary that does not contain the distinguished relation symbol $\leq$. We first prove that $\exists \text{SO}$ captures $\text{NP}$ on the class of ordered $\tau \cup \{\leq\}$-structures. We need to prove that for every property $\mathcal{P}$ of ordered $\tau \cup \{\leq\}$-structures the following two statements are equivalent:

(i) There is a sentence $\varphi \in \exists \text{SO}[\tau]$ such that for all ordered $\tau \cup \{\leq\}$-structures $A$ we have $A \models \varphi \iff A \in \mathcal{P}$.

(ii) There is a nondeterministic polynomial time algorithm $(\mathfrak{M}, p)$ such that for all ordered $\tau \cup \{\leq\}$-structures $A$ the Turing machine $\mathfrak{M}$ accepts $\langle A \rangle$ in at most $p(\langle A \rangle)$ steps if and only if $A \in \mathcal{P}$.

Preliminary Version
For the forward direction, let \( \varphi = \exists X_1 \ldots \exists X_k \psi \), where \( \psi \) is a first-order formula. Given a \( \tau \)-structure \( A \), the nondeterministic algorithm \( \mathfrak{A} \) first nondeterministically guesses an assignment \((P_1, \ldots, P_k) \in A^{(X_1, \ldots, X_\ell)}\) and then (deterministically) evaluates \( \psi \) in the interpretation \((A, P_1, \ldots, P_k)\). The latter is possible in polynomial time by Lemma 3.1.6.

For the backward direction, \((\mathfrak{R}, p)\) be a nondeterministic polynomial time algorithm. We describe configurations and computation paths of \( \mathfrak{R} \) in a similar way as we did for \( \mathfrak{M} \) in the proof of the Immerman-Vardi Theorem. In particular, we can describe a computation path of \( \mathfrak{R} \) of at length most \( p(|\langle A \rangle|) \) by a tuple \((Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell)\) of \( 2k \) and \( 3k \)-ary relations over \( V(A) \), for a suitable \( k \). Furthermore, we can construct a first-order formula \( \psi(X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell) \) such that for all \( \tau \)-structures \( A \) with \(|A| \geq 2 \) and all \((Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell) \in A^{(X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell)}\) we have \( A \models \psi[Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell] \) if and only if \((Q, H_1, \ldots, H_\ell, S_1, \ldots, S_\ell)\) is an accepting computation path of \( \mathfrak{R} \) of length at most \( p(|\langle A \rangle|) \). Then we let

\[
\varphi := \exists X \exists Y_1 \ldots \exists Y_\ell \exists Z_1 \ldots Z_\ell \psi(X, Y_1, \ldots, Y_\ell, Z_1, \ldots, Z_\ell).
\]

It remains to generalise the capturing result from ordered to arbitrary \( \tau \)-structures. Let \( \mathcal{P} \) be a property of \( \tau \)-structures. We need to prove that

\[
\mathcal{P} \text{ is } \exists \text{SO-definable } \iff \mathcal{L}(\mathcal{P}) \in \text{NP}. \tag{3.1.3}
\]

Let \( \mathcal{P}_\leq := \{(A, \leq^A) \mid A \in \mathcal{P}, \leq^A \text{ linear order of } V(A)\} \). Then \( \mathcal{P}_\leq \) is a property of ordered \( \tau \cup \{\leq\}\)-structures. Observe that, by the definition of \( \mathcal{L}(\mathcal{P}) \), we have \( \mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}_\leq) \).

To prove the forward direction of (3.1.3), let \( \varphi \in \exists \text{SO}[\tau] \) be a sentence that defines \( \mathcal{P} \). Then \( \varphi \) is also in \( \exists \text{SO}[\tau \cup \{\leq\}] \) and defines \( \mathcal{P}_\leq \). As \( \exists \text{SO} \) captures \( \text{NP} \) on the class of ordered \( \tau \cup \{\leq\}\)-structures, there is a nondeterministic polynomial time algorithm \( \mathfrak{R} \) that accepts \( \mathcal{L}(\mathcal{P}_\leq) = \mathcal{L}(\mathcal{P}) \).

For the backward direction of (3.1.3), suppose that \( \mathcal{L}(\mathcal{P}) \in \text{NP} \). Again as \( \exists \text{SO} \) captures \( \text{NP} \) on the class of ordered \( \tau \cup \{\leq\}\)-structures, there is a sentence \( \varphi_\leq \in \exists \text{SO}[\tau \cup \{\leq\}] \) such that for every ordered \( \tau \cup \{\leq\}\)-structure \((A, \leq^A)\) we have \( (A, \leq^A) \models \varphi_\leq \iff (A, \leq^A) \in \mathcal{P}_\leq \).

Let \( X_\leq \) be a fresh binary relation variable, and let \( \chi(X_\leq) \) be a first-order formula saying that \( X_\leq \) is a linear order. Let \( \psi(X_\leq) \in \exists \text{SO}[\tau] \) be the formula obtained from \( \varphi_\leq \) by replacing each atom \( x \leq y \) by \( X_\leq(x, y) \). Then for each linear order \( \leq^A \) of \( V(A) \) we have

\[
A \models \psi[\leq^A] \iff (A, \leq^A) \models \varphi_\leq \iff (A, \leq^A) \in \mathcal{P}_\leq \iff A \in \mathcal{P}.
\]

It follows that the sentence \( \varphi := \exists X_\leq (\chi(X_\leq) \land \psi(X_\leq)) \), which is easily seen to be equivalent to an \( \exists \text{SO}[\tau] \)-sentence, defines \( \mathcal{P} \).

We close this section by observing that the Immerman-Vardi Theorem 3.1.5 together with Lemma 3.1.6 implies:

**Corollary 3.1.7.** \( \text{IFP+C} \) captures \( \text{PTIME} \) on the class \( \mathcal{O} \) of ordered structures.

### 3.1.4 The Quest for a Logic Capturing Polynomial Time

The main open question in descriptive complexity theory asks whether there is a logic that captures \( \text{PTIME} \) on all and not just ordered structures. The question goes back to Chandra and Harel [19] and was formulated by Gurevich [59] in the way that is best known...
3.1. Logics Capturing Complexity Classes

today. Gurevich conjectured that there is no logic capturing \( \text{PTIME} \), which would imply that \( \text{PTIME} \neq \text{NP} \), because by Fagin’s Theorem there is a logic capturing \( \text{NP} \).

To make the question precise, we first need to specify what we mean by “logic” in this context. An abstract logic \( \mathcal{L} \) consists of a set of \( \mathcal{L}[\tau] \)-sentences for each vocabulary \( \tau \) and a mapping that associates a property \( \mathcal{P}_\varphi \) of \( \tau \)-structures with each \( \mathcal{L}[\tau] \)-sentence \( \varphi \). We say that \( \varphi \) defines \( \mathcal{P}_\varphi \) and that \( \mathcal{P}_\varphi \) is \( \mathcal{L} \)-definable. For every \( \tau \)-structure \( A \) we write \( A \models_\mathcal{L} \varphi \) (or just \( A \models \varphi \) if \( \mathcal{L} \) is clear from the context) if \( A \in \mathcal{P}_\varphi \).

It is easy to see that we can view the logics \( \text{FO} \), \( \text{IFP} \), \( \text{IFP+C} \) as abstract logics if we restrict our attention to their sentences.

Remark 3.1.8. Our definition of “abstract logic” is very liberal and only captures the bare minimum of what is required of a logic. Nevertheless, Dawar [22] has proved that if there is an abstract logic \( \mathcal{L} \) capturing \( \text{PTIME} \) then there also is a logic \( \mathcal{L}' \) capturing \( \text{PTIME} \) and satisfying much stronger conditions: it is regular in the sense of abstract model theory (see [30]). Actually, Dawar’s logic is obtained by adjoining the vectorised version of a single Lindström quantifier to \( \text{FO} \).

Our view of what constitutes a logic is purely semantic and stands in the tradition of abstract model theory [11]. This is the common perspective in descriptive complexity theory, but there also are more syntactical or proof theoretical views on logic and also on the relations between logic and complexity.

Let \( \mathcal{L} \) be an abstract logic and \( \mathcal{C} \) a class of structures. Furthermore, let \( \mathcal{P} \) be a property of \( \tau \)-structures. We say that a sentence \( \varphi \in \mathcal{L}[\tau] \) defines \( \mathcal{P} \) on \( \mathcal{C} \) if \( \mathcal{P}_\varphi \cap \mathcal{C} = \mathcal{P} \cap \mathcal{C} \), or equivalently, if for all \( A \in \mathcal{C}[\tau] \) we have \( A \models_\mathcal{L} \varphi \iff A \in \mathcal{P} \). We say that an algorithm \( \mathcal{A} \) decides \( \mathcal{P} \) on \( \mathcal{C} \) if for all \( A \in \mathcal{C}[\tau] \) we have: if \( A \in \mathcal{P} \) then \( \mathcal{A} \) accepts all \( x \in \mathcal{L}(A) \), and if \( A \notin \mathcal{P} \) then \( \mathcal{A} \) accepts no \( x \in \mathcal{L}(A) \).

Definition 3.1.9. Let \( \mathcal{L} \) be an abstract logic and \( \mathcal{C} \) a class of structures. Then \( \mathcal{L} \) captures \( \text{PTIME} \) on \( \mathcal{C} \) if it satisfies the following three conditions.

(C.1) For every vocabulary \( \tau \) the set of \( \mathcal{L}[\tau] \)-sentences is decidable.

(C.2) For every vocabulary \( \tau \) there is an algorithm that associates with every sentence \( \varphi \in \mathcal{L}[\tau] \) a polynomial time algorithm \( \mathcal{A} \) that decides \( \mathcal{P}_\varphi \) on \( \mathcal{C} \).

(C.3) For every vocabulary \( \tau \) and every polynomial time algorithm \( \mathcal{A} \) that decides a property \( \mathcal{P}_\mathcal{A} \) of \( \tau \)-structures on \( \mathcal{C} \), there is a sentence \( \varphi \in \mathcal{L} \) that defines \( \mathcal{P}_\mathcal{A} \) on \( \mathcal{C} \).

Furthermore, \( \mathcal{L} \) captures \( \text{PTIME} \) if \( \mathcal{L} \) captures \( \text{PTIME} \) on the class \( \mathcal{S} \) of all structures.

The following example illustrates why we cannot replace [C.2] by the weaker condition that for every vocabulary \( \tau \) and every sentence \( \varphi \in \mathcal{L}[\tau] \) there is a polynomial time algorithm \( \mathcal{A} \) that decides \( \mathcal{P}_\varphi \) on \( \mathcal{C} \). (This weaker condition was underlying the “working definition” of a logic capturing a complexity class used in the previous section.)

Example 3.1.10. As there are only countably many polynomial time algorithms, there are only countably many properties of \( \tau \)-structures decidable in polynomial time. Let \( \mathcal{P}_1^\tau, \mathcal{P}_2^\tau, \mathcal{P}_3^\tau, \ldots \) be an enumeration of all such properties. (We do not require this enumeration to be computable, but just an abstract list.) Now we let \( \mathcal{L} \) be the “logic” whose syntax is defined by \( \mathcal{L}[\tau] := \mathbb{N}^+ \) for every \( \tau \) and whose semantics is defined by letting sentence \( n \in \mathcal{L}[\tau] \) define the property \( \mathcal{P}_n^\tau \). Then \( \mathcal{L} \) satisfies the requirements [C.1] and [C.3], and for every sentence
\( \varphi \in L[\tau] \) there is a polynomial time algorithm \( \mathfrak{A} \) that decides \( \mathcal{P}_\varphi \). Yet \( L \) is clearly not a meaningful logic capturing \( \text{PTIME} \).

It is easy to see that the logics \( \text{FO} \), \( \text{IFP} \), and \( \text{IFP} + C \), viewed as abstract logics, all satisfy (C.1). It follows from Lemma 3.1.6 that they also satisfy (C.2) for the class of all structures and hence also for every other class \( \mathcal{C} \) of structures. Thus to prove that one of these logics captures \( \text{PTIME} \) on a class \( \mathcal{C} \) of structures, it suffices to verify (C.3). It is known that neither of the three logics captures \( \text{PTIME} \) (on the class of all structures). For \( \text{FO} \) and \( \text{IFP} \) this follows from the fact that they are less expressive than \( \text{IFP} + C \), which is actually not difficult to prove (see Example 3.4.6). It is surprisingly hard to prove that \( \text{IFP} + C \) does not capture polynomial time. For a while, \( \text{IFP} + C \) was actually viewed as a serious candidate for a logic that captures \( \text{PTIME} \), until the following result destroyed such hopes.

**Fact 3.1.11 (Cai, F"urer, and Immerman[16]).** There is a polynomial time decidable property \( \mathcal{P}_{\text{CFI}} \) of 3-regular graphs that is not definable in \( \text{IFP} + C \). Thus \( \text{IFP} + C \) does not capture \( \text{PTIME} \) on any class \( \mathcal{C} \) of structures that contains the class of all 3-regular graphs.

However, there are also several positive results. First of all, Hella, Kolaitis, and Luosto[64] proved that \( \text{IFP} + C \) captures \( \text{PTIME} \) on almost all graphs. The result can easily be generalised from graphs to arbitrary structures, but we are mostly interested in graphs here. To explain the result, let \( \mathcal{C} \) be a class of graphs. For every \( n \in \mathbb{N} \), we let \( p_n(\mathcal{C}) \) be the probability that a graph \( G \) with \( V(G) = [n] \) chosen uniformly at random is in \( \mathcal{C} \). We say that a class \( \mathcal{C} \) of graphs contains almost all graphs if \( \lim_{n \to \infty} p_n(\mathcal{C}) = 1 \). The precise formulation of Hella, Kolaitis, and Luosto’s result is that there is a class \( \mathcal{C} \) of graphs that contains almost all graphs such that \( \text{IFP} + C \) captures \( \text{PTIME} \) on \( \mathcal{C} \). The intuition behind this result is that in a random graph \( G \), with high probability any two vertices either have different degrees, or if they have the same degree the sequences of degrees of their neighbours are different. This had been proved earlier by Babai, Erd"os, and Selkow[8] in the context of the graph isomorphism problem (see Section 3.5). We can use the difference in the degrees to define a linear order on the vertices of a random graph and then apply the Immerman-Vardi theorem. This technique will be explained in the next section.

However, it may be said that almost all interesting graphs are not random (this is almost a tautology). As far as specific graph classes are concerned, Immerman and Landers[76] proved that \( \text{IFP} + C \) captures \( \text{PTIME} \) on the class of all trees. In the conference papers that lead to this book, I proved that \( \text{IFP} + C \) captures \( \text{PTIME} \) on the class of all planar graphs[42] (see Chapter 9), on the class of all graphs that exclude \( K_5 \) as a minor[45] (see Chapter 11), and, together with Julian Mariño[53], on all classes of graphs of bounded tree width (see Chapter 6). All these results are superseded by the main result of this book that \( \text{IFP} + C \) captures \( \text{PTIME} \) on all classes of graphs with excluded minors. There are also a few interesting classes that do not exclude any graph as a minor and on which \( \text{IFP} + C \) still captures \( \text{PTIME} \). In[46], I proved that the class of all chordal line graphs is an example. More significantly, Laubner[83] proved that \( \text{IFP} + C \) captures \( \text{PTIME} \) on the class of all interval graphs.

### 3.2 Definable Orders

The Immerman-Vardi Theorem can be used to prove that the logics \( \text{IFP} \) and \( \text{IFP} + C \) capture \( \text{PTIME} \) not only on ordered structures, but on all classes of structures that admit a definable order. Throughout this and the next section, we make the following assumption.
**Assumption 3.2.1.** \( L \) is one of the logics IFP, IFP+C.

**Definition 3.2.2.**
1. A formula \( \varphi(\overline{x}, y_1, y_2) \in L[\tau] \) defines an order on a \( \tau \)-structure \( A \) (with parameters \( \overline{x} \)) if there is a tuple \( \overline{y} \in A^{\overline{y}} \) such that the binary relation \( \varphi[A, \overline{y}, y_1, y_2] \) is a linear order on \( V(A) \).

2. A formula \( \varphi(\overline{x}, y_1, y_2) \in L[\tau] \) defines orders on a class \( C \) of \( \tau \)-structures (with parameters \( \overline{x} \)) if \( \varphi(\overline{x}, y_1, y_2) \) defines an order on all structures \( A \in C \).

3. A class \( C \) admits \( L \)-definable orders if for every vocabulary \( \tau \) there is an \( L[\tau] \)-formula that defines orders on \( C[\tau] \).

**Example 3.2.3.** The following IFP-formula defines orders on the class of all paths:

\[
\text{ord-paths}(x, y_1, y_2) := \text{ifp} \left( Y(y_1, y_2) \leftarrow y_1 = x \lor \exists z (Y(z, y_2) \land z \neq y_2 \land E(z, y_1)) \right)(y_1, y_2).
\]

**Remark 3.2.4.** Note that we only admit individual variables as parameters in formulae defining orders. If we would also admit relation variables, the a formula

\[
\varphi(X, y_1, y_2) := \text{order}(X) \land X(y_1, y_2)
\]

with a binary relation variable \( X \) as a parameter and subformula \( \text{order}(X) \) expressing that \( X \) is a linear order, would trivially define an order on all structures.

**Lemma 3.2.5.** For every formula \( \varphi(x_1, y_1, y_2) \in L[\tau] \), there is a sentence \( \text{ord}_\varphi \) that defines the class of all structures \( A \) such that \( \varphi(x_1, y_1, y_2) \) defines an order on \( A \).

**Proof.** Straightforward.

The following lemma is a fairly simple consequence of the Immerman-Vardi Theorem. We will prove it as a corollary of a more general result (Lemma 3.3.8) later.

**Lemma 3.2.6.** Let \( C \) be a class of \( \tau \)-structures that admits \( L \)-definable orders. Then \( L \) captures polynomial time on \( C \).

**Corollary 3.2.7.** Let \( \mathcal{P} \) be a polynomial time decidable property of \( \tau \)-structures that admits \( L \)-definable orders. Then \( \mathcal{P} \) is \( L \)-definable.

**Proof.** Let \( \varphi(\overline{x}, y_1, y_2) \) be an \( L \)-formula that defines orders on \( \mathcal{P} \). Let \( C \) be the class of all \( \tau \)-structures \( A \) such that \( \varphi(\overline{x}, y_1, y_2) \) defines an order on \( A \). Then by Lemma 3.2.5, there is an \( L \)-sentence \( \text{ord}_\varphi \) that defines \( C \). By Lemma 3.2.6, there is an \( L \)-sentence \( \varphi \) that defines \( \mathcal{P} \) on \( C \). Then \( \varphi \land \text{ord}_\varphi \) defines \( \mathcal{P} \).

The following lemma is an extension of the corollary from structures to interpretations.

**Lemma 3.2.8.** Let \( \xi \) be a signature that only contains plain relation variables and vertex variables. Let \( \mathcal{Q} \) a \((\tau, \xi)\)-query such that:

1. \( \mathcal{Q} \) is decidable in polynomial time.
2. The class \( \{ A \mid (A, \alpha) \in \mathcal{Q} \} \) of \( \tau \)-structures admits \( L \)-definable orders.

Then \( \mathcal{Q} \) is \( L \)-definable.
Proof. For every $k$-ary relation variable $X \in \xi$, we let $R_X$ be a fresh $k$-ary relation symbol, and for every vertex variables $x \in \xi$ we let $R_x$ be a fresh unary relation symbol. We let $\tau_\xi$ be the union of $\tau$ with all these relation symbols. For every $(\tau, \xi)$-interpretation $(A, \alpha)$ we let $A_\alpha$ be the $\tau_\xi$-expansion of $A$ with $R_X(A_\alpha) := \alpha(X)$ for all relation variables $X \in \xi$ and $R_x(A_\alpha) := \{\alpha(x)\}$ for all individual variables $x \in \xi$. We let $Q_\xi := \{A_\alpha \mid (A, \alpha) \in Q\}$. Note that $Q_\xi$ is a property of $\tau_\xi$-structures, because $Q$ is a query.

It follows from (i) that $Q_\xi$ is polynomial time decidable and from (ii) that $Q_\xi$ admits $L$-definable orders. Hence by Corollary 3.2.7, $Q_\xi$ is $L$-definable. This implies that $Q$ is $L$-definable as well.

Remark 3.2.9. The restriction to plain relation variables and vertex variables in Lemma 3.2.8 is not necessary, but it makes the proof simpler, and we only need this restricted version.  

Lemma 3.2.10 (Union Lemma for Definable Orders). Let $C, D$ be classes of $\tau$-structures that admit $L$-definable orders. Then the class $C \cup D$ admits $L$-definable orders.

Proof. Let $\varphi(\tau_1, y_1, y_2) \in L[\tau]$ define orders on $C$ and $\psi(\tau_1, y_1, y_2) \in L[\tau]$ on $D$. Choose the formula $\text{ord}_\varphi$ according to Lemma 3.2.5.

Then the formula

$$\chi(\tau_1 \tau_2, y_1, y_2) := (\text{ord}_\varphi \land \varphi(\tau_1, y_1, y_2)) \lor (\neg \text{ord}_\varphi \land \psi(\tau_2, y_1, y_2))$$

defines orders on $C \cup D$.

Lemma 3.2.11. Every class of $\tau$-structures that is finite up to isomorphism admits $L$-definable orders.

Proof. For every $k \in \mathbb{N}^+$, the formula $\varphi(x_1, \ldots, x_k, y_1, y_2) := \bigvee_{1 \leq i, j \leq k} (y_1 = x_i \land y_2 = x_j)$ defines orders on the class of all $k$-element structures.

Corollary 3.2.12. Let $C$ and $C^*$ be classes of $\tau$-structures such that the symmetric difference $C \Delta C^*$ is finite up to isomorphism. Then $C$ admits $L$-definable orders if and only if $C^*$ admits $L$-definable orders.

Recall that for every class $C$ of structures and every $k \in \mathbb{N}^+$, by $N_k(C)$ we denote the class of all $k$-enlargements of structures in $C$. That is, $N_k(C) = \{A \mid \exists W \subseteq V(A) : |W| \leq k \text{ and } A \setminus W \in C\}$.

Lemma 3.2.13 (Finite Extension Lemma for Definable Orders). Let $C$ be a class of $\tau$-structures that admits $L$-definable orders. Then for every $k \in \mathbb{N}$, the class $N_k(C)$ admits $L$-definable orders.

Proof. Let $\varphi(\tau, y_1, y_2)$ be an $L$-formula that defines orders on the class $C$, and let $k \in \mathbb{N}^+$. Let $\tau = (z_1, \ldots, z_k)$ be a tuple of vertex variables, and let $\Theta(\tau)$ be the simple IFP$[\tau, \tau]$-transduction with $\theta_{\text{dom}}(\tau) := \text{true}$, $\theta_V(\tau, y) = \bigwedge_{i=1}^k y \neq z_i$ and $\theta_R(\tau, \overline{y}) = R(\overline{y})$ for all $R \in \tau$. Then for every $\tau$-structure $A$ and every tuple $\overline{v} \in V(A)^k$, it holds that $\Theta[A, \overline{v}] = A \setminus \overline{w}$.

We apply the Transduction Lemma (Fact 2.4.6) to $\Theta(\tau)$ and $\varphi(\tau, y_1, y_2)$ and obtain a formula $\varphi^{\Theta}(\tau, \overline{v}, y_1, y_2)$ such that for every $\tau$-structure $A$ and for all $\overline{v} \in V(A)^k$, $\overline{w} \in A$ and $v_1, v_2 \in V(A)$ it holds that

$$A \models \varphi^{\Theta}(\overline{w}, \overline{v}, v_1, v_2) \iff (\overline{u} \cup \{v_1, v_2\}) \cap \overline{w} = \emptyset \text{ and } A \setminus \overline{w} \models \varphi(\overline{u}, v_1, v_2).$$  \hspace{1cm} (3.2.1)
3.3. Definable Canonisation

Now suppose that \( A \in \mathcal{N}_k(C) \setminus C \). Then there is a set \( W \subseteq V(A) \) with \( 1 \leq |W| \leq k \) such that \( A \setminus W \in C \). Let \( W = \{w_1, \ldots, w_\ell\} \) be such a set, where \( w_1, \ldots, w_\ell \) are pairwise distinct, and let \( \overline{w} \) be the \( k \)-tuple \((w_1, \ldots, w_{\ell-1}, w_\ell, w_\ell, \ldots, w_\ell)\), where \( w_\ell \) is repeated \((k-\ell+1)\) times. Then \( \tilde{w} = W \) and thus \( \Theta[A, \overline{w}] = A \setminus W \in C \). Hence the formula \( \varphi(\overline{x}, y_1, y_2) \) defines an order on \( A \setminus W \). This means that there is a tuple \( \overline{u} \in A^\ell \) with \( \overline{u} \cap \tilde{w} = \emptyset \) such that \( \varphi[A \setminus W, \overline{u}, y_1, y_2] \) is a linear order of \( V(A) \setminus W = V(A \setminus \tilde{w}) \). By \( \Theta[A, \overline{w}, \overline{u}, y_1, y_2] \), this implies that \( \varphi^{-\Theta}(\overline{x}, \overline{y}, y_1, y_2) \) is a linear order of \( V(A) \setminus \tilde{w} \). We let

\[
\psi(\overline{z}, \overline{x}, y_1, y_2) := \bigvee_{i,j \in [k]} (y_1 = z_i \land y_2 = z_j)
\]
\[
\land \bigvee_{i=1}^k (y_1 = z_i \land \bigwedge_{i=1}^k y_2 \neq z_i)
\]
\[
\land \varphi^{-\Theta}(\overline{x}, \overline{x}, y_1, y_2).
\]

Then \( \psi[A, \overline{w}, \overline{u}, y_1, y_2] \) is a linear order of \( V(A) \), and thus \( \psi(\overline{z}, \overline{x}, y_1, y_2) \) defines an order on \( A \) with parameters \( \overline{w} \). Thus the class \( \mathcal{N}_k(C) \setminus C \) admits \( \mathcal{L} \)-definable orders, and as the class \( C \) admits \( \mathcal{L} \)-definable orders as well, the statement of the lemma follows from the Union Lemma \ref{lemma:union-lemma}.

Example 3.2.14. As every cycle is a 1-enlargement of a path, it follows from the Finite Extension Lemma \ref{lemma:finite-extension} Lemma and Example \ref{example:finite-extension} that the class of cycles admits \( \mathcal{IFP} \)-definable orders. It is also easy to see this directly; we leave it as an exercise for the reader to construct an \( \mathcal{IFP} \)-formula \( \varphi(x_1, x_2, y_2, y_2) \) that defines an order on every cycle.

Example 3.2.15. The class of all connected graphs of maximum degree 2 admits \( \mathcal{IFP} \)-definable orders. To see this, note that all connected graphs of maximum degree 2 are paths or cycles. Thus the result follows from Examples \ref{example:finite-extension} and \ref{example:finite-extension} and the Union Lemma \ref{lemma:union-lemma}.

3.3 Definable Canonisation

Remember that in this section we still make Assumption \ref{assumption:tau} that is, \( \mathcal{L} \) is one of the logics \( \mathcal{IFP} \), \( \mathcal{IFP}+\mathcal{C} \). In addition, we make the following assumption.

Assumption 3.3.1. \( \tau \) is a vocabulary with \( \leq \not\in \tau \).

Before we introduce the notion of “definable canonisation”, let us first review the meaning of “canonisation” in the standard algorithmic and complexity theoretic context and then adapt it to our logical setting. We usually think of a canonisation mapping for a class \( C \) of \( \tau \)-structures as a mapping \( \mathcal{C} : C \to S \) such that for all \( A \in C \) we have \( \mathcal{C}(A) \cong A \), and for all \( A, B \in C \), if \( A \cong B \) then \( \mathcal{C}(A) = \mathcal{C}(B) \). A canonisation algorithm for \( C \) is an algorithm that computes a canonisation mapping for \( C \), that is, an algorithm \( \mathcal{C} \) that, given a string \( x \in \mathcal{L}(A) \) for some \( A \in C \), computes a string \( \mathcal{C}(x) \) such that \( \mathcal{C}(x) \in \mathcal{L}(A) \) and for all \( x, y \in \mathcal{L}(A) \) we have \( \mathcal{C}(x) = \mathcal{C}(y) \). We say that a class \( C \) of graphs admits polynomial time canonisation if there is a polynomial time canonisation algorithm for \( C \).

Since equality of structures is a problematic concept both in the logical context, where isomorphism is the more natural notion, and in the algorithmic context, where structures are
represented by binary strings and we can only speak of equality of these strings, it will be convenient to lift canonisation to the slightly more abstract level of structures and ordered structures. An ordered copy of a $\tau$-structure $A$ is an ordered $\tau \cup \{\leq\}$-structure $A'$ such that $A'|_\tau \cong A$. An abstract canonisation mapping for a class $C$ of $\tau$-structures is a mapping $\mathfrak{c} : C \rightarrow O[\tau \cup \{\leq\}]$ such that the following two conditions are satisfied:

**(CM.1)** For all $A \in C$ the $\tau \cup \{\leq\}$-structure $\mathfrak{c}(A)$ is an ordered copy of $A$.

**(CM.2)** For all $A,B \in C$, if $A \cong B$ then $\mathfrak{c}(A) \cong \mathfrak{c}(B)$.

Note that if $\mathfrak{c}$ is an abstract canonisation mapping for $C$ then the mapping $\mathfrak{c}'$ that maps each structure $A \in C$ to the unique $\tau$-structure $A'$ such that $V(A') = |A|$ and $(A', \leq) \cong \mathfrak{c}(A)$, where $\leq$ denotes the natural order on $|A|$, is a canonisation mapping. Conversely, if $\mathfrak{c}$ is a canonisation mapping for $C$ then we can obtain an abstract canonical mapping $\mathfrak{c}'$ for $C$ by choosing an arbitrary linear order $\leq^A$ of $V(A)$ for each $\tau$-structure $A$ and then letting $\mathfrak{c}'(A) := (\mathfrak{c}(A), \leq^{\mathfrak{c}(A)})$. Hence canonisation mappings and abstract canonical mappings are essentially the same thing. Observe that on the algorithmic level the difference between canonical mappings and abstract canonical mappings vanishes completely, because our representation of structures by binary strings does not distinguish between isomorphic ordered structures.

After these general remarks and definitions regarding canonisation, we now come to the key notion of definable canonisation.

**Definition 3.3.2.** Let $\Theta(\tau)$ be an $\mathcal{L}[\tau, \tau \cup \{\leq\}]$-transduction.

1. $\Theta(\tau)$ canonises a $\tau$-structure $A$ if there is at least one tuple $\overline{p} \in A^\tau$ such that $(A, \overline{p}) \in \mathcal{D}_{\Theta(\tau)}$, and for all tuples $\overline{p} \in A^\tau$ such that $(A, \overline{p}) \in \mathcal{D}_{\Theta(\tau)}$ the $\tau \cup \{\leq\}$-structure $\Theta[A, \overline{p}]$ is an ordered copy of $A$.

2. $\Theta(\tau)$ canonises a class $C$ of $\tau$-structures if it canonises all $A \in C$.

3. An L-canonisation of a class $C$ of $\tau$-structures is an $\mathcal{L}[\tau, \tau \cup \{\leq\}]$-transduction that canonises $C$.

4. A class $C$ of structures admits L-definable canonisation if for all vocabularies $\tau$ the class $\mathcal{C}[\tau]$ has an L-canonisation.

**Remark 3.3.3.** Note that in canonisations we only admit individual variables as parameters. If we would also admit relation variables, then it would be easy to define an IFP-canonisation for the class of all $\tau$-structures. We leave it as an exercise for the reader to prove this (cf. Remark 3.2.4).

It is not obvious that an L-canonisation yields an (abstract) canonisation mapping, but indeed it does, even a polynomial time computable one. For an L-canonisation $\Theta$ without parameters this is obvious: the mapping $A \mapsto \Theta[A]$ is an abstract canonisation mapping. If $\Theta(\tau)$ is an L-canonisation of a class $C$ of $\tau$-structures with parameters $\pi$, then we define an abstract canonisation mapping $\mathfrak{c}$ for $C$ by letting $\mathfrak{c}(A)$ be a structure $\Theta[A, \overline{p}]$ for some $\overline{p}$ such that $(\Theta[A, \overline{p}])$ is lexicographically minimal in

$$\{(\Theta[A, \overline{q}]) \mid \overline{q} \in A^\tau \text{ such that } (A, \overline{q}) \in \mathcal{D}_{\Theta(\pi)}\}.$$  

As there are only polynomially many choices for the tuple $\overline{p}$ and as IFP+C-formulae can be evaluated in polynomial time (by Lemma 3.1.6), there is a polynomial time algorithm that computes this canonisation mapping $\mathfrak{c}$.
Example 3.3.4. Let $\mathcal{K}$ be the class of all complete graphs. Note that the class $\mathcal{K}$ does not admit $\text{IFP}+\text{C}$-definable orders, because even if we fix $k$ vertices as parameters, a complete graph $K \in \mathcal{K}$ with $|K| \geq k + 2$ still has a nontrivial automorphism. This would be impossible if we could define an order on the graph with $k$ parameters. Actually, this argument shows that the class $\mathcal{K}$ does not admit $L$-definable orders in any reasonable logic $L$.

Let $\Theta$ be the simple numerical $\text{IFP}+\text{C}[\{E\}, \{E, \leq\}]$-transduction with

- $\theta_{\text{dom}} := \text{true}$;
- $\theta_V(y) := \neg \text{largest}(y)$ (see Example 2.3.5);
- $\theta_E(y_1, y_2) := y_1 \neq y_2$;
- $\theta_{\leq}(y_1, y_2) := y_1 \leq y_2$.

Then for every graph $G$ of order $n := |G|$, the $\{E\}$-restriction of $\Theta[G]$ is the complete graph with vertex set $[0, n - 1]$, and $\leq^{\Theta[G]}$ is the natural order on $V(\Theta[G]) = [0, n - 1]$. Hence if $G$ is a complete graph, then $\Theta[G]$ is an ordered copy of $G$. This means that $\Theta$ canonises $\mathcal{K}$. \[\]

Example 3.3.5. A star is a graph that has one vertex $v$ called the centre of the star such that all other vertices, called tips, are adjacent to $v$, but pairwise nonadjacent (cf. Figure 3.1). (Equivalently, a star is the underlying undirected graph of a directed tree of height at most 1.) Let $\mathcal{ST}$ be the class of all stars, and observe that $\mathcal{ST}$ does not admit $\text{IFP}+\text{C}$-definable orders.

Let $\Theta$ be the simple numerical $\text{IFP}+\text{C}[\{E\}, \{E, \leq\}]$-transduction with

- $\theta_{\text{dom}} := \text{true}$;
- $\theta_V(y) := \neg \text{largest}(y)$;
- $\theta_E(y_1, y_2) := (y_1 = 0 \land y_2 \neq 0) \lor (y_2 = 0 \land y_1 \neq 0)$;
- $\theta_{\leq}(y_1, y_2) := y_1 \leq y_2$.

It is easy to see that $\Theta$ canonises $\mathcal{ST}$. \[\]

It can be proved by standard techniques from finite model theory that the classes $\mathcal{K}$ of complete graphs and $\mathcal{ST}$ of stars do not admit $\text{IFP}$-definable canonisation. The following, more complicated example gives a class of graphs that admits $\text{IFP}$-definable canonisation, but does not admit $\text{IFP}$-definable orders.
Example 3.3.6. For every \( n \in \mathbb{N}^+ \), let \( Q_n \) be the graph with vertex set \( V(Q_n) := [n] \times [2] \) and edge set

\[
E(Q_n) := \{(i, 1)(i, 2) \mid i \in [n]\} \cup \{(i, j)(i + 1, j') \mid i \in [n - 1], j, j' \in [2]\}.
\]

Figure 3.2 shows the graph \( Q_{10} \). Let \( \mathcal{Q} \) be the class of all graphs isomorphic to a graph \( Q_n \) for some \( n \in \mathbb{N}^+ \). Again, for any choice of \( k \)-parameters in \( Q_n \), for \( n > k \), there is a nontrivial automorphism that keeps the parameters fixed. This shows that the class \( \mathcal{Q} \) does not admit IFP+C-definable orders.

It is fairly easy to see that \( \mathcal{Q} \) admits IFP+C-definable canonisation by defining a simple numerical IFP+C\([-\{E\}, \{E, \leq \}\]-transduction \( \Theta \) in such a way that for every \( n \in \mathbb{N}^+ \) it holds that \( V(\Theta[Q_n]) = [0, 2n - 1] \) and the mapping \( (i, j) \mapsto 2i - j - 1 \) is an isomorphism from \( Q_n \) to \( \Theta[Q_n] \).

We shall define an IFP-canonisation \( \Theta'(x_1, x_2) \) of \( \mathcal{Q} \). Let \( n \in \mathbb{N}^+ \); without loss of generality we assume that \( n \geq 3 \). Call two vertices \( (i, j), (i', j') \in V(Q_n) \) equivalent if \( i = i' \). Let \( \equiv \) denote this equivalence relation. The following IFP-formula defines \( \equiv \):

\[
equiv(x, x') := \forall y (y = x \lor y = x' \lor (E(x, y) \leftrightarrow E(x', y))).
\]

The ends of \( Q_n \) are the four vertices \((1, 1), (1, 2), (n, 1), (n, 2)\). Note that the ends are the only vertices of \( Q_n \) of degree 3. Hence there is an IFP-formula \( \text{end}(x) \) that defines the ends. Note that the ends fall into two \( \equiv \)-equivalence classes. The parameters \( x_1, x_2 \) of \( \Theta'(x_1, x_2) \) are to be interpreted by two equivalent ends. Hence we let

\[
\theta'_{\text{dom}}(x_1, x_2) := \equiv(x_1, x_2) \land x_1 \neq x_2 \land \text{end}(x_1) \land \text{end}(x_2).
\]

The transduction \( \Theta'(x_1, x_2) \) will be 2-dimensional. We let

\[
\theta'_{\text{V}}(x_1, x_2, y, y') := (y' = x_1 \lor y' = x_2)
\]

and

\[
\theta'_{\equiv}(x_1, x_2, y_1, y_1', y_2, y_2') := \equiv(y_1, y_2) \land y_1' = y_2'.
\]

Then if \( v_1, v_2 \) are two distinct equivalent ends, then we have

\[
V(\Theta'[Q_n, v_1, v_2]) = \left\{(i, 1), (i, 2)\} \mid i \in [n], j \in [2]\right\}
\]

We shall define \( \theta'_{\text{E}} \) in such a way that the mapping \( (i, j) \mapsto \{(i, 1), (i, 2)\} \) is an isomorphism. We achieve this by simply letting

\[
\theta'_{\text{E}}(x_1, x_2, y_1, y_1', y_2, y_2') := E(y_1, y_2) \land \neg \theta'_{\equiv}(x_1, x_2, y_1, y_1', y_2, y_2').
\]

It remains to define the order. We first define a formula \( \text{ord}(x_1, x_2, y, y') \) such that for \( (i, j), (i', j') \), \( v_1, v_2 \in V(Q_n) \),

\[
M. Grohe, Definable Graph Structure Theory
By the Transduction Lemma (in the version stated as Corollary 2.4.7) there exists a formula that captures polynomial time on $L$-structures. By the Immerman-Vardi Theorem, there exists an $L$-definable canonisation of $A$. Then $A$ admits $L$-definable orders. Then $θ_{\leq}(x_1, x_2, y_1, y_2) := \text{ord}(x_1, x_2, y_1, y_2) \land (\text{equiv}(y_1, y_2) \rightarrow (y'_1 = x_1 \lor y'_2 = x_2))$.

**Lemma 3.3.7.** Let $C$ be a class of $τ$-structures that admits $L$-definable orders. Then $C$ admits $L$-definable canonisation.

**Proof.** Let $ϕ(\bar{x}, z_1, z_2)$ be an $L$-formula that defines orders on $C$, and let $\text{ord}_ϕ(\bar{x})$ be a formula expressing that $ϕ(\bar{x}, y_1, y_2)$ is linear order, that is, for all $(τ, \bar{x})$-interpretations $(A, \bar{p})$ we have $A \models \text{ord}_ϕ(\bar{x})$ if and only if $ϕ[A, \bar{p}, y_1, y_2]$ is a linear order of $V(A)$.

Let $Θ(\bar{x})$ be the simple 1-dimensional $L[τ, τ \cup \{≤\}]$-transduction defined by

- $θ_{\text{dom}}(\bar{x}) := \text{ord}_ϕ(\bar{x})$;
- $θ_{\text{dom}}(\bar{x}, y) := \text{true}$;
- $θ_R(\bar{x}, y_1, \ldots, y_k) := R(y_1, \ldots, y_k)$ for all $k$-ary $R ∈ τ$;
- $θ_{≤}(\bar{x}, y_1, y_2) := ϕ(\bar{x}, y_1, y_2)$.

It is easy to see that $Θ(\bar{x})$ canonises $C$. □

The following lemma is the main result of this section.

**Lemma 3.3.8 (Otto [94]).** If a class $C$ of structures admits $L$-definable canonisation, then $L$ captures polynomial time on $C$.

**Proof.** We have to prove that for every polynomial time decidable property $P$ of $τ$-structures there is a sentence $ψ ∈ L[τ]$ such that for all structures $A ∈ C[τ]$ it holds that

$$A ∈ P \iff A \models ψ. \tag{3.3.1}$$

Let $P$ be a polynomial time decidable property of $τ$-structures. Let $P_{≤}$ be the class of ordered $τ \cup \{≤\}$-structures $A$ with $A|τ ∈ P$. Then $P_{≤}$ is a polynomial time decidable property of $τ \cup \{≤\}$-structures. By the Immerman-Vardi Theorem, there exists an $L$-sentence $ϕ_{≤}$ that defines $P_{≤}$.

Let $Θ(\bar{x})$ be an $L$-canonisation of $C$. Then for all $(A, \bar{p}) ∈ D_{Θ(\bar{x})}$ with $A ∈ C$ we have:

$$Θ[A, \bar{p}] \models ϕ_{≤} \iff Θ[A, \bar{p}] ∈ P_{≤} \iff Θ[A, \bar{p}]|τ ∈ P \iff A ∈ P$$

(by $A \cong Θ[A, \bar{p}]|τ$).

By the Transduction Lemma (in the version stated as Corollary 2.4.7) there exists a formula $ϕ(\bar{x}) ∈ L[τ]$ such that for all $(A, \bar{p}) ∈ D_{Θ(\bar{x})}$ we have

$$A \models ϕ(\bar{p}) \iff Θ[A, \bar{p}] \models ϕ_{≤}.$$  

Thus $A \models ϕ(\bar{p}) \iff A ∈ P$, and the sentence $ψ := ⊃ (θ_{\text{dom}}(\bar{x}) ∧ ϕ(\bar{x}))$ satisfies (3.3.1). □
3.3.1 Parameter Independent Transductions

Definition 3.3.9. An L-transduction \( \Theta(\overline{X}) \) is parameter independent if for all structures \( A \) and all \( \overline{P}, \overline{Q} \in G^{\overline{X}} \) such that \( (A, \overline{P}), (A, \overline{Q}) \in D_{\Theta(\overline{X})} \) we have \( \Theta[A, \overline{P}] \cong \Theta[A, \overline{Q}] \).

Parameter-independence is mainly of interest for transductions that serve as canonisations, but it will be useful for us to have it defined for all transductions, at least for all transductions mapping structures to ordered structures. An L-transduction from \( \tau \)-structures to ordered \( \tau \cup \{ \leq \} \)-structures is an L[\( \tau, \tau \cup \{ \leq \} \)]-transduction \( \Theta(\overline{X}) \) such that \( \Theta[G, \overline{P}] \in O[\tau \cup \{ \leq \}] \) for all \( (G, \overline{P}) \in D_{\Theta(\overline{X})} \). Of course the most important example of such transductions are transductions defining a canonisation. The main result of this subsection is the following lemma, which will be proved at the end of the subsection.

Lemma 3.3.10. For every L-transduction \( \Theta(\overline{X}) \) from \( \tau \)-structures to ordered \( \tau \cup \{ \leq \} \)-structures there exists an IFP+C-transduction \( \Theta'(\overline{X}) \) such that the following conditions are satisfied.

(i) \( \Theta'(\overline{X}) \) is parameter independent.

(ii) For all \( (A, \overline{p}) \in D_{\Theta(\overline{X})} \) there exists a \( \overline{q} \in A^\overline{X} \) such that \( (A, \overline{q}) \in D_{\Theta'(\overline{X})} \).

(iii) For all \( (A, \overline{p}) \in D_{\Theta(\overline{X})} \) we have \( (A, \overline{p}) \in D_{\Theta(\overline{X})} \) and \( \Theta'[A, \overline{p}] = \Theta[A, \overline{p}] \).

Corollary 3.3.11. Let \( C \) be a class of \( \tau \)-structures that admits L-definable canonisation. Then there exists a parameter independent transduction \( \Theta(\overline{X}) \) that canonises \( C \).

To prove Lemma 3.3.10, we introduce the lexicographical ordering on \( \tau \)-structures and prove that it is L-definable. This will be relevant beyond the proof of Lemma 3.3.10. Recall the definition of the lexicographical ordering on \( \{0,1\}^* \): for strings \( a_1 \ldots a_m, b_1 \ldots b_n \in \{0,1\}^* \) we let \( a_1 \ldots a_m \leq_{\text{lex}} b_1 \ldots b_n \) if either \( m \leq n \) and \( a_i = b_i \) for all \( i \in [m] \) or if there is an \( i \leq \min\{m, n\} \) such that \( a_i = 0 \) and \( b_i = 1 \) and \( a_j = b_j \) for all \( j \in [i-1] \). We define the lexicographical ordering \( \leq_{s-\text{lex}} \) on the class \( O[\tau \cup \{ \leq \}] \) of ordered \( \tau \cup \{ \leq \} \)-structures by letting

\[
A \leq_{s-\text{lex}} B \iff \langle A \rangle \leq_{\text{lex}} \langle B \rangle,
\]

where the \( \leq_{\text{lex}} \) on the right-hand side denotes the lexicographical order on \( \{0,1\}^* \). Note that, up to isomorphism, \( \leq_{s-\text{lex}} \) is a linear order on \( O[\tau \cup \{ \leq \}] \), that is, it is reflexive, transitive, and total, and for all \( A, B \in O[\tau \cup \{ \leq \}] \) it holds that \( A \leq_{s-\text{lex}} B \) and \( B \leq_{s-\text{lex}} A \Rightarrow A \cong B \).

The following lemma shows that \( \leq_{s-\text{lex}} \) is L-definable (in a certain sense).

Lemma 3.3.12. Let \( \Theta(\overline{X}) \) be an L-transduction from \( \tau \)-structures to ordered \( \tau \cup \{ \leq \} \)-structures. Then there is an formula \( \text{lex}_{\Theta}(\overline{X}, \overline{Q}) \in L[\tau] \) such that for all \( \tau \)-structures \( A \) and all tuples \( \overline{p}, \overline{q} \in A^\overline{X} \) such that \( (A, \overline{p}), (A, \overline{q}) \in D_{\Theta(\overline{X})} \),

\[
A \models \text{lex}_{\Theta}(\overline{p}, \overline{q}) \iff \Theta[A, \overline{p}] \leq_{s-\text{lex}} \Theta[A, \overline{q}].
\]

Proof. Recall the definition of our representation scheme \( \langle \cdot \rangle \) for ordered \( \tau \cup \{ \leq \} \)-structures. Let \( R_1, \ldots, R_\ell \) be the enumeration of \( \tau \) that is used to define \( \langle \cdot \rangle \), and for all \( i \in [\ell] \), let \( r_i \) be the arity of \( R_i \).

M. Grohe, Definable Graph Structure Theory
Let $B, B' \in \mathcal{O}[\tau \cup \{\leq\}]$. Let $n := |B|$ and $n' := |B'|$, and suppose that $V(B) = \{v(1), \ldots, v(n)\}$ with $v(1) \leq^B v(2) \leq^B \ldots \leq^B v(n)$ and $V(B') = \{v'(1), \ldots, v'(n')\}$ with $v'(1) \leq^{B'} \ldots \leq^{B'} v'(n')$.

Observe that $(B) \leq_{\text{lex}} (B)'$ if and only if

(A) $1^n 0 <_{\text{lex}} 1^{n'} 0$,

(B) or $1^n 0 = 1^{n'} 0$ and there is a $k \in [\ell]$ such that $(R_k(B)) \leq_{\text{lex}} (R_k(B'))$ and $(R_j(B)) \leq_{\text{lex}} (R_j(B'))$ for all $j < k$,

(C) or $B \cong B'$.

Note that $1^n 0 <_{\text{lex}} 1^n 0$ if and only if $n < n'$. If $n = n'$, we have $(R_k(B)) \leq_{\text{lex}} (R_k(B'))$ if and only if there is a tuple $(i_1, \ldots, i_{r_k}) \in [n]^{r_k}$ such that $(v(i_1), \ldots, v(i_{r_k})) \notin R_k(B)$ and $(v'(i_1), \ldots, v'(i_{r_k})) \in R_k(B')$ and $(v(j_1), \ldots, v(j_{r_k})) \in R_k(B)$ $\iff$ $(v'(j_1), \ldots, v'(j_{r_k})) \in R_k(B')$ for all $(j_1, \ldots, j_{r_k}) \in [n]^{r_k}$ with $(j_1, \ldots, j_{r_k}) \leq_{\text{lex}} (i_1, \ldots, i_{r_k})$. Hence conditions (A)–(C) are equivalent to the following three conditions.

(D) $n < n'$,

(E) or $n = n'$ and there is a $k \in [\ell]$ and a tuple $(i_1, \ldots, i_{r_k}) \in [n]^{r_k}$ such that

- $(v(i_1), \ldots, v(i_{r_k})) \notin R_k(B)$ and $(v'(i_1), \ldots, v'(i_{r_k})) \in R_k(B')$,
- and $(v(j_1), \ldots, v(j_{r_k})) \in R_k(B')$ $\iff$ $(v'(j_1), \ldots, v'(j_{r_k})) \in R_k(B')$ for all $(j_1, \ldots, j_{r_k}) \in [n]^{r_k}$ with $(j_1, \ldots, j_{r_k}) \leq_{\text{lex}} (i_1, \ldots, i_{r_k})$,
- and $(v(j_1), \ldots, v(j_{r_i})) \in R_i(B)$ $\iff$ $(v'(j_1), \ldots, v'(j_{r_i})) \in R_i(B')$ for all $i < k$ and $(j_1, \ldots, j_{r_i}) \in [n]^{r_i}$,

(F) or $B \cong B'$.

It is easy to see that the conditions (D)–(F) applied to the structures $B := \Theta[A, \overline{p}], B' := \Theta[A, \overline{q}]$ can be formalised in L.

\textbf{Proof of Lemma 3.3.10.} Let $\Theta(\overline{x})$ be an $L$-transduction from $\tau$-structures to ordered $\tau \cup \{\leq\}$-structures. Let

$$\theta'_{\text{dom}}(\overline{x}) := \theta_{\text{dom}}(\overline{x}) \land \forall \overline{x}'(\theta_{\text{dom}}(\overline{x}') \rightarrow \text{lex}_\Theta(\overline{x}, \overline{x}')),$$

where the formula $\text{lex}_\Theta$ is taken from Lemma 3.3.12. Then for every $\tau$-structure $A$ the set $\theta'_{\text{dom}}[A, \overline{x}]$ consists of all $\overline{p}$ that yield a lexicographically minimal $\Theta[A, \overline{p}]$.

We let $\Theta'(\overline{x})$ be the transduction obtained from $\Theta(\overline{x})$ by replacing $\theta_{\text{dom}}(\overline{x})$ by $\theta'_{\text{dom}}(\overline{x})$. \hfill \Box

\subsection{3.3.2 Arithmetic in $\text{IFP}+\text{C}$}

From now on, we focus on the logic $\text{IFP}+\text{C}$. Before we continue developing the theory of definable canonisation, it will be convenient to extend our repertoire of $\text{IFP}+\text{C}$-definable arithmetic. Recall Example 2.3.5 where we defined $\text{IFP}+\text{C}$-formulas for the constants $0, 1, 2, \ldots$, the successor relation, addition, and multiplication. Instead of writing down further tedious $\text{IFP}+\text{C}$-definitions of arithmetical predicates, we can apply the following powerful fact, which easily follows from the Immerman-Vardi Theorem 3.1.5.

Preliminary Version
Fact 3.3.13. Let $R \subseteq \mathbb{N}^k$ be polynomial time decidable if the input numbers are given in unary. Then there is an IFP+C-formula $\varphi_R(x_1, \ldots, x_k)$ with free number variables $x_1, \ldots, x_k$, such that for all structures $A$

$$\varphi_R[A, x_1, \ldots, x_k] = R \cap \text{Num}(A)^k.$$ 

All arithmetic we have defined in IFP+C is restricted to the range $\text{Num}(A) = [0, n]$ in a structure $A$ of order $n$. Occasionally, we need to carry out calculations that involve higher numbers than $n$. For this purpose, we view $k$-tuples $\overline{m} \in \text{Num}(G)^k$ as the base-$(n+1)$ encodings of numbers in $[0, (n+1)^k - 1]$. To be precise, the number encoded by $\overline{m} = (m_1, \ldots, m_k)$ is

$$\langle\langle \overline{m} \rangle\rangle_n := \sum_{i=1}^k m_i \cdot (n + 1)^{k-i}.$$ 

This way, we can expand the range of our calculations at least to polynomials in $n$, and this will be sufficient. It follows from Fact 3.3.13 that for all $k, k_1, k_2, k_3 \in \mathbb{N}^+$ and tuples $\overline{x}, \overline{x}_1, \overline{x}_2, \overline{x}_3$ of number variables of the respective lengths there are IFP+C-formulae $\text{add}_{k_1, k_2, k_3}^{\overline{x}}$, $\text{null}_{k_1}^{\overline{x}}$, $\text{one}_{k}^{\overline{x}}$, $\text{mult}_{k_1, k_2, k_3}^{\overline{x}}$, $\text{null}_{k}^{\overline{x}}$, $\text{one}_{k}^{\overline{x}}$ with the obvious meaning. For example, for all structures $A$ of size $n := |A|$ and all tuples $\overline{m} \in \text{Num}(A)^k$ we have

$$A \models \text{add}_{k_1, k_2, k_3}^{\overline{x}}[\overline{m}_1, \overline{m}_2, \overline{m}_3] \iff \langle\langle \overline{m}_1 \rangle\rangle_n + \langle\langle \overline{m}_2 \rangle\rangle_n = \langle\langle \overline{m}_3 \rangle\rangle_n,$$

$$A \models \text{one}_{k_1}^{\overline{x}}[\overline{m}_1] \iff \langle\langle \overline{m}_1 \rangle\rangle_n = 1.$$ 

We simplify the notation by writing $\overline{x}_1 + \overline{x}_2 = \overline{x}_3$, $\overline{x}_1 \cdot \overline{x}_2 = \overline{x}_3$, $\overline{x} = 0$, $\overline{x} = 1$.

We will also have to extend the counting formulae from single vertex variables to arbitrary tuples of variables.

Fact 3.3.14 ([40]). For all IFP+C-$[\overline{\tau}]$-formulae $\varphi(\overline{X}, \overline{x})$, where $\overline{X}$ is an arbitrary tuple of variables and $\overline{x}$ is a tuple of individual variables, and all $k \in \mathbb{N}^+$ there is an IFP+C-$[\overline{\tau}]$-formula $\text{count}_{\varphi, \overline{\tau}, k}(\overline{X}, \overline{y})$, where $\overline{y}$ is a $k$-tuple of number variables, such that for all $\overline{\tau}$-structures $A$ of order $n := |A|$ and all $\overline{P} \in A^{\overline{X}}$, $\overline{m} \in \text{Num}(A)^k$ we have

$$A \models \text{count}_{\varphi, \overline{\tau}, k}^{\overline{P}, \overline{m}} \iff \langle\langle \overline{m} \rangle\rangle_n = |\{\overline{p} \in A^{\overline{P}} \mid A \models \varphi(\overline{P}, \overline{p})\}|.$$ 

The idea of the proof is expressed in the following equations.

$$\#(x_1, x_2) \varphi(x_1, x_2) = \sum_{x_1} \#x_2 \varphi(x_1, x_2) = \sum_{y} y \cdot \#x_1(\#x_2 \varphi(x_1, x_2) = y).$$

We write $\#\overline{x} \varphi(\overline{X}, \overline{x}) = \overline{y}$ instead of $\text{count}_{\varphi, \overline{\tau}, k}(\overline{X}, \overline{y})$ from now on.

We will tacitly use all these and similar arithmetical formulae in the following.

3.3.3 Normalising IFP+C-Transductions

Recall that a transduction $\Theta(\overline{X})$ is numerical if its domain variables, that is, the variables in the tuple $\overline{y}$ of the formula $\theta_\overline{y}(\overline{X}, \overline{y})$, are number variables.

Definition 3.3.15. (1) An ordered $\overline{\tau}$-structure $A$ is normal if $V(A)$ is an initial segment of the nonnegative integers and $\leq^A$ is the natural linear order on $V(A)$, that is, $\leq^A := \leq_{|V(A)|}$.
3.3. Definable Canonisation

(2) An IFP+C-transduction $\Theta(\bar{x})$ from $\tau$-structures to ordered $\tau \cup \{\leq\}$-structures is normal if it is simple, numerical, and 1-dimensional and for all $(G,\bar{p}) \in \mathcal{D}_{\Theta(\bar{x})}$ the ordered graph $\Theta[G,\bar{p}]$ is normal.

Note that if two normal ordered graphs are isomorphic then they are equal. Also note that for a normal IFP+C-transduction $\Theta(\bar{x})$ we may always assume that $\theta_{\leq}(x,y_1,y_2) = y_1 \leq y_2$.

**Example 3.3.16.** The IFP+C-transductions of Examples 3.3.4 and 3.3.5 which canonise the classes of complete graphs and stars, are both normal.

**Lemma 3.3.17.** Let $\Theta(\bar{x})$ be an IFP+C-transduction from $\tau$-structures to ordered $\tau \cup \{\leq\}$-structures such that for all $(A,\bar{p}) \in \mathcal{D}_{\Theta(\bar{x})}$ it holds that $|\Theta[A,\bar{p}]| \leq |A| + 1$. Then there is a normal IFP+C-transduction $\Theta'(\bar{x})$ such that $\mathcal{D}_{\Theta(\bar{x})} = \mathcal{D}_{\Theta'(\bar{x})}$ and

$$\Theta[A,\bar{p}] \cong \Theta'[A,\bar{p}]$$

for all $(A,\bar{p}) \in \mathcal{D}_{\Theta(\bar{x})}$.

**Proof.** The main part of the proof is to define, for all $(A,\bar{p}) \in \mathcal{D}_{\Theta(\bar{x})}$, a bijection from $V(\Theta[A,\bar{p}])$ to an initial segment of $\text{Num}(A)$ in IFP+C. Note that such a bijection exists, because $|\Theta[A,\bar{p}]| \leq |A| + 1 = |\text{Num}(A)|$. Once we have defined this bijection, it will be easy to define $\Theta'(\bar{x})$.

Let $\text{eqrel}_{\theta_{\leq}}(\bar{x},\bar{y}_1,\bar{y}_2)$ be an IFP+C-formula that defines the equivalence relation $\equiv$ generated by $\theta_{\equiv}(\bar{x},\bar{y}_1,\bar{y}_2)$. The following formula defines the linear order defined by $\theta_{\leq}(\bar{x},\bar{y}_1,\bar{y}_2)$ on the domain of the structure defined by $\Theta(\bar{x})$.

$$\text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y}_1,\bar{y}_2) := \theta_{\equiv}(\bar{x},\bar{y}_1) \land \theta_{\equiv}(\bar{x},\bar{y}_2) \land \exists \bar{y}' \exists \bar{y}_2' \left( \theta_{\equiv}(\bar{x},\bar{y}_1') \land \text{eqrel}_{\theta_{\leq}}(\bar{x},\bar{y}_1,\bar{y}_1') \land \theta_{\equiv}(\bar{x},\bar{y}_2') \land \text{eqrel}_{\theta_{\leq}}(\bar{x},\bar{y}_2,\bar{y}_2') \land \theta_{\equiv}(\bar{x},\bar{y}_1',\bar{y}_2') \right)$$

Next, we define formulae $\text{succe}_{\theta_{\equiv}}(\bar{x},\bar{y}_1,\bar{y}_2)$ and $\text{min}_{\theta_{\equiv}}(\bar{x},\bar{y})$ defining the successor relation and the minimum of the order defined by $\text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y}_1,\bar{y}_2)$. We let

$$\text{succe}_{\theta_{\equiv}}(\bar{x},\bar{y}_1,\bar{y}_2) := \theta_{\equiv}(\bar{x},\bar{y}_1) \land \theta_{\equiv}(\bar{x},\bar{y}_2) \land \text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y}_1,\bar{y}_2) \land \neg \text{eqrel}_{\theta_{\leq}}(\bar{x},\bar{y}_1,\bar{y}_2) \land \forall \bar{y}' \left( \theta_{\equiv}(\bar{x},\bar{y}') \land \text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y},\bar{y}') \land \text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y}_2,\bar{y}') \right)$$

$$\text{min}_{\theta_{\equiv}}(\bar{x},\bar{y}) := \theta_{\equiv}(\bar{x},\bar{y}) \land \forall \bar{y}' \left( \theta_{\equiv}(\bar{x},\bar{y}') \land \text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y},\bar{y}') \land \text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y}_2,\bar{y}') \right) \land \forall \bar{y}' \left( \theta_{\equiv}(\bar{x},\bar{y}',\bar{y}_2) \land \text{leq}_{\theta_{\equiv}}(\bar{x},\bar{y},\bar{y}') \right)$$

Now we are ready to define the bijection between the vertex set of the ordered graph defined by $\Theta(\bar{x})$ and an initial segment of the numerical part of a structure by stepping through all pairs $(\bar{y},z)$ along the successor relations, starting with the minima. We let

$$\text{bij}(\bar{x},\bar{y},z) := \text{ifp} \left( X(\bar{y},z) \leftarrow \left( \text{min}_{\theta_{\equiv}}(\bar{x},\bar{y}) \land z = 0 \right) \lor \exists \bar{y}' \exists z' \left( X(\bar{y}',z') \land \text{succe}_{\theta_{\equiv}}(\bar{x},\bar{y}',\bar{y}_2) \land z' + 1 = z \right) \right) \right)(\bar{y},z).$$

Suppose that $V(\Theta[A,\bar{p}]) = \{\bar{v}_0/\equiv, \ldots, \bar{v}_{n-1}/\equiv\}$, where $\bar{v}_0 <^{\Theta[A,\bar{p}]} \bar{v}_1 <^{\Theta[A,\bar{p}]} \ldots <^{\Theta[A,\bar{p}]} \bar{v}_{n-1}$. Then for all $\bar{v} \in A^7$ and all $i \in \text{Num}(A)$ it holds that

$$A \models \text{bij}[\bar{p},\bar{v},i] \iff \bar{v} \equiv \bar{v}_i.$$

Now we define the IFP+C-$\tau \cup \{\leq\}$-transduction $\Theta'(\bar{x})$ in such a way that the mapping $\bar{v}_i/\equiv \mapsto i$ is an isomorphism from $\Theta[A,\bar{p}]$ to $\Theta'[A,\bar{p}]$. We let

Preliminary Version
• $\theta'_\text{dom}(\bar{x}) := \theta_{\text{dom}}(\bar{x})$,  
• $\theta'_V(\bar{x}, z) := \exists \bar{y} \, \text{bij}(\bar{x}, \bar{y}, z)$,  
• $\theta'_\leq(\bar{x}, z_1, z_2) := z_1 = z_2$,  
• $\theta'_R(\bar{x}, z_1, \ldots, z_k) := \exists \bar{y}_1 \ldots \exists \bar{y}_k (\text{bij}(\bar{x}, \bar{y}_1, z_1) \land \ldots \land \text{bij}(\bar{x}, \bar{y}_k, z_k) \land \theta_R(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k))$ for each $k$-ary $R \in \tau$,  
• $\theta'_<(\bar{x}, z_1, z_2) := (z_1 \leq z_2)$.

Clearly, $\Theta'(\bar{x})$ has the desired properties.

We can combine parameter-independence and normality to eliminate the parameters from $\text{IFP}+\text{C}$-canonisations.

**Lemma 3.3.18.** For every class $C$ of $\tau$-structures that admits $\text{IFP}+\text{C}$-definable canonisation there is a normal $\text{IFP}+\text{C}$-transduction without parameters that canonises $C$.

**Proof.** Let $\Theta(\bar{x})$ be an $\text{IFP}+\text{C}$-transduction from $\tau$-structures to ordered $\tau \cup \{\leq\}$-structures that canonises a class $C$ of $\tau$-structures. Then for all $(A, \bar{p}) \in D_{\Theta(\bar{x})}$ with $A \in C$ we have $|\Theta[A, \bar{p}]| = |A| + 1$, and as we are only interested in structures $A \in C$, we may modify the formula $\theta_{\text{dom}}(\bar{x})$ such that $|\Theta[A, \bar{p}]| = |A|$ for all $(A, \bar{p}) \in D_{\Theta(\bar{x})}$. Thus Lemmas 3.3.10 and 3.3.17 we may assume that $\Theta(\bar{x})$ is parameter-independent and normal.

This means that for all $\tau$-structures $A$ and all $\bar{p}, \bar{q} \in A^\tau$ such that $(A, \bar{p}), (A, \bar{q}) \in D_{\Theta(\bar{x})}$ we have

$$\Theta[A, \bar{p}] = \Theta[A, \bar{q}].$$

To see this, note that $\Theta[A, \bar{p}] \cong \Theta[A, \bar{q}]$ by parameter-independence. By normality, this implies $\Theta[A, \bar{p}] = \Theta[A, \bar{q}]$.

We define a simple $\text{IFP}+\text{C}[\tau, \tau \cup \{\leq\}]$-transduction $\Theta'$ by

• $\theta'_{\text{dom}} := \exists \bar{x} \, \theta_{\text{dom}}(\bar{x})$,  
• $\theta'_V(y) := \exists \bar{x} \, (\theta_{\text{dom}}(\bar{x}) \land \theta_V(\bar{x}, y))$,  
• $\theta'_R(y_1, \ldots, y_k) := \exists \bar{x} \, (\theta_{\text{dom}}(\bar{x}) \land \theta_R(\bar{x}, y_1, \ldots, y_k))$ for every $k$-ary $R \in \tau$,  
• $\theta'_<(y_1, y_2) := y_1 \leq y_2$.

Then $\Theta'[A] = \Theta[A, \bar{p}]$ for all $(A, \bar{p}) \in D_{\Theta(\bar{x})}$. Thus $\Theta'$ is a normal $\text{IFP}+\text{C}$-transduction that canonises $C$. $\square$

### 3.3.4 Components and Disjoint Unions

In this section, we will show how to lift definable canonisations from the connected components of a graph to the whole graph. This is not only a basic result about definable canonisations, but its proof is also the foundation of the proof of the important Canonisation Theorem 7.4.1. Several key ideas of the proof of the Canonisation Theorem appear here in a more transparent form. Therefore, it is worthwhile to make an effort to digest these ideas here. We will give fairly detailed proofs here, down to the level of writing many actual $\text{IFP}+\text{C}$-definitions. We restrict our attention to graphs, although the result can easily be extended to arbitrary relational structures.
An \textit{IFP+C-transduction from graphs to ordered graphs} is an \textit{IFP+C-}\textit{-transduction }\Theta(\overline{X})\textit{ such that for all }\langle G, \overline{P} \rangle \in \mathcal{D}_{\Theta(\overline{X})}\textit{ the }\{E, \leq\}\textit{-structures }G\textit{ is a graph and the }\{E, \leq\}\textit{-structure }\Theta[G, \overline{P}]\textit{ is an ordered graph. The \textit{ordered union} of two disjoint ordered graphs }G, H\textit{ is the ordered graph }G \uplus H\textit{ with }V(G \uplus H) := V(G) \cup V(H)\textit{ and }E(G \uplus H) := E(G) \cup E(H)\textit{ and }\leq_{G \uplus H} \textit{defined by }v \leq_{G \uplus H} w\textit{ if either }v, w \in V(G)\textit{ and }v \leq_{G} w\textit{ or }v \in V(G)\textit{ and }w \in V(H)\textit{ or }v, w \in V(H)\textit{ and }v \leq_{H} w\textit{. For a normal ordered graph }G\textit{ and a }k \in \mathbb{N}\textit{, we let }G + k\textit{ be the image of }G\textit{ under the mapping }i \mapsto i + k\textit{. Moreover, we let }G \times 0\textit{ be the empty graph, and for }k \geq 1\textit{ we let }

\[ G \times k := G \uplus \underbrace{(G + n) \uplus \cdots \uplus (G + (k - 1)n)}_{n := |G|}, \]

where } n := |G| \textit{. The following lemma states that }G + k \textit{ and }G \times k\textit{ are IFP+C-definable.}

\textbf{Lemma 3.3.19.} \textit{Let }\Theta(\overline{x})\textit{ be a normal IFP+C-transduction from graphs to ordered graphs.}

1. \textit{There is a normal IFP+C-transduction }\Theta^{+}(\overline{x}, y)\textit{ from graphs to ordered graphs such that}
   \begin{itemize}
   \item \(\mathcal{D}_{\Theta^{+}(\overline{x}, y)}\) consists of all }\langle G, \overline{p}, k \rangle\textit{ such that }\langle G, \overline{p} \rangle \in \mathcal{D}_{\Theta(\overline{x})}\textit{ and }|\Theta[G, \overline{p}]| + k \leq |\text{Num}(G)|;
   \item for all }\langle G, \overline{p}, k \rangle \in \mathcal{D}_{\Theta^{+}(\overline{x}, y)}\textit{ we have }\Theta^{+}[G, \overline{p}, k] = \Theta[G, \overline{p}] + k.
   \end{itemize}

2. \textit{There is a normal IFP+C-transduction }\Theta^{\times}(\overline{x}, y)\textit{ from graphs to ordered graphs such that}
   \begin{itemize}
   \item \(\mathcal{D}_{\Theta^{\times}(\overline{x}, y)}\) consists of all }\langle G, \overline{p}, k \rangle\textit{ such that }\langle G, \overline{p} \rangle \in \mathcal{D}_{\Theta(\overline{x})}\textit{ and }|\Theta[G, \overline{p}]| \cdot k \leq |\text{Num}(G)|;
   \item for all }\langle G, \overline{p}, k \rangle \in \mathcal{D}_{\Theta^{\times}(\overline{x}, y)}\textit{ we have }\Theta^{\times}[G, \overline{p}, k] = \Theta[G, \overline{p}] \times k.
   \end{itemize}

\textit{Proof.} We let }\text{ord}_{\Theta}(\overline{x}, y) := \#z \theta_{V}(\overline{x}, z) = y.\textit{ Then for all }\langle G, \overline{p} \rangle \in \mathcal{D}_{\Theta(\overline{x})}\textit{ and all }n \in \text{Num}(G)\textit{ we have }

\[ G \models \text{ord}_{\Theta}[\overline{p}, n] \iff |\Theta[G, \overline{p}]| = n. \]

We define the domain of }\Theta^{+}(\overline{x}, y)\textit{ by the following formula:

\[ \theta^{+}_{\text{dom}}(\overline{x}, y) := \theta_{\text{dom}}(\overline{x}) \land \exists y'(\text{ord}_{\Theta}(\overline{x}, y') \land \exists y'' y' + y' \leq y'' + 1). \]

Here the subformula }\exists y'' y' + y' = y'' + 1\textit{ makes sure that }k + k' \leq |\text{Num}(G)|\textit{ if }y\text{ is }k\text{ and }y'\text{ is }k'.\textit{ Now we can define the vertex and edge set of the image of the transduction by appropriately shifting vertex and edge set of the input graph. We let }

\[ \theta_{V}^{+}(\overline{x}, y, z) := \exists z'(z' + y = z \land \theta_{V}(\overline{x}, z')), \]

\[ \theta_{E}^{+}(\overline{x}, y, z_1, z_2) := \exists z_1 \exists z_2 (z_1 + y = z_1 \land z_2 + y = z_2 \land \theta_{E}(\overline{x}, z_1', z_2')). \]

Finally, we define the order of the image of the transduction to be just the natural order. Hence we let

\[ \theta^{+}_{<}(\overline{x}, y, z_1, z_2) := z_1 \leq z_2. \]

This completes the definition of }\Theta^{+}(\overline{x}, y)\textit{. We proceed similarly to define }\Theta^{\times}(\overline{x}, y)\textit{. We let }

\[ \theta^{\times}_{\text{dom}}(\overline{x}, y) := \theta_{\text{dom}}(\overline{x}) \land \exists y'(\text{ord}_{\Theta}(\overline{x}, y') \land \exists y'' y \cdot y' \leq y'' + 1). \]
To define the vertex and edge set, we just have to express these sets as the union of the vertex and edge sets of appropriately shifted copies of the input graph. We let

\[
\begin{align*}
\theta_V^j(x, y, z) &:= \exists y_1 \exists y_2 \exists y_3 (\text{ord}(x, y_1) \land y_2 < y \land y_1 \cdot y_2 = y_3 \land \theta_V^{j+1}(x, y_3, z)), \\
\theta_E^j(x, y, z_1, z_2) &:= \exists y_1 \exists y_2 \exists y_3 (\text{ord}(x, y_1) \land y_2 < y \land y_1 \cdot y_2 = y_3 \land \theta_E^{j+1}(x, y_3, z_1, z_2)).
\end{align*}
\]

Here the subformula \(\text{ord}(x, y_1) \land y_2 < y \land y_1 \cdot y_2 = y_3\) makes sure that \(y_3\) is \(|\Theta[G, \overline{p}]| \cdot j\) for some \(j < k\) (if \(x = p\) and \(y = k\)). Thus the formulas define the unions of the sets \(V(\Theta^+[\overline{p}], |\Theta[G, \overline{p}]| \cdot j)\) and \(E(\Theta^+[\overline{p}], |\Theta[G, \overline{p}]| \cdot j)\) in \(\Theta^+[\overline{p}], |\Theta[G, \overline{p}]| \cdot j\) for \(j \in [0, k - 1]\). These unions are precisely the vertex set and the edge set of the product \(\Theta[G, \overline{p}] \times k\).

To complete the definition of \(\Theta^\times(x, y)\), we let

\[
\theta_E^\times(x, y, z_1, z_2) := z_1 \leq z_2.
\]

The following lemma says, in a complicated way, that if we can canonise the connected components of a graph then we can canonise the whole graph. The reason for phrasing the lemma this way is that this directly leads to a generalisation needed later (Lemma 7.4.4).

**Lemma 3.3.20.** Let \(\Theta(x)\) be an \(\text{IFP+C}\)-transduction from graphs to ordered graphs and \(\chi(x, y)\) an \(\text{IFP+C}\)-formula. Then there is an \(\text{IFP+C}\)-transduction \(\Theta'\) from graphs to ordered graphs such that the following holds. Let \(G\) be a graph such that

- (i) for every connected component \(A\) of \(G\) there is a tuple \(\overline{p} \in G^\times\) such that \((G, \overline{p}) \in D_{\Theta(x)}\) and \(\chi(G, \overline{p}, y) = V(A)\);
- (ii) for every connected component \(A\) of \(G\) and every tuple \(\overline{p} \in G^\times\) such that \((G, \overline{p}) \in D_{\Theta(x)}\) and \(\chi(G, \overline{p}, y) = V(A)\) the structure \(\Theta[G, \overline{p}]\) is an ordered copy of \(A\).

Then \(\Theta'\) canonises \(G\).

**Proof.** We start by normalising the transduction \(\Theta(x)\) as far as we can.

**Claim 1.** There is a normal transduction \(\Theta^2(x)\) such that for all graphs \(G\) satisfying (i) and (ii) the following conditions are satisfied.

- (A) For every connected component \(A\) of \(G\) there is a tuple \(\overline{p} \in G^\times\) such that \((G, \overline{p}) \in D_{\Theta^2(x)}\) and \(\chi(G, \overline{p}, y) = V(A)\).
- (B) For every tuple \(\overline{p} \in G^\times\) such that \((G, \overline{p}) \in D_{\Theta^2(x)}\) there is a connected component \(A\) of \(G\) such that \(\chi(G, \overline{p}, y) = V(A)\).
- (C) For every connected component \(A\) of \(G\) and every tuple \(\overline{p} \in G^\times\) such that \((G, \overline{p}) \in D_{\Theta^2(x)}\) and \(\chi(G, \overline{p}, y) = V(A)\) the structure \(\Theta^2[G, \overline{p}]\) is an ordered copy of \(A\).
- (D) For every connected component \(A\) of \(G\) and all tuples \(\overline{p}, \overline{p}' \in G^\times\) with \((G, \overline{p}), (G, \overline{p}') \in D_{\Theta^2(x)}\) and \(\chi(G, \overline{p}, y) = \chi(G, \overline{p}', y) = V(A)\) we have \(\Theta^2[G, \overline{p}] = \Theta^2[G, \overline{p}']\).

**Proof.** Note that conditions (A) and (C) correspond to (i) and (ii), which we assume to be satisfied by \(\Theta(x)\). To enforce (B) and (D), we first restrict the domain of the transduction appropriately. Then in a second step, we normalise the transduction.
For (B), we define a formula \( \comp \) satisfied by all \( \overline{p} \) such that \( \chi[G, \overline{p}, y] \) is the vertex set of a connected component of \( G \). We let

\[
\comp := \exists y \left( \chi(\overline{x}, y) \land \forall y' (\chi(\overline{x}, y') \leftrightarrow \path(y, y')) \right).
\]

For (D), we use the same trick as we used in Lemma 3.3.10 to achieve parameter independence: we just leave those \( \overline{p} \) in the domain that yield a \( \Theta[G, \overline{p}] \) that is lexicographically minimal among all \( \overline{p}' \) with the same component. We let

\[
\samecomp(\overline{x}, \overline{x}') := \forall y (\chi(\overline{x}, y) \leftrightarrow \chi(\overline{x}', y)),
\]

\[
\lexmin(\overline{x}) := \theta_{\text{dom}}(\overline{x}) \land \forall \overline{x}' \left( (\theta_{\text{dom}}(\overline{x}'') \land \samecomp(\overline{x}, \overline{x}'')) \rightarrow \lex(\overline{x}, \overline{x}') \right).
\]

Now we let \( \theta_{\text{dom}}^1(\overline{x}) := \comp(\overline{x}) \land \lexmin(\overline{x}) \), and we let \( \Theta^1(\overline{x}) \) be the transduction obtained from \( \Theta(\overline{x}) \) by replacing \( \theta_{\text{dom}}(\overline{x}) \) by \( \theta_{\text{dom}}^1(\overline{x}) \).

The transduction \( \Theta^1(\overline{x}) \) satisfies (A)-(D), but is not necessarily normal. However, an application of Lemma 3.3.17 to \( \Theta^1(\overline{x}) \) yields a normal transduction \( \Theta^2(\overline{x}) \) that still satisfies (A)-(D).

For the rest of the proof, we fix a graph \( G \) satisfying (i) and (ii). Of course the transduction \( \Theta' \) we shall define will not depend on \( G \). Let \( n := |G| \), and let \( A_1, \ldots, A_m \) be the connected components of \( G \). We choose a transduction \( \Theta^2(\overline{x}) \) according to Claim 1. For every \( i \in [m] \) we let \( P_i \) be the set of all \( \overline{p} \in G^\overline{x} \) such that \( (G, \overline{p}) \in \Delta_{\theta_2(\overline{x})} \) and \( \chi[G, \overline{p}, y] = V(A_i) \). By (A) we have \( P_i \neq \emptyset \). We let \( B_i := \Theta^2(\overline{x}, \overline{p}) \) for some \( \overline{p} \in P_i \). By (D), \( B_i \) does not depend on the choice of \( \overline{p} \). As \( \Theta^2(\overline{x}) \) is normal, \( B_i \) is a normal ordered graph, and thus we have \( B_i \cong B_j \iff B_i = B_j \). Without loss of generality we may assume that we have chosen the indices in our enumeration of the components of \( G \) in such a way that \( B_1 \leq_s \text{lex} B_2 \leq_s \text{lex} \ldots \leq_s \text{lex} B_m \). Let \( 1 = i_1 < i_2 < \ldots < i_\ell \leq i_{\ell+1} = m + 1 \) such that

\[
B_{i_1} = B_{i_1+1} = \ldots = B_{i_2-1} <_{\text{lex}} B_{i_2} = \ldots = B_{i_3-1} <_{\text{lex}} \ldots <_{\text{lex}} B_{i_\ell} = \ldots = B_{i_{\ell+1}-1}.
\]

For every \( j \in [\ell] \), let \( J_j := \{i_j, i_j + 1, \ldots, i_{j+1} - 1\} \) and \( B^j := B_{i_j} \) and \( n_j := |B^j| \) and \( k_j := i_{j+1} - i_j \) and

\[
s_j := \sum_{i=1}^{j} n_i \cdot k_i.
\]

Furthermore, we let \( s_0 := 0 \). Let \( C_j := (B^j \times k_j) + s_{j-1} \) and observe that \( C_j \) is an ordered copy of the structure \( \bigcup_{i \in J_j} A_i \). Note, furthermore, that the vertex sets of the \( C_j \)'s are disjoint consecutive intervals of integers with \( \bigcup_{j=1}^\ell V(C_j) = [0, n-1] = \text{Num}(G) \setminus \{n\} \). Let

\[
C := C_1 \uplus < C_2 \uplus < \ldots \uplus < C_\ell.
\]

Then \( C \) is an ordered copy of the graph \( G \). Indeed, \( C \) is the desired canonical copy of \( G \). It remains to prove that \( C \) is \( \text{IFP} + \text{C-definable} \), that is, to construct an \( \text{IFP} + \text{C} \)-transduction \( \Theta' \) (not depending on \( G \)) such that \( \Theta'[G] = C \).

Claim 2. For all \( j \in [\ell] \) and \( i, i' \in J_j \) we have \( |P_i| = |P_{i'}| \).
Proof. Let \( j \in [\ell] \) and \( i, i' \in J_j \). Then \( B_i = B_{i'} \) and thus \( A_i \cong A_{i'} \). Hence there is an automorphism \( f \) of \( G \) that swaps \( A_i \) and \( A_{i'} \) and keeps all other components fixed. Since the set \( \bigcup_{i=1}^{m} P_i = \theta_{\text{dom}}^2(\bar{x}) \) is invariant under \( f \), we have \( f(P_i) = P_{i'} \). This proves the claim.

For every \( j \in [\ell] \), let \( p_j := |P_i| \) for some (and hence for all) \( i \in J_j \) and

\[
P^j := \bigcup_{i \in J_j} P_i.
\]

Observe that \(|P^j| = p_j \cdot k_j\).

**Claim 3.** There is an \( \text{IFP+C} \)-formula \( P(\bar{x}, y) \) such that for all \( \bar{p} \in G^\bar{x} \) and \( j \in \text{Num}(G) \)

\[
G \models P[\bar{p}, j - 1] \iff j \in [\ell] \text{ and } \bar{p} \in P^j.
\]

**Proof.** We first define \( \text{IFP+C} \)-formulae \( \min^2(\bar{x}) \) and \( \text{succ}^2(\bar{x}, \bar{x}') \) defining the minimum and the successor relation of the quasi-order \( \leq^2 \) on the set of all \( \bar{p} \) such that \( (G, \bar{p}) \in D_{\text{G}^2(\bar{x})} \) that is defined by \( \bar{p} \leq^2 \bar{p}' \iff \Theta^2[G, \bar{p}] \leq_{\text{s-lex}} \Theta^2[G, \bar{p}'] \). Then we can define \( P^j \) for each index \( j \) by a simple induction. We let

\[
P(\bar{x}, y) := \text{ifp}
\begin{align*}
\text{min}^2(\bar{x}) \land y = 0 \\
\land \exists \bar{x}' \exists y'(X \bar{x}' y \land \text{succ}^2(\bar{x}', \bar{x}) \land y' + 1 = y)
\end{align*}(\bar{x}, y).
\]

**Claim 4.** There is an \( \text{IFP+C} \)-formula \( k(z, y) \) such that for all \( k, j \in \text{Num}(G) \)

\[
G \models k[k, j - 1] \iff j \in [\ell] \text{ and } k = k_j.
\]

**Proof.** By Claim 2 we have \( P^j = P[G, \bar{x}, j - 1] \). Thus \( k_j \cdot p_j = |P^j| \) is the number of \( \bar{p} \) such that \( G \models P[G, \bar{p}, j - 1] \). We let

\[
\text{Psiz}e(\bar{x}, y) := \# \bar{x} \ P(\bar{x}, y) = \bar{z}.
\]

Then \( G \models \text{Psiz}e[\bar{q}, j - 1] \iff j \in [\ell] \) and \( \langle \langle \bar{q} \rangle \rangle_n = |P^j| \).

To calculate \( p_j \), we take an arbitrary \( \bar{p} \in P^j \). Then \( p_j \) is the number of \( \bar{p}' \in P^j \) such that \( \chi[G, \bar{p}, y] = \chi[G, \bar{p}', y] \). We let

\[
p(\bar{x}, y) := \exists \bar{x}' \ P(\bar{x}, y) \land \# \bar{x}' \ (P(\bar{x}', y) \land \text{samecomp}(\bar{x}, \bar{x}')) = \bar{z},
\]

where \( \text{samecomp}(\bar{x}, \bar{x}') \) is defined as in the proof of Claim 1. Then \( G \models p[\bar{q}, j - 1] \iff j \in [\ell] \) and \( \langle \langle \bar{q} \rangle \rangle_n = p_j \).

Putting things together, we let

\[
k(\bar{z}, y) := \exists \bar{x}'' \exists \bar{z}''(p(\bar{x}', y) \land \text{Psiz}e(\bar{z}'', y) \land z \cdot \bar{z}' = \bar{z}'').
\]

As \( k_j \cdot p_j = |P^j| \), this formula has the desired meaning.
3.3. Definable Canonisation

Claim 5. There is an IFP+C-formula \( s(z, y) \) such that for all \( s, j \in \text{Num}(G) \)

\[
G \models s[s, j - 1] \iff j \in [\ell] \text{ and } s = s_j.
\]

Proof. Observe that \( n_j \) is the number of \( v \) such that \( G \models \chi[p, v] \) for some (and hence any) \( p \in P^3 \), for all \( j \in [\ell] \). This allows us to define the numbers \( n_j \) by an IFP+C-formula \( n(z, y) \).

Then we can define the sums \( s_j \) by a simple inductive definition. We omit the details. \( \blacksquare \)

Claim 6. There is an IFP+C-transduction \( \Theta^4(y) \) from graphs to ordered graphs such that

\[
\forall j \in \text{Num}(G) \: (G, j) \in D_{\Theta^4(y)} \iff j \in [0, \ell - 1], \text{ and for all } j \in [\ell] \text{ we have } \Theta^4[G, j - 1] = C_j.
\]

Proof. This follows easily from Lemma 3.3.19. We first define a transduction \( \Theta^3(y) \) such that \( \Theta^3[G, j] = B^j \times k_j \). This is easy, because we have \( \Theta^3[G, j] = B^j \times k_j \).

Similarly, we can now use the transduction \( (\Theta^3)^+(y, y') \) obtained from Lemma 3.3.19(1) and the formula \( s(z, y) \) of Claim 5 to define \( \Theta^4(y) \) in such a way that for all \( j \in [\ell] \) we have

\[
\Theta^4[G, j - 1] = \Theta^3[G, j - 1] + s_{j-1} = (B^j \times k^j) + s_{j-1} = C_j.
\]

It remains to define the transduction \( \Theta' \) such that \( \Theta'[G] = C \). This is easy, because we have \( V(C) = \bigcup_{j=1}^{\ell} V(C_j) \) and \( E(C) = \bigcup_{j=1}^{\ell} E(C_j) \) and \( \leq_C \leq |V(C) \). We let

\[
\begin{align*}
\theta'_\text{dom} &:= \text{graph}, \\
\theta'_V(z) &:= \exists y \theta'_E(y, z), \\
\theta'_E(z_1, z_2) &:= \exists y \theta'_E(y, z_1, z_2), \\
\theta'_\leq(z_1, z_2) &:= z_1 \leq z_2.
\end{align*}
\]

\( \blacksquare \)

Corollary 3.3.21. Let \( \mathcal{C} \) be a class of graphs such the class of all connected components of the graphs in \( \mathcal{C} \) admits IFP+C-definable canonisation. Then \( \mathcal{C} \) admits IFP+C-definable canonisation.

Proof. Let \( \mathcal{A} \) be the class of all connected components of the graphs in \( \mathcal{C} \), and let \( \Theta^1 \) be an IFP+C-transduction that canonises all graph in \( \mathcal{A} \). Without loss of generality we may assume that \( \Theta^1 \) is normal and parameter-free.
Let $\Theta^2(x)$ be the simple transduction that maps a graph to the connected component that contains $x$. That is, we let

$$
\begin{align*}
\theta^2_{\text{dom}}(x) & := \text{graph}, \\
\theta^2_V(x, y) & := \text{path}(x, y), \\
\theta^2_E(x, y_1, y_2) & := E(y_1, y_2).
\end{align*}
$$

Now let $\Theta(x) := (\Theta^1)^{-\Theta^2}$ be the simple transduction with $\theta_{\text{dom}}(x) := (\theta^1_{\text{dom}})^{-\Theta^2}(x)$ and $\theta_V(x, y) := (\theta^1_V)^{-\Theta^2}(x, y)$ et cetera. Then by the Transduction Lemma (Fact 2.4.6), for all graphs $G$, all connected components $A$ of $G$, and all $v \in V(A)$ we have $\Theta[G, v] = \Theta^1[A]$.

Hence all graphs $G \in \mathcal{C}$ satisfy conditions (i) and (ii) of Lemma 3.3.20, and the transduction $\Theta'$ of the lemma canonises $G$.

Example 3.3.22. Recall from Example 3.2.15 that the class $\mathcal{C}$ of all connected graphs of maximum degree 2 admits IFP-definable orders. Hence by Lemma 3.3.7, the class $\mathcal{C}$ admits IFP+C-definable canonisation. Now it follows from Corollary 3.3.21 that the class of all graphs of maximum degree 2 admits IFP+C-definable canonisation. By Lemma 3.3.8, this implies that IFP+C captures polynomial time on the class of all graphs of maximum degree 2.

By Fact 3.1.11, IFP+C does not capture polynomial time on the class of all graphs of maximum degree 3. Thus the class of all graphs of maximum degree at most 3 does not admit IFP+C-definable canonisation.

3.4 Finite Variable Logics and Pebble Games

If we restrict the number of variables allowed in a formula, we obtain a very interesting family of fragments of FO. These finite variable fragments of FO and other, related “finite variable logics” have come to play a very important role in finite model theory and descriptive complexity theory. The two most important aspects of these logics for us are that they can be used as a tool to prove inexpressibility results for fixed-point logics and that they have a close connection to certain combinatorial algorithms for the graph isomorphism problem.

3.4.1 The Logics $L_k$ and $L_k^\omega$

Let $k \in \mathbb{N}^+$. Then $L_k$ is the fragment of first-order logic FO consisting of all formulae that contain at most $k$ distinct variables. The fragment $L_k$, for $k \geq 2$, is more interesting than one might think at first sight, mainly because it is allowed to “re-use” variables, as the following example illustrates.

Example 3.4.1. The diameter of a graph is the maximum distance between any two vertices. For every $n \in \mathbb{N}$, the following $L^3$-sentence says that a graph has diameter at most $n$.

$$
diam_n := \forall x \forall y \bigvee_{i=0}^n \text{walk}_i(x, y)
$$

with $\text{walk}_0(x, y) := (x = y)$, $\text{walk}_1(x, y) := E(x, y)$, and $\text{walk}_{i+1}(x, y) := \exists z (E(x, z) \land \text{walk}_i(z, y))$ for all $i \in \mathbb{N}^+$, where $\text{walk}_i(z, y)$ is the formula obtained from $\text{walk}_i(x, y)$ by exchanging $x$ and $z$ everywhere. For example,

$$
\text{walk}_5(x, y) = \exists z (E(x, z) \land \exists x (E(z, x) \land \exists z (E(x, z) \land \exists x (E(z, x) \land E(x, y))))).
$$

M. Grohe, Definable Graph Structure Theory
It has turned out to be very fruitful to also work with “infinitary” versions of the logics $L^k$. We start by introducing the unconstrained infinitary logic $L_{\infty\omega}$: it is the extension of first-order logic that, in addition to the usual Boolean connectives and quantifiers, also allows infinitary conjunctions and disjunctions of the form $\land \Phi$ and $\lor \Phi$, where $\Phi$ is an arbitrary set of formulae. The logic $L_{\infty\omega}$ is far too expressive to be interesting. Indeed, every property of finite structures is $L_{\infty\omega}$-definable. To see this, we first observe that for every $\tau$-structure $A$ there is a sentence $id_A \in FO[\tau]$ such that for all $\tau$-structures $B$ we have $B \models id_A \iff B \cong A$. Then if $P$ is a property of $\tau$-structures, the sentence $\forall A \in P id_A$ defines $P$. More interesting are the finite variable fragments of $L_{\infty\omega}$. For every $k \in \mathbb{N}^+$, we let $L^k_{\infty\omega}$ be the fragment of $L_{\infty\omega}$ consisting of all formulae that contain at most $k$ distinct variables.

**Example 3.4.2.** The $L^3_{\infty\omega}$-sentence

$$\forall x \forall y \bigvee_{n \in \mathbb{N}} \text{walk}_n(x, y)$$

says that a graph is connected.

**Lemma 3.4.3 (Kolaitis and Vardi [78, 79]).** For every IFP-sentence $\varphi$ there is a $k$ and an $L^k_{\infty\omega}$-sentence $\varphi'$ that is equivalent to $\varphi$.

**Proof sketch.** This lemma is proved by induction on $\varphi$. We let $\{x_1, \ldots, x_\ell\}$ be the set of individual variables occurring in $\varphi$. We let $k := \ell + 2m$, where $m$ is the maximum arity of a relation variable in $\varphi$, and we choose fresh variables $x_{\ell+1}, \ldots, x_k$. The variables of the formula $\varphi'$ will be among $x_1, \ldots, x_k$. The crucial step in the inductive construction is to transform a fixed-point formula $\varphi = \text{ifp}(X \leftarrow \psi)\psi'$ into an $L^k_{\infty\omega}$-formula $\varphi'$. We inductively construct formulae $\psi^i$ defining the stages of the fixed-point process: we let $\psi^0 := \text{false}$, and for $i \in \mathbb{N}$ we let $\psi^{i+1}$ be the formula obtained from $\psi^i$ by replacing each atom of the form $XY \phi$ by the formula $\exists \bar{z} (\bar{z} = \bar{y} \land \exists \bar{x} (\bar{x} = \bar{z} \land \psi^i))$, where $\bar{z}$ is a tuple of variables in $x_{\ell+1}, \ldots, x_k$ that is disjoint from $\bar{y}$. The tuple $\bar{z}$ is also disjoint from $\bar{x}$ and all free variables of $\psi^i$, because they are among $\{x_1, \ldots, x_\ell\}$. Our choice of $k$ guarantees that such a tuple $\bar{z}$ exists. The reason for this rather complicated formula is that this way we can avoid renaming variables in $\psi^i$. Then we let

$$\varphi' := \exists \bar{z} \left( \bar{z} = \bar{x}' \land \exists \bar{x} (\bar{x} = \bar{z} \land \bigvee_{i \in \mathbb{N}} \psi^i) \right).$$

We let $L^\omega_{\infty\omega}$ be the union of all the fragments $L^k_{\infty\omega}$ for $k \in \mathbb{N}^+$. Then we have

$$FO = \bigcup_{k \in \mathbb{N}^+} L^k \subseteq \text{IFP} \subseteq L^\omega_{\infty\omega}, \quad (3.4.1)$$

where $L \subseteq L'$ denotes that $L$ is at most as expressive as $L'$, that is, for every sentence $\varphi \in L$ there is a sentence $\varphi' \in L'$ such that $\varphi$ and $\varphi'$ are equivalent. We write $L \equiv L'$ if $L$ and $L'$ have the same expressive power, that is, if $L \equiv L'$ and $L' \equiv L$, and we write $L \not\equiv L'$ if $L'$ is more expressive than $L$, that is, $L \equiv L'$ and $L \not\equiv L'$.

As a matter of fact, both inequalities in (3.4.1) are strict. We have $FO \not\equiv \text{IFP}$, because graph connectivity is not expressible in $FO$ (see any of the textbooks [31, 75, 80, 39] for a proof), and $\text{IFP} \not\equiv L^\omega_{\infty\omega}$, because we can easily use the infinite conjunctions and disjunctions.

---

\[1\]Remember that structures are always finite in this book.
in $\mathcal{L}^{\omega}_{\omega}$ to define undecidable properties of structures, whereas all properties definable in $\mathcal{IFP}$ are decidable in polynomial time.

By (3.4.1), to prove that some property is not definable in $\mathcal{IFP}$ it suffices to prove that it is not definable in $\mathcal{L}^{\omega}_{\omega}$. We have a fairly powerful technique for proving inequivalence in $\mathcal{L}^{\omega}_{\omega}$ based on the following games. Let $k \in \mathbb{N}^+$, and let $A, B$ be $\tau$-structures. The $k$-pebble game on $A$ and $B$ is played by two players called the Spoiler and the Duplicator. Both players have $k$ pebbles labelled $1, \ldots, k$. A play of the game consists of a (possibly infinite) sequence of rounds. In each round, the Spoiler picks up one of his pebbles and places it on an element of one of the structures $A, B$. (The pebble he picks up may have been either unused in the game so far or placed on some other element in an earlier round.) The Duplicator answers by picking up her pebble with the same label and placing it on an element of the other structure.

Note that after each round $r$ there is a subset $I \subseteq [k]$, consisting of the labels of the pebbles currently used, such that for each $i \in I$ there is an $v_i \in V(A)$ and a $w_i \in V(B)$ on which a pebble with label $i$ is placed. We call the pair $(\overline{v}, \overline{w})$ for $\overline{v} = (v_i \mid i \in I)$ and $\overline{w} = (w_i \mid i \in I)$ the position after round $r$. We say that a position $(\overline{v}, \overline{w})$ is a partial isomorphism if $v_i \mapsto w_i$ defines an injective mapping, that is, we have $v_i = v_j \iff w_i = w_j$, and this mapping is an isomorphism from the induced substructure $A[\overline{v}]$ to the induced substructure $B[\overline{w}]$.

The Duplicator wins the play if all positions are partial isomorphisms. Otherwise, the Spoiler wins. We can now define winning strategies for the Spoiler and the Duplicator in the usual way.

We only give one, almost trivial example showing pebble games in action. Many more and more interesting examples can be found in the textbooks [31] [39] [75] [86].

Example 3.4.4. For some $k \in \mathbb{N}^+$, let $G$ and $H$ be complete graphs of (possibly different) cardinalities $|G|, |H| \geq k$. Then the Duplicator has a winning strategy for the $k$-pebble game on $G, H$. She simply has to make sure that the position after each round is an injective mapping.

It will be convenient to extend $k$-pebble games from pairs of structures to pairs of interpretations $(A, \overline{\tau}_0), (B, \overline{\tau}_0)$, where $\overline{\tau}_0 = (v_1, \ldots, v_j) \in V(A)^j$ and $\overline{\tau}_0 = (w_1, \ldots, w_j) \in V(B)^j$ for some $j \in [k]$. In this version of the game, for all $i \in [j]$ the two pebbles with label $i$ are placed on $v_i$ and $w_i$ before the game starts, so that the initial position is $(\overline{v}_0, \overline{w}_0)$ instead of $((), ())$. Then the game continues as the game on $A, B$. Note that we require all positions, including the initial position $(\overline{v}_0, \overline{w}_0)$, to be partial isomorphisms in order for the Duplicator to win a play.

The following fact links pebble games and finite variable logics. For a logic $\mathcal{L}$, we say that two $\tau$-structures $A$ and $B$ are $\mathcal{L}$-equivalent if for all sentences $\varphi \in \mathcal{L}[\tau]$ it holds that $A \models \varphi \iff B \models \varphi$. More generally, we say that two $(\tau, \xi)$-interpretations $(A, \alpha)$ and $(B, \beta)$ are $\mathcal{L}$-equivalent if for all formulae $\varphi \in \mathcal{L}[\tau]$ with free variables in $\xi$ it holds that $(A, \alpha) \models \varphi \iff (B, \beta) \models \varphi$.

Fact 3.4.5 (Barwise [10], Immerman [71]). For all $k \in \mathbb{N}^+$, all $\tau$-structures $A, B$, all $j \in [0, k]$, and all $\overline{\tau}_0 \in V(A)^j, \overline{\tau}_0 \in V(B)^j$ the following three statements are equivalent.

(i) $(A, \overline{\tau}_0)$ and $(B, \overline{\tau}_0)$ are $\mathcal{L}^k$-equivalent.

(ii) $(A, \overline{\tau}_0)$ and $(B, \overline{\tau}_0)$ are $\mathcal{L}^k_{\omega\omega}$-equivalent.

(iii) The Duplicator has a winning strategy for the $k$-pebble game on $(A, \overline{\tau}_0)$ and $(B, \overline{\tau}_0)$.

M. Grohe, Definable Graph Structure Theory
Example 3.4.6. There is no $L_{\infty\omega}^{\omega}$-sentence and thus no $\text{IFP}$-sentence saying that a graph has an even number of vertices.

To see this, suppose for contradiction that $\varphi \in L_{\infty\omega}^{\omega}$ was such a sentence. Let $k \in \mathbb{N}^+$ such that $\varphi \in L_k^{\infty\omega}$. Then $K_k \models \varphi$ and $K_{k+1} \not\models \varphi$ (if $k$ is even) or $K_k \not\models \varphi$ and $K_{k+1} \models \varphi$ (if $k$ is odd). Thus $K_k$ and $K_{k+1}$ are not $L_{\infty\omega}^{\omega}$-equivalent. However, by Example 3.4.4, the Duplicator has a winning strategy for the $k$-pebble game on $K_k$ and $K_{k+1}$. This contradicts Fact 3.4.5.

Note that there is an $\text{IFP}+\text{C}$-sentence saying that a graph has an even number of vertices (cf. Example 2.3.5). Thus $\text{IFP}+\text{C} \not\subseteq L_{\infty\omega}^{\omega}$ and $\text{IFP} \not\subseteq \text{IFP}+\text{C}$. 

We say that a sentence $\varphi$ of some logic $L$ identifies a $\tau$-structure $A$ if for every $\tau$-structure $B$ it holds that $B \models \varphi \iff A \cong B$. Recall that $\text{FO}$ identifies every structure. Example 3.4.4 shows that $L^k$ does not identify the complete graph $K_k$.

Fact 3.4.7 (Dawar, Lindell, Weinstein [25], Poizat [100]). For every $k \in \mathbb{N}$ and every $\tau$-structure $A$ there is an $L_k^{\tau}$-sentence $\varphi^k_A$, that characterises $A$ up to $L_k$-equivalence, that is, for all $\tau$-structures $B$ we have $B \models \varphi^k_A$ if and only if $A$ and $B$ are $L_k$-equivalent.

It follows from Facts 3.4.5 and 3.4.7 that to prove that $L_k$ identifies a structure $A$ it suffices to prove either that $L_{\infty\omega}^k$ identifies $A$, or that for all structures $B$ not isomorphic to $A$ the Spoiler has a winning strategy for the $k$-pebble game on $A,B$.

Lemma 3.4.8. $L^2$ identifies every ordered graph.

Proof. Let $G = (V(G), E(G), \leq^G)$ be an ordered graph. As we have just noted, it suffices to prove that for every $\{E, \leq\}$-structure $H = (V(H), E(H), \leq^H)$ that is not isomorphic to $G$, the Spoiler has a winning strategy for the 2-pebble game on $G, H$. So let $H = (V(H), E(H), \leq^H)$ be an $\{E, \leq\}$-structure that is not isomorphic to $G$.

Suppose first that $\leq^H$ is not a linear order. If there is a $w \in V(H)$ such that $w \not\prec^H w$, then the Spoiler wins the game by placing a pebble on $w$. The Duplicator has to answer with some $v \in V(G)$ for which $v \not\leq^G w$, because $\leq^G$ is reflexive. But then the resulting position $(v, w)$ is not a partial isomorphism. Similarly, if there is a pair of distinct vertices $w, w' \in V(H)$ such that $w \not\leq^H w'$ and $w' \not\leq^H w$, then the Spoiler can win in two rounds. Hence we may assume that $\leq^H$ is reflexive and total. Then it is a linear order if and only if there is no cycle of length greater than one. Suppose that $w_1 \leq^H w_2 \leq^H \cdots \leq^H w_k \leq^H w_1$ for some $k \geq 2$ and mutually distinct $w_1, \ldots, w_k$. Then the spoiler wins the game as follows: he places his first pebble on $w_1$, then his second pebble on $w_2$, then his first pebble on $w_3$, 

... , eventually his $i$th pebble on $w_i$, then his $(3-i)$th pebble on $w_1$, his $i$th pebble on $w_2$, et cetera, walking round the cycle until the Duplicator can no longer answer. Hence we may assume that $\leq^H$ is a linear order.

Suppose that $V(G) = \{v_1, \ldots, v_m\}$ with $v_1 <^G v_2 <^G \cdots <^G v_m$ and $V(H) = \{w_1, \ldots, w_n\}$ with $w_1 <^H w_2 <^H \cdots <^H w_n$. If $m \neq n$ then the Spoiler wins the game as follows. Say, $m > n$. Then he places his first pebble on $v_1$, then his second pebble on $w_2$, then his first pebble on $v_3$, et cetera, and once he reaches $v_{n+1}$ the Duplicator is lost. Hence we may assume that $m = n$.

As $G$ and $H$ are not isomorphic, there must be $i, j \in [n]$ such that $v_i v_j \in E(G)$ and $w_i w_j \not\in E(H)$ or vice-versa. The Spoiler wins the game by placing his two pebbles on $v_i$ an

---

2Immerman and Lander [78] use the term “characterise” instead of “identify.”
Remark 3.4.9. The lemma can be generalised to arbitrary structures as follows. Let \( \tau \) be a vocabulary that contains \( \leq \), and let \( k \) be the maximum of the arities of the relation symbols in \( \tau \). Then \( L^k \) identifies all ordered \( \tau \)-structures.

Lemma 3.4.10. Let \( C \) be an \( \text{IFP} \)-definable class of graphs that admits \( \text{IFP} \)-definable canonisation. Then there is a \( k \in \mathbb{N} \) such that \( L^k \) identifies all graphs \( G \in C \).

Proof. Let \( \psi \) be an \( \text{IFP} \)-sentence that defines \( C \), and let \( \Theta(\bar{x}) \) be an \( \text{IFP} \)-transduction that canonises \( C \). By Lemma 3.3.10, we may assume without loss of generality that \( \Theta(\bar{x}) \) is parameter-independent. Let \( G \in C \) and \( \bar{p} \in G^\tau \) such that \( (G, \bar{p}) \in \mathcal{D}_\Theta(\bar{x}) \). Let \( \varphi \) be an \( L^2 \)-sentence that identifies the ordered graph \( \Theta[G, \bar{p}] \). We apply the Transduction Lemma (Corollary 2.4.7) and obtain an \( \text{IFP}\{\{E\}\} \)-formula \( \varphi^-\Theta(\bar{x}) \) such that for all \( (H, \bar{q}) \in \mathcal{D}_\Theta(\bar{x}) \) we have

\[
H \models \varphi^-\Theta[\bar{q}] \iff \Theta[H, \bar{q}] \models \varphi.
\]

Since \( \varphi \) identifies \( \Theta[G, \bar{p}] \), it follows that \( \Theta[H, \bar{q}] \cong \Theta[G, \bar{p}] \). If \( H \in C \) and thus \( \Theta(\bar{x}) \) canonises \( H \), this implies \( H \cong G \). Choose \( k \in \mathbb{N} \) sufficiently large such that there is an \( L^k_{\infty\omega} \)-formula \( \varphi'(\bar{x}) \) equivalent to \( \varphi^-\Theta(\bar{x}) \) and an \( L^k_{\infty\omega} \)-sentence \( \psi' \) equivalent to \( \varphi \) and an \( L^k_{\infty\omega} \)-formula \( \theta'(\bar{x}) \) equivalent to \( \theta_{\text{dom}}(\bar{x}) \). Then

\[
\chi := \psi' \land \exists\bar{x}(\theta'(\bar{x}) \land \varphi'(\bar{x})).
\]

is an \( L^k_{\infty\omega} \)-sentence that identifies \( G \). Hence (by the remarks after Fact 3.4.7) there is also an \( L^k \)-sentence that identifies \( G \).

Remark 3.4.11. For many more results on the number of variables to identify a structure, and also other logical “resources” such as the quantifier rank, I refer the reader to the recent survey [99].

3.4.2 The Logics \( C^k \) and \( C^k_{\infty\omega} \)

As Example 3.4.6 shows, \( \text{IFP}^+ \subseteq L^\omega_{\infty\omega} \), and thus the logic \( \text{IFP}^+ \) does not fit within the framework of the finite variable logics \( L^k \) and \( L^\omega_{\infty\omega} \). In this section, we will develop a similar framework of finite variable counting logics together with matching pebble games and use it to analyse the expressive power of \( \text{IFP}^+ \).

We let \( C \) be the extension of \( \text{FO} \) by \textit{counting quantifiers} \( \exists^{\geq i} \). That is, \( C \)-formulæ are formed from atomic formulæ with the usual Boolean connectives, standard existential and universal quantifiers, and the new counting quantifiers. The semantics of the counting quantifiers is the obvious one: an interpretation \( (A, \alpha) \) satisfies a formulæ \( \exists^{\geq i} x \varphi \) if there are at least \( i \) elements \( v \in V(A) \) such that \( (A, \alpha(v/x)) \) satisfies \( \varphi \). Note that, like \( \text{FO} \) and \( \text{IFP} \), the logic \( C \) lives in the usual one-sorted logical framework and not in the two-sorted framework of the logics \( \text{FO}^+ \) and \( \text{IFP}^+ \). Also observe that, while syntactically an extension of \( \text{FO} \), the
logic $C$ has the same expressive power as $FO$, because the formula $\exists x^1 x \varphi(x)$ is equivalent to

$$\exists x_1 \ldots \exists x_i \left( \bigwedge_{1 \leq j, j' \leq i} x_j \neq x_{j'} \wedge \bigwedge_{j=1}^i \varphi(x_j) \right).$$

We let $C_{\omega}$ be the extension of $C$ by infinitary conjunctions and disjunctions. Finally, and most importantly, for every $k \in \mathbb{N}^+$ we let $C^k$ and $C^k_{\omega}$ be the $k$-variable fragments of $C$ and $C_{\omega}$, respectively, and we let $C^\omega := \bigcup_{k \in \mathbb{N}} C^k_{\omega}$.

**Example 3.4.12.** For every $n \in \mathbb{N}$, the $C^1$-sentence $\text{ord}_n := \exists x^n x = x \wedge \exists x^{n+1} x = x$ says that a structure has order $n$. Hence the $C^1_{\omega}$-sentence $\bigvee_{n \in \mathbb{N}} \text{ord}_{2n}$ says that a structure has an even order.

While $FO = \bigcup_{k \geq 1} L^k$ and $C = \bigcup_{k \geq 1} C^k$ have the same expressive power, this is not the case for the individual levels of the hierarchies. As a matter of fact, for every $k \in \mathbb{N}$ we have $C^1 \not\preceq L^k_{\omega}$, and we have $C^k_{\omega} \not\preceq L^k_{\omega}$. To see this, recall from Examples 3.4.4 and 3.4.6 that there is no $L^k_{\omega}$-sentence saying that the order of a graph is exactly $k$ and no $L^k_{\omega}$-sentence saying that the order of a graph is even and compare this with Example 3.4.12.

**Lemma 3.4.13 (Grädel and Otto [40]).** For every $IFP+C$-sentence $\varphi$ there is a $k$ and a $C^k_{\omega}$-sentence $\varphi'$ that is equivalent to $\varphi$.

**Proof sketch.** In the first step, we translate every $IFP+C$-formula $\varphi$ into an equivalent formula $\varphi^*$ of an intermediate logic that is essentially the finite-variable infinitary logic $L^\omega_{\omega}$, but lives in the two-sorted framework of the logic $IFP+C$ and may still contain counting operators $\# x \varphi = y$. This translation is analogous to the translation of an $IFP$-formula into an $L^\omega_{\omega}$-formula in the proof of Lemma 3.4.3.

In the second step, we associate with every formula $\psi(x_1, \ldots, x_\ell, y_1, \ldots, y_m)$ of the intermediate $k$-variable infinitary logic with free vertex variables among $x_1, \ldots, x_\ell$ and free number variables among $y_1, \ldots, y_m$ a family of $C^k_{\omega}$-formulae $\psi_{i_1, \ldots, i_m}(x_1, \ldots, x_\ell)$, for all $i_1, \ldots, i_m \in \mathbb{N}$, such that for all structures $A$, $v_1, \ldots, v_\ell \in V(A)$, and $i_1, \ldots, i_m \in \text{Num}(A)$ we have

$$A \models \psi[v_1, \ldots, v_\ell, i_1, \ldots, i_m] \iff A \models \psi_{i_1, \ldots, i_m}[v_1, \ldots, v_\ell].$$

Clearly, this suffices to prove the lemma, because for formulæ $\psi$ without free number variables we obtain a single equivalent $C^k_{\omega}$-formula.

We construct $\psi_{i_1, \ldots, i_m}(x_1, \ldots, x_\ell)$ inductively, the only interesting steps being quantifiers over number variables and counting operators:

- If $\psi(x_1, \ldots, x_\ell, y_1, \ldots, y_m) = \exists y_{m+1} \chi(x_1, \ldots, x_\ell, y_1, \ldots, y_m, y_{m+1})$, then we let

$$\psi_{i_1, \ldots, i_m}(x_1, \ldots, x_\ell) := \bigvee_{j \in \mathbb{N}} \chi_{i_1, \ldots, i_m,j}(x_1, \ldots, x_\ell).$$

- If $\psi(x_1, \ldots, x_\ell, y_1, \ldots, y_m) = \# x_{\ell+1} \chi(x_1, \ldots, x_{\ell+1}, y_1, \ldots, y_{m-1}) = y_m$, then we let

$$\psi_{i_1, \ldots, i_m}(x_1, \ldots, x_\ell) := \exists x_{\ell+1} \chi_{i_1, \ldots, i_m-1}(x_1, \ldots, x_\ell, x_{\ell+1}),$$

where we use $\exists x \chi$ as an abbreviation for $\exists x \chi \land \neg \exists x_{\ell+1} \chi$. □
Hence we can extend (3.4.1) to the following diagram:

\[
\begin{align*}
\text{FO} &= \bigcup_{k \in \mathbb{N}^+} \mathbb{L}^k \leq \mathbb{IFP} \leq \mathbb{L}_{\omega}^\omega, \\
\equiv &= \leq \leq \equiv \\
\mathbb{C} &= \bigcup_{k \in \mathbb{N}^+} \mathbb{C}^k \leq \mathbb{IFP+C} \leq \mathbb{C}_{\omega}^\omega.
\end{align*}
\] (3.4.2)

It is not hard to prove that all inequalities in this diagram are strict.

Let us now turn to an extension of the pebble games suitable to analyse the finite variable counting logics. We actually have two such extensions.

Let \( k \in \mathbb{N}^+ \), and let \( A, B \) be \( \tau \)-structures of the same order. The cardinality \( k \)-pebble game on \( A \) and \( B \) is played by two players called the \textit{Spoiler} and the \textit{Duplicator} in the same way as the (ordinary) \( k \)-pebble game, except that the rounds of a play are different.

- The Spoiler picks up one of his pebbles and chooses a subset \( S \) of the vertices of one of the structures \( A, B \).
- The Duplicator answers by picking up her pebble with the same label and choosing a subset \( T \) of the vertices of the other structure such that \( |S| = |T| \). Note that this is always possible because we assumed \( |A| = |B| \).
- The Spoiler places his pebble on an element of \( T \).
- The Duplicator places her pebble on an element of \( S \).

The winning conditions remain the same as in the \( k \)-pebble game.

The bijective \( k \)-pebble game on \( A \) and \( B \) is also played in the same way as the \( k \)-pebble game and the cardinality \( k \)-pebble game, except that in each round the players proceed as follows.

- The Spoiler picks up one of his pebbles.
- The Duplicator picks up her pebble with the same label, and she chooses a bijective mapping \( f : V(A) \rightarrow V(B) \).
- The Spoiler places his pebble on an element \( a \in V(A) \).
- The Duplicator places her pebble on \( f(a) \in V(B) \).

Again, the winning conditions remain the same.

To simplify the notation, we extend the definition of both games to arbitrary structures, not necessarily of the same cardinality, by saying that the Spoiler wins the games immediately if \( |A| \neq |B| \). As we did for ordinary \( k \)-pebble games, we extend cardinality and bijective \( k \)-pebble games from pairs of structures to pairs of interpretations \((A, \tau_0), (B, \tau_0)\), where \( \tau_0 \in V(A)^j \) and \( \tau_0 \in V(B)^j \) for some \( j \in [k] \) by starting the game from the initial position \((\tau_0, \tau_0)\).

**Example 3.4.14.** Let \( G \) be a cycle of length 6, and let \( H \) be the disjoint union of two cycles of length 3. Then the Duplicator has winning strategies for both the cardinality 2-pebble game and the bijective 2-pebble game on \( G \) and \( H \). I leave it to the reader to find such strategies.
Fact 3.4.15 (Immerman and Lander [76], Hella [63]). For all $k \in \mathbb{N}^+$, all $\tau$-structures $A, B$, all $j \in [0, k]$, and all $v_0 \in V(A)^j, w_0 \in V(B)^j$, the following four statements are equivalent.

(i) $(A, v_0)$ and $(B, w_0)$ are $C_k$-equivalent.

(ii) $(A, v_0)$ and $(B, w_0)$ are $C_{\infty \omega}$-equivalent.

(iii) The Duplicator has a winning strategy for the cardinality $k$-pebble game on $(A, v_0)$ and $(B, w_0)$.

(iv) The Duplicator has a winning strategy for the bijective $k$-pebble game on $(A, v_0)$ and $(B, w_0)$.

Having established the correspondence between logics and games, we can use the games to prove inexpressibility results for the logics $C_k^{\infty \omega}$ and $C_{\infty \omega}^\omega$ and, via Lemma 3.4.13, $\text{IFP+C}$ in a similar way as we did with the ordinary $k$-pebble game for $L_k^{\infty \omega}$ and $L_{\infty \omega}^\omega$ and $\text{IFP}$ in Examples 3.4.4 and 3.4.6. However, the Spoiler is more powerful (or the Duplicator more constrained) in the cardinality $k$-pebble game and the bijective $k$-pebble game than in the ordinary $k$-pebble game, and it is usually much harder to find winning strategies for the Duplicator in these games. Nevertheless, Cai, Fürer, and Immerman proved the following powerful theorem.

Fact 3.4.16 (CFI-Theorem; Cai, Fürer, and Immerman [16]). For all $k \in \mathbb{N}$ there are nonisomorphic graphs $G_k$ and $H_k$ with the following properties:

(i) The Duplicator has a winning strategy for the cardinality $k$-pebble game on $G_k, H_k$.

(ii) There is a polynomial time algorithm that accepts all $G_k$ and rejects all $H_k$.

(iii) $|G_k|, |H_k| = O(k)$.

(iv) $G_k$ and $H_k$ are 3-regular.

It is easy to see that Fact 3.1.11 follows from the CFI-Theorem (combined with Lemma 3.4.13, Fact 3.4.15).

Let us now turn to the issue of identifying structures in $C_k$. We start with an analogue of Fact 3.4.17.

Fact 3.4.17 (Otto [94]). For every $k \in \mathbb{N}$ and every $\tau$-structure $A$ there is an $C_k[\tau]$-sentence $\varphi_A^k$ that characterises $A$ up $C_k$-equivalence.

Note that, since $L^2 \subseteq C^2$, Lemma 3.4.8 implies that $C^2$ identifies every ordered graph.

Lemma 3.4.18. Let $\mathcal{C}$ be an $\text{IFP+C}$-definable class of graphs that admits $\text{IFP+C}$-definable canonisation. Then there is a $k \in \mathbb{N}$ such that $C_k$ identifies all graphs $G \in \mathcal{C}$.

Proof. This can be proved completely analogously to Lemma 3.4.10 (Actually, the proof is even simpler than the proof of Lemma 3.4.10 because for $\text{IFP+C}$-canonisations we can always assume that they are normal and parameter-free.)
3.5 Isomorphism Testing and the Weisfeiler-Leman Algorithm

The graph isomorphism problem (GI) is the algorithmic problem of deciding whether two given graphs are isomorphic. The question of whether GI is in PTIME was already mentioned in Karp’s seminal paper on NP-completeness [77] forty years ago. There is evidence that GI is not NP-complete: if it was then the polynomial hierarchy PH would collapse to its second level [15, 115]. The best known complexity theoretic lower bound due to Toran [123] states that GI is hard for the class DET, which includes the more familiar class NL (“nondeterministic logarithmic space”). In a recent breakthrough, Babai [7] has proved that GI is in quasipolynomial time \( n^{\log n^{O(1)}} \). Polynomial time algorithms are known for the restrictions of GI to many natural classes of graphs, among them the class of planar graphs [67, 68], all classes of graphs embeddable in a fixed surface [35, 89], all classes graphs of bounded tree width [14], all classes of graphs with excluded minors [101], all classes of graphs of bounded degree [87], and all classes of graphs with excluded topological subgraphs [54]. The isomorphism test for graphs of bounded degree due to Luks [87] involves some nontrivial group theory, and many later isomorphism algorithms built on the group theoretic techniques developed by Babai, Luks, and others (e.g. [6, 9]) in the early 1980s. In particular, Ponomarenko’s [101] isomorphism algorithm for classes of graphs with excluded minors builds on a generalisation of Luks’s bounded degree isomorphism test due to Miller [90]. The isomorphism algorithm for classes of graphs with excluded topological subgraphs [54] builds on both Luks’s and Pomonomarenko’s algorithms and the treelike decompositions developed in this book.

As a matter of fact, for all these classes of graphs not only polynomial time isomorphism algorithms are known, but even polynomial time canonisation algorithms. (Observe that a canonisation algorithm immediately yields an isomorphism test.) For graphs of bounded degree, the canonisation algorithm is due to Babai and Luks [9]; the later papers [101, 54] directly gave a canonisation algorithm.

It is not hard to see that if a class \( \mathcal{C} \) of graphs admits polynomial time canonisation then there is a logic that captures PTIME on \( \mathcal{C} \). However, the logics derived from canonisation algorithms are usually not very natural, they are just artificial abstract logics satisfying conditions (C.1)–(C.3).

We will see now that there is a close connection between the, arguably natural, logic \( \text{IFP} + \mathcal{C} \) and a very simple generic family of combinatorial isomorphism algorithms.

3.5.1 Colour Refinement

One of the simplest approaches to the graph isomorphism problem is the colour refinement algorithm, which is also known as naive vertex classification. The colour refinement algorithm iteratively colours the vertices of a graph according to their “iterated degree sequence”. Initially, all vertices get the same colour. Then the algorithm proceeds in a sequence of refinement rounds until a stable colouring is reached. In each refinement round, the colouring is refined by assigning different colours to vertices that have a different number of neighbours of at least one colour assigned in the previous round. Thus after the first round, two vertices have the same colour if and only if they have the same degree. After the second round, two vertices have the same colour if and only if they have the same degree and for each \( d \) the same number of neighbours of degree \( d \). The algorithm stops if the colouring is stable, that is, if no further refinement is achieved. This happens after at most \( |G| \) refinement rounds. A naive implementation of the refinement procedure will require at least a quadratic running time in
3.5. Isomorphism Testing and the Weisfeiler-Leman Algorithm

the worst case, but using a trick that goes back to Hopcroft’s algorithm for minimising deterministic finite automata [66], the stable colouring can be computed in time $O(|G| \log |G|)$ [17] (also see [97, 12]).

To use colour refinement as an isomorphism test, we run it on the disjoint union of two graphs. If the stable colouring differs on the two graphs, that is, for some colour $c$ the graphs have a different number of vertices of colour $c$, then we say that colour refinement distinguishes the graphs. Clearly, if two graphs are isomorphic, then colour refinement does not distinguish them. The converse does not always hold: there are nonisomorphic graph that colour refinement fails to distinguish. Note that colour refinement distinguishes any two graphs of distinct orders.

We say that colour refinement identifies a graph $G$ if it distinguishes $G$ from all graphs not isomorphic to $G$.

**Example 3.5.1.** Colour refinement distinguishes the two graphs shown in Figure 3.3. Note that the stable colouring on these two graphs is reached after two refinement rounds.

**Example 3.5.2.** Colour refinement does not distinguish any two $k$-regular graphs of the same order. An example for $k = 2$ is a cycle of length 6 vs the disjoint union of two triangles.

**Example 3.5.3 (Immerman and Lander [76]).** Colour refinement identifies all forests.

**Example 3.5.4 (Babai, Erdös, and Selkow [8]).** Colour refinement identifies almost all graphs. That is, an $n$-vertex graph chosen uniformly at random is identified with probability approaching 1 as $n$ goes to infinity.

For the following proof as well as the generalisation in the next section, it will be convenient to take a more formal view on colour refinement. Recall that a *multiset* is an unordered collection of not necessarily distinct objects, whose multiplicities count. Formally, a multiset may be viewed as a mapping from a set to the positive integers, assigning to each element its multiplicity. To distinguish multisets from normal sets, we denote them using double brackets $\{\ldots\}$.

We say that a colouring $D$ refines a colouring $C$, or $C$ is at least as coarse as $D$, if the partition into colour classes induced by $D$ refines the partition induced by $C$. Let us call a colouring $C$ of the vertices of a graph $G$ stable if for all vertices $v, w \in V(G)$ and all colours $c$ in the range of $C$, if $C(v) = C(w)$ then

$$|N^G(v) \cap C^{-1}(c)| = |N^G(w) \cap C^{-1}(c)|.$$ 

Here $C^{-1}(c)$ denotes the set of all vertices $v \in V(G)$ with $C(u) = c$.

We think of colour refinement as computing a sequence $C_t$ of vertex colourings of the input graph $G$. The initial colouring assigns the same colour, say, 1 to all vertices: $C_0(v) := 1$ for

---

**Figure 3.3.** Two graphs distinguished by the colour refinement algorithm
all \(v \in V(G)\). The colouring \(C_{i+1}\) obtained in the \(i\)th refinement round associates with each node \(v\) its previous colour \(C_i(v)\) and the multiset of the previous colours of its neighbours:

\[
C_{i+1}(v) := \left( C_i(v), \{ C_i(w) \mid w \in N^G(v) \} \right).
\] (3.5.1)

Note that \(C_{i+1}(v) = C_{i+1}(w)\) if and only if \(C_i(v) = C_i(w)\) and for all colours \(c\) in the range of \(C_i\), the vertices \(v\) and \(w\) have the same number of neighbours of colour \(c\). This is exactly what was required in the original definition of colour refinement. Note that for all \(i\) the colouring \(C_{i+1}\) refines the colouring \(C_i\). Hence there is an \(i \leq |G|\) such that \(C_i\) and \(C_{i+1}\) induce the same partition into colour classes, that is, \(C_i(v) = C_i(w) \iff C_{i+1}(v) = C_{i+1}(w)\) for all \(v, w\). Note that this is the case if and only if the colouring \(C_i\) is stable. We denote \(C_i\) for the smallest such \(i\) such that \(C_i\) is stable by \(C_\infty\). It is easy to see that \(C_\infty\) is the coarsest stable colouring of \(G\). Note that if we run colour refinement on two graphs to use it as an isomorphism test, then \(C_\infty\) is the first \(C_i\) that is stable on both graphs. Hence colour refinement distinguishes graphs \(G\) and \(H\) if for some colour \(c\) in the range of \(C_\infty\),

\[
|V(G) \cap C_\infty^{-1}(c)| \neq |V(H) \cap C_\infty^{-1}(c)|.
\]

Of course we would not use the complicated objects consisting of nested multisets as colours in an actual implementation of colour refinement. The formal definition just serves as a convenient mathematical abstraction.

**Theorem 3.5.5 (Immerman and Lander [76]).** Let \(G\) and \(H\) be graphs. Then colour refinement does not distinguish \(G\) and \(H\) if and only if \(G\) and \(H\) are \(C^2\)-equivalent.

**Proof.** For the forward direction of the proof we use the characterisation of \(C^2\)-equivalence by the bijective 2-pebble game.

**Claim 1.** Suppose that colour refinement does not distinguish \(G\) and \(H\). Then the Duplicator has a winning strategy for the bijective 2-pebble game on \(G, H\).

**Proof.** Let us call a position \((v_1, v_2, w_1, w_2)\) of the bijective 2-pebble game on \(G, H\) good if it is a partial isomorphism (that is, \(v_1 = v_2 \iff w_1 = w_2\) and \(v_1v_2 \in E(G) \iff w_1w_2 \in E(H)\)) and \(C_\infty(v_i) = C_\infty(w_i)\) for \(i = 1, 2\). We call a position \((v, w)\) good if \(C_\infty(v) = C_\infty(w)\). We call the empty position \(((),())\) good as well.

We shall prove that the Duplicator can play the game in such a way that if before some round of a play the position is good then the position reached after the round is still good. So let \((v_1, v_2, w_1, w_2)\) be a good position. (The argument is similar, but simpler for good positions of the form \((v, w)\) and the empty position.) Say, the Spoiler picks up his pebble with label 2. Then the Duplicator picks up her pebble with label 2 and needs to define a bijective mapping \(f : V(G) \to V(H)\).

As colour refinement does not distinguish \(G\) and \(H\), for all colours \(c\) in the range of \(C_\infty\) we have

\[
|V(G) \cap C_\infty^{-1}(c)| = |V(H) \cap C_\infty^{-1}(c)|.
\]

Since the colouring is stable and \(C_\infty(v_1) = C_\infty(w_1)\), we have

\[
|N^G(v_1) \cap C_\infty^{-1}(c)| = |N^H(w_1) \cap C_\infty^{-1}(c)|.
\]

Thus the Duplicator can define her bijection \(f\) in such a way that for all colours \(c\) we have

\[
f(N^G(v_1) \cap C_\infty^{-1}(c)) = N^H(w_1) \cap C_\infty^{-1}(c)\]

and

\[
f((V(G) \setminus N^G(v_1)) \cap C_\infty^{-1}(c)) = (V(H) \setminus N^H(w_1)) \cap C_\infty^{-1}(c).
\]
\[ N^H(w_1) \cap C^{-1}_\infty(c) \]. Then for each \( v \in V(G) \) the Spoiler may choose, the new position \( ((v_1, v), (w_1, f(v))) \) is still good.

Since good positions are partial isomorphisms, this gives a winning strategy for the Duplicator.

By Fact \[\text{3.4.15}\] Claim 1 implies the forward direction of the lemma.

To prove the backward direction, we need the following claim.

**Claim 2.** For every \( i \in \mathbb{N} \) and every colour \( c \) in the range of \( C_i \) there is a \( C^2 \)-formula \( \varphi^c_i(x) \) that defines the colour class \( C^{-1}_i(c) \) in the sense that \( \varphi^c_i[G, x] = V(G) \cap C^{-1}_i(c) \) and \( \varphi^c_i[H, x] = V(H) \cap C^{-1}_i(c) \).

**Proof.** The proof is by induction on \( i \). The base step \( i = 0 \) is trivial. For the inductive step, let \( c \) be a colour in the range of \( C_{i+1} \). Note that \( C^{-1}_{i+1}(c) \subseteq C^{-1}_i(d_c) \) for some \( d_c \), because \( C_{i+1} \) refines \( C_i \). By the induction hypothesis, for every colour \( d \) in the range of \( C_i \), there is a formula \( \varphi^d_i(x) \) that defines \( C^{-1}_i(d) \). By the definition of \( C_{i+1} \), for all \( d \) in the range of \( C_i \), all vertices \( u \in C^{-1}_{i+1}(c) \) have the same number \( n_d \) of neighbours in \( C_{i+1}(d) \). We let

\[
\varphi^c_{i+1}(x) := \varphi^{d_c}_i(x) \land \bigwedge_d \exists y \left( E(x, y) \land \exists x (y = x \land \varphi^d_i(x)) \right),
\]

where the conjunction ranges over all colours \( d \) in the range of \( C_i \).

Claim 2 implies that for every colour \( c \) in the range of \( C_\infty \) there is a \( C^2 \)-formula \( \varphi^c_\infty(x) \) that defines \( C^{-1}_\infty(c) \) on \( G \) and \( H \). To complete the proof of the backward direction, suppose that colour refinement distinguishes \( G \) and \( H \). Then there is a colour \( c \) such that \( \ell := |V(G) \cap C^{-1}_\infty(c)| \neq |V(H) \cap C^{-1}_\infty(c)| \). Then \( G \models \exists^\ell x \varphi^c_\infty(x) \) and \( H \models \exists^\ell x \varphi^c_\infty(x) \). Thus \( G \) and \( H \) are not \( C^2 \)-equivalent. \(\square\)

### 3.5.2 The Weisfeiler-Leman Algorithm

The \textit{k-dimensional Weisfeiler-Leman algorithm} (for short: \( k \)-WL) is a straightforward generalisation of the colour refinement algorithm. Instead of vertices, it colours \( k \)-tuples of vertices. Given a graph \( G \), \( k \)-WL iteratively computes a colouring of \( V(G)^k \). Initially, two tuples \( \bar{v}, \bar{w} \in V(G)^k \) get the same colour if \( (\bar{v}, \bar{w}) \) is a partial isomorphism. In each round of the algorithm, the colouring is refined by assigning different colours to tuples \( \bar{v}, \bar{w} \) such that for some “type” of \( (k + 1) \)-tuples, \( \bar{v} \) has a different number of extensions \( \bar{v}x \) of this type than \( \bar{w} \) has extensions \( \bar{w}y \) of this type. Instead of making the required notion of “type” precise, we immediately give a formal definition of the colourings computed \( k \)-WL along the lines of the definition [3.5.1] of the colourings \( C_i \) computed by colour refinement.

We will actually define \( k \)-WL not only on graphs, but on arbitrary relational structures. We say that two tuples \( \bar{v} = (v_1, \ldots, v_k), \bar{w} = (w_1, \ldots, w_k) \) of elements of \( \tau \)-structures \( A, B \) have the same atomic type if the mapping \( v_i \mapsto w_i \) is a partial isomorphism. Then we define the atomic type \( \text{atp}(A, \bar{v}) \) of a tuple \( \bar{v} \) in a structure \( A \) to be the equivalence class of the tuple with respect to the equivalence relation of “having the same atomic type”. From a logical perspective, we can simply define \( \text{atp}(A, \bar{v}) \) to be the set of all atomic formulas \( \chi(\bar{v}) \) of vocabulary \( \tau \) such that \( A \models \chi(\bar{v}) \).

Now we can describe \( k \)-WL as the algorithm that, given a \( \tau \)-structure \( A \), computes the following sequence of colourings \( C^k_i \) of \( V(A)^k \) until it returns \( C^k_\infty = C^k_i \) for the smallest \( i \) such...
that for all \( \bar{v}, \bar{w} \in V(A)^k \) it holds that \( C_k^k(\bar{v}) = C_k^k(\bar{w}) \iff C_{i+1}^k(\bar{v}) = C_{i+1}^k(\bar{w}) \). The initial colouring \( C_0^k \) is defined by

\[
C_0^k(\bar{v}) := \text{atp}(A, \bar{v}),
\]

and the colouring \( C_{i+1}^k \) is defined by

\[
C_{i+1}^k(v_1, \ldots, v_k) := (C_i^k(v_1, \ldots, v_k), M_i(v_1, \ldots, v_k)),
\]

where \( M_i(v_1, \ldots, v_k) \) is the multiset

\[
\left\{ \left( \text{atp}(A, (v_1, \ldots, v_k, w)), \quad \begin{align*}
C_i^k(v_1, \ldots, v_{k-1}, w), C_i^k(v_1, \ldots, v_{k-2}, w, v_k), \ldots, C_i^k(w, v_2, \ldots, v_k) \end{align*} \right) \mid w \in V \right\}.
\]

Observe that if the arity of all relations in \( \tau \) is less than or equal to \( k \), then we can omit the entry \( \text{atp}(A, (v_1, \ldots, v_k, w)) \) from the tuples in the multiset, because all the information it contains is also contained in the entries \( C_i^k(\ldots) \).

Let us call a colouring \( C \) of \( V(A)^k \) stable if a refinement round applied to \( C \) yields a colouring that does not refine \( C \) properly. (We omit a more formal definition.) Then it is easy to see that \( C_\infty^k \) is a coarsest stable colouring.

To use \( k \)-WL as an isomorphism test, given \( \tau \)-structures \( A \) and \( B \), we run the refinement process in parallel on both structures until the colouring is stable on both structures\(^\text{3}\). In this case, \( C_\infty \) denotes the first colouring that is stable on both graphs. We say that \( k \)-WL distinguishes \( A \) and \( B \) if for some colour \( c \) in the range of \( C_\infty^k \),

\[
|V(A)^k \cap (C_\infty^k)^{-1}(c)| \neq |V(B) \cap (C_\infty^k)^{-1}(c)|,
\]

and \( k \)-WL identifies \( A \) if it distinguishes \( A \) from all \( \tau \)-structure \( B \) that are not isomorphic to \( A \).

It is easy to see that for every \( i \geq 0 \) the colouring \( C_i^1 \) computed by \( 1 \)-WL induces the same partition of the vertex set as the colourings \( C_i \) computed by colour refinement. Hence \( 1 \)-WL distinguishes two graphs if and only if colour refinement distinguishes the graphs.

**Example 3.5.6.** Figure 3.4 shows two nonisomorphic graphs that are not distinguished by \( 2 \)-WL. The only difference between the two graphs is in the grey area.

The graphs are built up from four identical gadgets which have three pairs of exit nodes \( x_0, x_1, y_0, y_1, z_0, z_1 \) connecting them to the other gadgets and four inner nodes. The crucial property of the gadgets is that each automorphism that keeps the three sets \( \{x_0, x_1\}, \{y_0, y_1\}, \{z_0, z_1\} \) fixed either keeps all exit nodes fixed or swaps precisely two of the three pairs (for example, it may map \( x_0 \) to \( x_1 \), \( x_1 \) to \( x_0 \), \( y_0 \) to \( y_1 \), \( y_1 \) to \( y_0 \), and fix \( z_0 \) and \( z_1 \)). Moreover, for each of these actions on the exit nodes there is an automorphism that realises it.

The gadgets were first introduced in [67], and they are also used in the proof of the CFI-Theorem (Fact 3.4.16). As a matter of fact, the graphs in Figure 3.4 are the graphs \( G_3 \) and \( H_3 \) of the CFI-Theorem.

**Theorem 3.5.7 (Immerman and Lander [76]).** Let \( k \in \mathbb{N} \), and let \( A \) and \( B \) be \( \tau \)-structures. Then \( k \)-WL does not distinguish \( A \) and \( B \) if and only if \( A \) and \( B \) are \( C^{k+1} \)-equivalent.

---

\(\text{3}\) Running it “in parallel” on both structures is better than running it on the disjoint union of the structures, because the latter would involve colouring mixed tuples that contain elements of both structures, which only complicates matters without giving any new insights.
3.5. Isomorphism Testing and the Weisfeiler-Leman Algorithm

Figure 3.4. Two nonisomorphic graphs not distinguished by 3-WL

Proof. The proof is very similar to the proof of Theorem 3.5.5.

Claim 1. Suppose that \( k \)-WL does not distinguish \( A \) and \( B \). Then the Duplicator has a winning strategy for the bijective \((k + 1)\)-pebble game on \( A, B \).

Proof. It is convenient to define for all \( \ell \in [k] \) a colouring \( C^{k,\ell}_\infty \) of \( V(A)\ell \cup V(B)\ell \) by

\[
C^{k,\ell}_\infty (u_1, \ldots, u_\ell) := C^k_\infty(u_1, \ldots, u_\ell, u_\ell, \ldots, u_\ell) \text{ \( k-\ell \) times}
\]

Note that tuples of the same colour under \( C^k_\infty \) have the same atomic type, because \( C^k_\infty \) refines \( C^k_0 \), which just colours each tuple by its atomic type. Hence for all colours \( c \) in the range of \( C^{k,\ell}_\infty \), all tuples \( \bar{u} = (u_1, \ldots, u_k) \in V(A)^k \cup V(B)^k \) with \( C^k_\infty(\bar{u}) = c \) satisfy \( u_\ell = u_{\ell+1} = \cdots = u_k \). As the colouring \( C^k_\infty \) does not distinguish \( A \) and \( B \), it follows that

\[
|V(A)^\ell \cap (C^{k,\ell}_\infty)^{-1}(c)| = |V(B) \cap (C^{k,\ell}_\infty)^{-1}(c)| \tag{3.5.2}
\]

for all colours \( c \) in the range of \( C^{k,\ell}_\infty \).

Let us call a position \((u_1, \ldots, u_\ell), (w_1, \ldots, w_\ell)\) of the bijective \((k + 1)\)-pebble game on \( A, B \) good if

(A) either \( \ell = 0 \)

(B) or \( 1 \leq \ell \leq k \) and \( C^{k,\ell}_\infty(v_1, \ldots, v_\ell) = C^{k,\ell}_\infty(w_1, \ldots, w_\ell) \)

(C) or \( \ell = k + 1 \) and \( \text{atp}(A, (v_1, \ldots, v_{k+1})) = \text{atp}(B, (w_1, \ldots, w_{k+1})) \) and

\[
C^k_\infty(v_1, \ldots, v_k) = C^k_\infty(w_1, \ldots, w_k),
\]

\[
C^k_\infty(v_{k+1}, v_2, \ldots, v_k) = C^k_\infty(w_{k+1}, w_2, \ldots, w_k),
\]

\[
C^k_\infty(v_1, v_{k+1}, v_3, \ldots, v_k) = C^k_\infty(w_1, w_{k+1}, w_3, \ldots, w_k),
\]

\[
\vdots
\]

\[
C^k_\infty(v_1, \ldots, v_{k-1}, v_{k+1}) = C^k_\infty(w_1, \ldots, w_{k-1}, w_{k+1}).
\]
We shall prove that the Duplicator can play the game in such a way that if before some round of a play the position is good then the position reached after the round is still good. So let \(((v_1, \ldots, v_\ell), (w_1, \ldots, w_\ell))\) be a good position.

**Case 1: \(\ell = 0\).**

It follows from (3.5.2) that there is a bijection \(f : V(A) \rightarrow V(B)\) such that \(C^1_\infty(v) = C^1_\infty(f(v))\) for all \(v \in V(A)\). The Duplicator chooses such a bijection. Then if the Spoiler places his pebble on \(v\), the new position \((v, f(v))\) satisfies (B) and thus is good.

**Case 2: \(1 \leq \ell < k\).**

Without loss of generality we may assume that the Spoiler picks up his \((\ell + 1)\)st pebble. As the colouring \(C^k_\infty\) is stable and \(C^k_\infty(v_1, \ldots, v_\ell) = C^k_\infty(w_1, \ldots, w_\ell)\), for each colour \(c\) we have

\[
\left| \{ v \in V(A) \mid C^k_\infty(v_1, \ldots, v_\ell, v, v_\ell, \ldots, v_1) = c \} \right| = \left| \{ w \in V(B) \mid C^k_\infty(w_1, \ldots, w_\ell, w, w_\ell, \ldots, w_1) = c \} \right|.
\]

Hence the Duplicator can choose a bijection \(f : V(A) \rightarrow V(B)\) such that for all \(v \in V(A)\),

\[
C^k_\infty(v_1, \ldots, v_\ell, v, v_\ell, \ldots, v_1) = C^k_\infty(w_1, \ldots, w_\ell, f(v), w_\ell, \ldots, w_1). \tag{3.5.3}
\]

Suppose the Spoiler places his pebble on \(v\). Then the new position is

\[
((v_1, \ldots, v_\ell, v), (w_1, \ldots, w_\ell, f(v))).
\]

If \(\ell = k - 1\), then (3.5.3) immediately implies that the new position satisfies (B). Suppose that \(\ell < k - 1\). Let \(c := C^k_\infty(v_1, \ldots, v_\ell, v, v_\ell, \ldots, v_1)\). Then

\[1 = \left| \{ v' \in V(A) \mid C^k_\infty(v_1, \ldots, v_\ell, v, v', v_\ell, \ldots, v_1) = c \} \right| = \left| \{ w' \in V(B) \mid C^k_\infty(w_1, \ldots, w_\ell, f(v), w', w_\ell, \ldots, w_1) = c \} \right|.
\]

Here the first equality holds because \(v' = v\) is the only element of \(V(A)\) such that \(\text{atp}(A, (v_1, \ldots, v_\ell, v, v', v_\ell, \ldots, v_1)) = \text{atp}(A, (v_1, \ldots, v_\ell, v, v, v_\ell, \ldots, v_1))\), and the second equality holds because \(C^k_\infty\) is stable. Thus there is exactly one \(w' \in V(B)\) such that

\[
C^k_\infty(v_1, \ldots, v_\ell, v, v, v_\ell, \ldots, v_1) = C^k_\infty(w_1, \ldots, w_\ell, f(v), w', w_\ell, \ldots, w_1),
\]

and as \(w' = f(v)\) is the only \(w'\) that yields the correct atomic type, we have

\[
C^k_\infty(v_1, \ldots, v_\ell, v, v, v_\ell, \ldots, v_1) = C^k_\infty(w_1, \ldots, w_\ell, f(v), f(v), w_\ell, \ldots, w_1).
\]

Repeatedly applying this argument, we see that

\[
C^k_\infty(v_1, \ldots, v_\ell, v, v, v_\ell, \ldots, v_1) = C^k_\infty(w_1, \ldots, w_\ell, f(v), \ldots, f(v), w_\ell, \ldots, w_1).
\]

for all \(i \leq k - \ell\), and for \(i = k - \ell\), this implies (B).

M. Grohe, *Definable Graph Structure Theory*
3.5. Isomorphism Testing and the Weisfeiler-Leman Algorithm

Case 3: \( \ell = k \) or \( \ell = k + 1 \).

Without loss of generality we may assume that the Spoiler picks up his \((k+1)\)st pebble. As the colouring \( C_\infty^k \) is stable and \( C_\infty^k(v_1, \ldots, v_k) = C_\infty^k(w_1, \ldots, w_k) \), for each atomic type \( a \) and all colours \( c_1, \ldots, c_k \) we have

\[
\left| \left\{ v \in V(A) \mid \text{atp}(A, (v_1, \ldots, v_k)) = a, \right. \right.
\]
\[
C_\infty^k(v, v_2, \ldots, v_k) = c_1,
\]
\[
C_\infty^k(v_1, v, v_3, \ldots, v_k) = c_2,
\]
\[
\vdots
\]
\[
C_\infty^k(v_1, v_2, \ldots, v_{k-1}, v) = c_k \left. \right| \right.
\]
\[
= \left| \left\{ w \in V(B) \mid \text{atp}(B, (w_1, \ldots, w_k)) = a, \right. \right.
\]
\[
C_\infty^k(w, w_2, \ldots, w_k) = c_1,
\]
\[
C_\infty^k(w_1, w, w_3, \ldots, w_k) = c_2,
\]
\[
\vdots
\]
\[
C_\infty^k(w_1, w, \ldots, w_{k-1}, w) = c_k \left. \right| \right.
\]

Hence the Duplicator can choose a bijection \( f : V(A) \to V(B) \) such that

\[
\text{atp}(A, (v_1, \ldots, v_k, v)) = \text{atp}(B, (w_1, \ldots, w_k, f(v))),
\]
\[
C_\infty^k(v, v_2, \ldots, v_k, v) = C_\infty^k(f(v), w_2, \ldots, w_k),
\]
\[
\vdots
\]
\[
C_\infty^k(v_1, v_2, \ldots, v_{k-1}, v) = C_\infty^k(w_1, w_2, \ldots, w_{k-1}, f(v)),
\]

Thus if the Spoiler places his pebble on \( v \), the new position

\[
((v_1, \ldots, v_k, v), (w_1, \ldots, w_k, f(v)))
\]

satisfies (C).

By Fact 3.4.15, Claim 1 implies the forward direction of the lemma.

Claim 2. For every \( i \in \mathbb{N} \) and every colour \( c \) in the range of \( C_i^k \) there is a \( C_i^{k+1} \)-formula \( \varphi_i^c(x_1, \ldots, x_k) \) that defines the colour class \((C_i^k)^{-1}(c)\) in the sense that \( \varphi_i^c[A, x_1, \ldots, x_k] = V(A)^k \cap (C_i^k)^{-1}(c) \) and \( \varphi_i^c[B, x_1, \ldots, x_k] = V(B)^k \cap (C_i^k)^{-1}(c) \).

Proof. The proof is by induction on \( i \). For the base step, let \( a \) be a colour in the range of \( C_0^k \), that is, an atomic type of a \( k \)-tuple. We view \( a \) as a set of atomic formulas with variables in \( \{x_1, \ldots, x_k\} \). Then we let

\[
\varphi_0^a(x_1, \ldots, x_k) := \bigwedge_{\chi \in a} \chi \land \bigwedge_{\xi \notin a} \neg \xi,
\]

where the second conjunction ranges over all atomic formulas with variables in \( \{x_1, \ldots, x_k\} \) that are not in \( a \).

For the inductive step, let \( c \) be a colour in the range of \( C_{i+1}^k \). Note that \( (C_{i+1}^k)^{-1}(c) \subseteq (C_i^k)^{-1}(d_c) \) for some \( d_c \), because \( C_{i+1}^k \) refines \( C_i^k \). For all tuples \( t = (a, c_1, \ldots, c_k) \) consisting of
Remark 3.5.9. Otto [94] proved that if \( G \) is an atomic type \( a \) of \((k+1)\)-tuples and colours \( c_1, \ldots, c_k \) in the range of \( C_i^k \) there is a number \( n_t \) such that for all tuples \( \bar{u} = (u_1, \ldots, u_k) \in (\mathbb{C}_i^k)^{-1}(c) \) there are exactly \( n_t \) elements \( u \) of the respective structure such that \((u_1, \ldots, u_k, u)\) has atomic type \( a \) and the \( k \)-tuples obtained by substituting \( u \) in the \( j \)-th position of \( \bar{u} \) have \( C_i^k \)-colour \( c_j \). To define the atomic type \( a \), we use the formula
\[
\psi_a(x_1, \ldots, x_k, y) := \bigwedge_{\chi \in a} \chi \wedge \bigwedge_{\xi \in a} \neg \xi.
\]
Here we view the atomic type \( a \) a set of atomic formulas with variables in \( \{x_1, \ldots, x_k, y\} \). The first conjunction ranges over all formulas \( \chi \) in this set and the second conjunction ranges over all atomic formulas \( \xi \) with variables in \( \{x_1, \ldots, x_k\} \) that are not in \( a \).

Now we let
\[
\varphi_{i+1}^c(x_1, \ldots, x_k) := \varphi_i^{dc}(x_1, \ldots, x_k)
\]
\[
\wedge \bigwedge_{t=(a, c_1, \ldots, c_k)} \exists^{n_t} y \left( \psi_a(x_1, \ldots, x_k, y) \wedge \bigwedge_{j=1}^k \exists x_j \left( y = x_j \wedge \varphi_i^c(x_1, \ldots, x_k) \right) \right)
\]

Claim 2 implies the backward direction of the theorem. \hfill \square

As a corollary of Theorem 3.5.7 and Lemma 3.4.18 we obtain:

**Corollary 3.5.8.** Let \( C \) be an \( \text{IFP}+\text{C} \)-definable class of graphs that admits \( \text{IFP}+\text{C} \)-definable canonisation. Then there is a \( k \in \mathbb{N} \) such that \( k\text{-WL} \) identifies all graphs \( G \in C \).

**Remark 3.5.9.** Otto [94] proved that if \( \text{IFP}+\text{C} \) captures \( \text{PTIME} \) on a class \( C \) of graphs closed under disjoint unions, then there is a \( k \in \mathbb{N} \) such that \( k\text{-WL} \) distinguishes any two graphs in \( C \). His arguments can also be used to prove Corollary 3.5.8.

### 3.5.3 A Linear Programming Approach to the Isomorphism Problem

In addition to the characterisations of \( C_i^k \)-equivalence by the cardinality and bijective \( k \)-pebble games, and by the \((k-1)\)-dimensional Weisfeiler-Leman algorithm, in this section we will see another very intriguing characterisation of the equivalence in terms of a family of linear programs.

Throughout the section, we make the following assumptions.

**Assumption 3.5.10.** \( G \) and \( H \) are two graphs of the same order, and \( V := V(G), W := V(H) \) are their vertex sets. We assume that \( V \) and \( W \) are disjoint. Furthermore, \( A \in \{0,1\}^{V \times V} \) and \( B \in \{0,1\}^{W \times W} \) are the adjacency matrices of \( G \) and \( H \).

The notation may require some explanation. For finite sets \( U, U' \) and an arbitrary set \( S \), we let \( S^{U \times U'} \) be the set of all \(|U| \times |U'|\)-matrices with entries from \( S \) and rows indexed by the elements of \( U \) and columns indexed by the elements of \( U' \). The order in which the rows and columns appear is arbitrary (and irrelevant for us). We denote the entries of a matrix \( M \in S^{U \times U'} \) by \( M_{uvv'} \).

Thus the matrix \( A \) has entries \( A_{vv'} \) for \( v, v' \in V = V(G) \), and we have \( A_{vv'} = 1 \) if \( vv' \in E(G) \) and \( A_{vv'} = 0 \) otherwise. Similarly, the matrix \( B \) has entries \( B_{ww'} \) for \( w, w' \in W = V(H) \).
Observe that the graphs $G$ and $H$ are isomorphic if and only if there is a permutation matrix $X \in \{0, 1\}^{V \times W}$ such that $X^tAX = B$, or equivalently,

$$AX = XB.$$  \hfill (3.5.4)

(Remember that a permutation matrix is a square matrix with $\{0, 1\}$-entries that has exactly one 1-entry in each row and column.) Hence the $G$ and $H$ are isomorphic if the following system $(L)$ of linear equations and inequalities in the variables $X_{vw}$ has an integer solution:

$$\sum_{w \in W} X_{vw} = 1 \quad \text{for all } v \in V, \quad (3.5.5)$$

$$\sum_{v \in V} X_{vw} = 1 \quad \text{for all } w \in W, \quad (3.5.6)$$

$$\sum_{v' \in V} A_{vv'}X_{v'w} = \sum_{w' \in W} X_{v'w}B_{w'w} \quad \text{for all } v, w \in W, \quad (3.5.7)$$

$$X_{vw} \geq 0 \quad \text{for all } v, w \in W. \quad (3.5.8)$$

We may ask whether the system has a real, not necessarily integral, solution. Such a solution is sometimes called a fractional isomorphism from $G$ to $H$. Equivalently, we may ask whether there is a doubly stochastic matrix $X$ satisfying (3.5.4). (A matrix $X$ is doubly stochastic if it has nonnegative real entries and the entries of each row and column sum up to 1.) Surprisingly, the colour refinement algorithm answers precisely this question.

**Theorem 3.5.11 (Tinhofer [122]).** The following two statements are equivalent.

(i) Colour refinement does not distinguish the graphs $G$ and $H$.

(ii) There is a fractional isomorphism from $G$ to $H$, that is, the system $(L)$ of linear equalities and inequalities has a real solution.

**Proof sketch.** The forward direction of the proof is not difficult. Suppose that colour refinement does not distinguish $G$ and $H$, and let $C_\infty$ denote the stable colouring computed by the colour refinement algorithm (see page 78 for the definition). Then for all colours $c$ in the range of $C_\infty$ it holds that $|V \cap C^{-1}_\infty(c)| = |W \cap C^{-1}_\infty| =: k_c$, and for all colours $c, d$ there is a number $\ell_{cd}$ such that each for all $u \in C^{-1}(c)$ we have $|N(u) \cap C^{-1}_\infty(d)| = \ell_{cd}$. Let $X$ be the $V \times W$-matrix defined by

$$X_{vw} := \begin{cases} 1/k_c & \text{if } C_\infty(v) = C_\infty(w) = c \text{ for some color } c, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $X$ is a doubly stochastic matrix satisfying (3.5.4).

Let us turn to the backward direction. Suppose that $X$ is a fractional isomorphism from $G$ to $H$. Let $G_X$ be the graph with vertex set $V(G_X) = V \cup W$ and edge set $E(G_X) = \{vw \mid X_{vw} > 0\}$, and let $U_1, \ldots, U_m$ be the vertex sets of the connected components of $G_X$. For each $i \in [m]$, let $V_i = V \cap U_i$ and $W_i = W \cap U_i$.

**Claim 1.** For all $i \in [m]$ we have $|V_i| = |W_i|$.

**Proof.** The proof is a straightforward calculation that only uses equations (3.5.5) and (3.5.6):

$$|V_i| \sum_{v \in V_i} \sum_{w \in W} X_{vw} = \sum_{v \in V_i} \sum_{w \in W_i} X_{vw} = \sum_{w \in W_i} \sum_{v \in V} X_{vw} |W_i| \sum_{v \in V_i} \sum_{w \in W} X_{vw} \quad \text{for all } i \in [m].$$

Preliminary Version
Here the two equations \((\ast)\) hold because \(X_{vw} \neq 0\) only if \(v\) and \(w\) belong to the same connected component of the graph \(G_X\).

We define a coloring \(C : V \cup W \to [m]\) by \(C(u) = i\) if \(u \in U_i\), so that for every color \(i \in [m]\) we have \(C^{-1}(i) = U_i\). We shall prove in Claim 3 that this coloring is stable.

Recall that a convex combination of numbers \(x_1, \ldots, x_k\) is a linear combination \(\sum_{i=1}^{k} a_i x_i\) where all the \(a_i\) are nonnegative and their sum is 1. If all the \(a_i\) are positive, we call \(\sum_{i=1}^{k} a_i x_i\) a positive convex combination of \(x_1, \ldots, x_k\). The following claim is the core of the proof.

\[
\text{Claim 3. Let } i, j \in [m].
\]

(1) For all \(v \in V_i\), the number \(|N(v) \cap V_j|\) is a positive convex combination of the numbers \(|N(w) \cap W_j|\) for all \(w \in N^{G_X}(v)\).

(2) For all \(w \in W_i\), the number \(|N(w) \cap V_j|\) is a positive convex combination of the numbers \(|N(v) \cap V_j|\) for all \(v \in N^{G_X}(w)\).

\[\text{Proof.}\] We only prove (1); (2) follows by symmetry. Let \(v \in V_i\). Then

\[
|N(v) \cap V_j| = \sum_{w' \in V_j} A_{vv'} \sum_{w \in W} X_{vw} \sum_{w' \in V_j} X_{v'w} = \sum_{w \in W} \sum_{w' \in V_j} A_{vw} X_{vw} \sum_{w' \in V_j} X_{v'w} \tag{3.5.5}
\]

\[
= \sum_{w \in W} \sum_{w' \in W} X_{vw'} B_{w'w} = \sum_{w \in W} X_{vw'} \sum_{w' \in W} B_{w'w} \tag{\ast\ast}
\]

Here equation \((\ast)\) holds, because \(AX = XB\). Equation \((\ast\ast)\) holds because \(X_{vw'} \neq 0\) only if \(w'\) is a neighbour of \(v\) in the graph \(G_X\). As \(X_{vw'} > 0\) for all \(w' \in N^{G_X}(v)\) and \(\sum_{w' \in N^{G_X}(v)} X_{vw'} = \sum_{w' \in W} X_{vw'} = 1\) by (3.5.5), this gives us the desired positive convex combination.

\[
\text{Claim 3. The coloring } C \text{ is stable.}
\]

\[\text{Proof.}\] We need to prove that for all \(i, j \in [m]\) and all \(v \in V_i, w \in W_i\) it holds that \(|N(v) \cap V_j| = |N(w) \cap W_j|\).

We use the following simple fact: If \(A\) is a connected graph and \(f : V(A) \to \mathbb{R}\) such that for all \(v \in V(A)\) the value \(f(v)\) is a positive convex combination of the values \(f(w)\) for \(w \in N(v)\), then \(f\) is constant. We apply this fact to the graph \(A := G_X[U_i]\) and the function \(f : U_i \to \mathbb{R}\) defined by

\[
f(u) := |N(u) \cap U_j| = \begin{cases} |N(u) \cap V_j| & \text{if } u \in V_i, \\ |N(u) \cap W_j| & \text{if } u \in W_i. \end{cases}
\]

As the coloring \(C_{\infty}\) computed by colour refinement is the coarsest stable colouring, \(C\) refines \(C_{\infty}\). Thus every colour class \(C_{\infty}^{-1}(c)\) of \(C_{\infty}\) is a union of colour classes \(U_i\) of \(C\). Hence it follows from Claim 1 that \(|V \cap C_{\infty}^{-1}(c)| = |W \cap C_{\infty}^{-1}(c)|\). Hence colour refinement does not distinguish \(G\) and \(H\).

\[\text{Remark 3.5.12.}\] The proof of the backward direction of Theorem [3.5.11] presented here is due to Erkal Selman [118].

\[\text{Corollary 3.5.13.}\] The following two statements are equivalent.

M. Grohe, Definable Graph Structure Theory
(i) \( G \) and \( H \) are \( C^2 \)-equivalent.

(ii) The system \((L)\) of linear equations and inequalities has a real solution.

There is an analogue of this corollary for \( C^k \)-equivalence for all \( k \geq 3 \), which we will briefly sketch in the following. For each integer linear program, there is a well-known hierarchy of linear-programming relaxations that was introduced by Sherali and Adams [119]. We skip the general definition and directly consider the Sherali-Adams hierarchy for the integer linear program for graph isomorphism, which is our system \((L)\) of linear equalities and inequalities (we ignore the objective functions here and are only interested in feasible solutions), in a streamlined formulation due to Atserias and Maneva [5]: for all \( k \in \mathbb{N}^+ \), the \( k \)th level of the Sherali-Adams hierarchy for the linear program \((L)\) is the following linear program \((L_k)\) in the variables \( X_p \) for all \( p \subseteq V \times W \) of size \( |p| \leq k \):

\[
\sum_{w \in W} X_{p \cup vw} = X_p \\
\text{for all } v \in V \text{ and } p \subseteq V \times W \text{ of size } |p| \leq k - 1,
\]  

(3.5.9)

where we write \( p \cup vw \) instead of \( p \cup \{(v, w)\} \),

\[
\sum_{v \in V} X_{p \cup vw} = X_p \\
\text{for all } w \in W \text{ and } p \subseteq V \times W \text{ of size } |p| \leq k - 1,
\]  

(3.5.10)

\[
X_\emptyset = 1,
\]  

(3.5.11)

\[
\sum_{v' \in V} A_{v'w'} X_{p \cup v'w'} = \sum_{w' \in W} X_{p \cup vw'} B_{w'w} \\
\text{for all } v, w \in W \text{ and } p \subseteq V \times W \text{ of size } |p| \leq k - 1,
\]  

(3.5.12)

\[
X_p \geq 0 \\
\text{for all } p \subseteq V \times W \text{ of size } |p| \leq k.
\]  

(3.5.13)

Note that \((L^1)\) is equivalent to \((L)\) if we identify the variables \( X_\{(v, w)\} \) and \( X_{vw} \). Moreover, since we only add (in)equalities when increasing \( k \), every solution of \((L^k)\) projects to a solution for \((L^{k'})\) for all \( k' \leq k \). In particular, for \( k' = 1 \) this implies if \((L^k)\) has a real solution then \( G \) and \( H \) are \( C^2 \)-equivalent. But actually we have a much stronger result. Let \((L^C_k)\) be the linear program consisting of the equations \((3.5.9), (3.5.10), (3.5.11), \) and \((3.5.13)\) as stated and the equations \((3.5.12)\) only for \( p \subseteq V \times W \text{ of size } |p| \leq k - 2 \). Equivalently, we can characterise \((L^C_k)\) as the linear program obtained from \((L^k)\) by deleting all equations \((3.5.12)\) for \( p \subseteq V \times W \text{ of size } |p| = k - 1 \), or as the linear program obtained from \((L^{k-1})\) by adding all equations \((3.5.9), (3.5.10)\) for \( p \subseteq V \times W \text{ of size } |p| = k - 1 \) and equations \((3.5.13)\) for \( p \subseteq V \times W \text{ of size } |p| = k \). Hence \((L^C_k)\) is situated between \((L^k)\) and \((L^{k-1})\), in the sense that if \((L^k)\) has a solution then \((L^C_k)\) has, and if \((L^C_k)\) has a solution then \((L^{k-1})\) has.

**Fact 3.5.14** (Atserias and Maneva [5], Grohe and Otto [55], Malkin [88]). For all \( k \geq 2 \), the following two statements are equivalent.

(i) \( G \) and \( H \) are \( C^k \)-equivalent.

(ii) \((L^C_k)\) has a real solution.
Chapter 4

Treelike Decompositions

In this chapter, we introduce treelike decompositions, one of the core concepts of this book. Before we do so, in the first section we review tree decompositions. After that we define treelike decompositions and consider a few examples. Then we start to develop the theory of treelike decompositions. In Section 4.3, we give normal forms into which treelike decomposition can be transformed without losing any relevant properties. In Section 4.4, we consider particularly well-behaved decompositions, which we call tight. Tight treelike decomposition will play an important role in the second part of this book. In Section 4.5, we briefly discuss different notions of “equivalence” or “similarity” between treelike decompositions. In the final section of this chapter, we study the relation between treelike decompositions and tree decompositions.

4.1 Tree Decompositions

For reasons that will become apparent later, it will be convenient for us to define tree decompositions over directed trees instead of (undirected) trees. Recall that a subset \( U \subseteq V(T) \) of the node set of a directed tree \( T \) induces a subtree of \( T \) if and only if \( U \) is nonempty and connected in the undirected tree underlying \( T \).

**Definition 4.1.1.** A tree decomposition of a graph \( G \) is a pair \( \Delta = (T^\Delta, \beta^\Delta) \), where \( T^\Delta \) is a directed tree and \( \beta^\Delta : V(T^\Delta) \to 2^V(G) \) such that the following two axioms are satisfied.

(T.1) For every \( v \in V(G) \) the set \( \{ t \in V(T^\Delta) \mid v \in \beta^\Delta(t) \} \) induces a subtree of \( T^\Delta \).

(T.2) For every \( e \in E(G) \) there is a \( t \in V(T^\Delta) \) such that \( e \subseteq \beta^\Delta(t) \).

The sets \( \beta^\Delta(t) \), for \( t \in V(T^\Delta) \), are called the bags of the decomposition.

We introduce some additional notation. Let \( \Delta \) be a tree decomposition of a graph \( G \). We let \( V(\Delta) := V(T^\Delta) \), \( E(\Delta) := E(T^\Delta) \), and for every \( t \in V(\Delta) \) we let \( N^\Delta_+(t) := N^T_+(t) \). We denote the partial order induced by \( T^\Delta \) by \( \leq^\Delta \) instead of \( \leq^T \). Recall that \( \leq^T \) denotes the transitive closure of \( E(T^\Delta) \), that is, the natural partial order in which the root is the minimum. For every node \( t \in V(\Delta) \) we let:

\[
\gamma^\Delta(t) := \bigcup_{u \in V(\Delta) \text{ with } t \leq^\Delta u} \beta^\Delta(u), \tag{4.1.1}
\]

\(89\)
\[ \sigma^\Delta(t) := \begin{cases} \emptyset & \text{if } t \text{ is the root of } T^\Delta, \\ \beta^\Delta(t) \cap \beta^\Delta(s) & \text{if } t \in N_+^\Delta(s), \end{cases} \] (4.1.2)

\[ \alpha^\Delta(t) := \gamma^\Delta(t) \setminus \sigma^\Delta(t). \] (4.1.3)

The sets \( \gamma^\Delta(t) \), \( \sigma^\Delta(t) \), and \( \alpha^\Delta(t) \) are called the cone, separator, and component of \( \Delta \) at \( t \).

The separators of \( \Delta \) at \( t \) are the subsets \( \sigma^\Delta(t) \) and \( \sigma^\Delta(u) \) for all \( u \in N_+^\Delta(t) \), and a separator \( S \) at \( t \) is maximal if it is not a proper subset of any other separator at \( t \). The design of \( \Delta \) at \( t \) is the set

\[ \delta^\Delta(t) := \{ S \subseteq \beta(t) \mid S \text{ nonempty maximal separator of } \Delta \text{ at } t \}. \] (4.1.4)

Recall that for an arbitrary finite set \( S \), by \( K[S] \) we denote a complete graph with vertex set \( S \). The torso of \( \Delta \) at \( t \) is the graph

\[ \tau^\Delta(t) := G[\beta^\Delta(t)] \cup \bigcup_{S \in \delta^\Delta(t)} K[S] \]
\[ = G[\beta^\Delta(t)] \cup K[\sigma^\Delta(t)] \cup \bigcup_{u \in N_+^\Delta(t)} K[\sigma^\Delta(u)]. \] (4.1.5)

We omit the index \( \Delta \) if \( \Delta \) is clear from the context. As a matter of fact, we often leave tree decompositions unnamed and just introduce them as pairs \((T, \beta)\).

Note that \( \gamma, \beta \) are definable in terms of \( \sigma \) and \( \alpha \): for all \( t \in V(T) \) we have \( \gamma(t) = \sigma(t) \cup \alpha(t) \) and

\[ \beta(t) = \gamma(t) \setminus \bigcup_{u \in N_+^\Delta(t)} \alpha(u). \] (4.1.6)

A tree decomposition \( \Delta \) is over a class \( \mathcal{C} \) of graphs if all its torsos belong to \( \mathcal{C} \). The class of all graphs that have a tree decomposition over \( \mathcal{C} \) is denoted by \( \mathcal{T}(\mathcal{C}) \). The adhesion of a tree decomposition \( \Delta = (T, \beta) \) is

\[ \text{ad}(\Delta) := \max \{|\sigma(t)| \mid t \in V(T)\}, \]

and the width of \( \Delta \) is

\[ \text{wd}(\Delta) := \max \{|\beta(t)| \mid t \in V(T)\} - 1. \]

The tree width of a graph \( G \), denoted by \( \text{tw}(G) \), is the minimum of the widths of all tree decompositions of \( G \). For every integer \( k \geq -1 \), the class of all graphs of tree width at most \( k \) is denoted by \( \mathcal{T}_k \). Note that \( \mathcal{T}_{-1} = \{\emptyset\} \) and that \( \mathcal{T}_0 \) is the class of all graphs without edges. It is not hard to see that \( \mathcal{T}_1 \) is the class of all forests. For all \( k \geq -1 \) we have \( \mathcal{T}_k = \mathcal{T}(G_{k+1}) \).

**Example 4.1.2.** Figure 4.1 shows a graph \( G \) and a tree decomposition \((T, \beta)\) of this graph; the nodes are labelled by their bags.

Let \( t \) be the (highlighted) node with bag \( \beta(t) = \{3, 4, 6, 8\} \). For this node \( t \) we have

\[ \gamma(t) = \{3, 4, 6, 7, 8, 11, 12\}, \]
\[ \sigma(t) = \{3\}, \]
\[ \alpha(t) = \{4, 6, 7, 8, 11, 12\}, \]
\[ \delta(t) = \{\{3\}, \{4, 6, 8\}\}, \]
Figure 4.1. A graph with a tree decomposition

$$\tau(t) = (\{3, 4, 6, 8\}, \{34, 36, 46, 68\})$$

The width of the decomposition \((T, \beta)\) is 3. It is not hard to see that the tree width of \(G\) is 3 as well.

We collect a few simple facts about tree decompositions that we will often use implicitly in our arguments. Proofs can be found in [29].

**Fact 4.1.3.** Let \((T, \beta)\) be a tree decomposition of a graph \(G\).

1. For every \(t \in V(T)\) the set \(\sigma(t)\) separates \(\gamma(t)\) from \(V(G) \setminus \alpha(t)\).

2. For every connected subset \(W \subseteq V(G)\) the set \(\{t \in V(T) \mid W \cap \beta(t) \neq \emptyset\}\) induces a subtree of \(T\), and the set \(\{t \in V(T) \mid W \subseteq \beta(t)\}\) is either empty or it induces a subtree of \(T\).

3. Let \(H_1, \ldots, H_m\) be connected subgraphs of \(G\) such that for all \(i, j \in [m]\), either \(V(H_i) \cap V(H_j) \neq \emptyset\) or there is an edge \(e = vw \in E(G)\) with \(v \in V(H_i)\) and \(w \in V(H_j)\). Then there is a \(t \in V(T)\) such that \(V(H_i) \cap \beta(t) \neq \emptyset\) for all \(i \in [m]\).

4. For every clique \(X \subseteq V(G)\) there is a node \(t \in V(T)\) such that \(X \subseteq \beta(t)\).
4.2 Treelike Decompositions

In this section we will introduce a general notion of decomposition over directed graphs that are not necessarily trees. It will be convenient to base these decompositions on “components” \( \alpha(t) \) and “separators” \( \sigma(t) \) of the nodes \( t \) instead of “bags” \( \beta(t) \).

**Definition 4.2.1.** A decomposition of a graph \( G \) is a triple \( \Delta = (D^\Delta, \sigma^\Delta, \alpha^\Delta) \) where \( D^\Delta \) is a directed graph and \( \sigma^\Delta, \alpha^\Delta \) are mappings from \( V(D) \) to \( 2^{|V(G)|} \).

We introduce similar notation and terminology as for tree decompositions. Let \( \Delta \) be a decomposition of a graph \( G \). We let \( V(\Delta) := V(D^\Delta) \), \( E(\Delta) := E(D^\Delta) \), and for every \( t \in V(\Delta) \) we let \( N^\Delta_+(t) := N^D_+(t) \). If \( D \) is acyclic, we denote the partial order induced by \( D^\Delta \) by \( \preceq^\Delta \) instead of \( \preceq^D \). For every node \( t \in V(D^\Delta) \) we let:

\[
\gamma^\Delta(t) := \alpha^\Delta(t) \cup \sigma^\Delta(t), \\
\beta^\Delta(t) := \gamma^\Delta(t) \setminus \bigcup_{u \in N^\Delta_+(t)} \alpha^\Delta(u). 
\]

The sets \( \alpha^\Delta(t), \beta^\Delta(t), \gamma^\Delta(t), \) and \( \sigma^\Delta(t) \) are called the component, bag, cone, and separator, respectively, of \( \Delta \) at \( t \). From these sets we define the design and torso at \( t \) as for tree decompositions: The separators of \( \Delta \) at \( t \) are the sets \( \sigma^\Delta(u) \) for all \( u \in N^\Delta_+(t) \), and a separator \( S \) at \( t \) is maximal if it is not a proper subset of any other separator at \( t \). The design of \( \Delta \) at \( t \) is the set

\[
\delta^\Delta(t) := \{ S \subseteq \beta^\Delta(t) \mid S \text{ nonempty maximal separator of } \Delta \text{ at } t \}.
\]

The torso of \( \Delta \) at \( t \) is the graph

\[
\tau^\Delta(G) := G[\beta^\Delta(t)] \cup \bigsqcup_{S \in \delta^\Delta(t)} K[S].
\]

Two nodes \( t, t' \in V(\Delta) \) are parallel in \( \Delta \) (we write \( t \parallel^\Delta t' \)) if \( \sigma^\Delta(t) = \sigma^\Delta(t') \) and \( \alpha^\Delta(t) = \alpha^\Delta(t') \). More generally, if \( \Delta \) and \( \Delta' \) are decompositions of the same graph \( G \) and \( t \in V(\Delta), t' \in V(\Delta') \), then \( t \) and \( t' \) are parallel in \( \Delta, \Delta' \) (we write \( t \parallel^\Delta \Delta' t' \)) if \( \sigma^\Delta(t) = \sigma^\Delta(t') \) and \( \alpha^\Delta(t) = \alpha^\Delta(t') \). Note that \( t \parallel^\Delta \Delta' t' \) implies \( \gamma^\Delta(t) = \gamma^\Delta(t') \), but not \( \beta^\Delta(t) = \beta^\Delta(t') \) or \( \delta^\Delta(t) = \delta^\Delta(t') \) or \( \tau^\Delta(t) = \tau^\Delta(t') \). Two nodes \( t, t' \in V(\Delta) \) are orthogonal in \( \Delta \) (we write \( t \perp^\Delta t' \)) if

\[
\gamma^\Delta(t) \cap \gamma^\Delta(t') = \sigma^\Delta(t) \cap \sigma^\Delta(t').
\]

We omit the index \( ^\Delta \) if \( \Delta \) is clear from the context, and we occasionally leave decompositions unnamed and just introduce them as triples \( (D, \alpha, \sigma) \). We frequently use implicit naming conventions such as the following: for a decomposition \( \Delta' = (D', \sigma', \alpha') \) we let \( \beta' := \beta^\Delta, \gamma' := \gamma^\Delta, \) et cetera.

The adhesion of a decomposition \( \Delta \) of a graph \( G \) is \( \text{ad}(\Delta) := \max \{|\sigma(t)| \mid t \in V(\Delta)\} \), and the width of \( \Delta \) is \( \text{wd}(\Delta) := \max \{|\beta(t)| \mid t \in V(\Delta)\} - 1 \). If \( V(D) = \emptyset \), which is only possible if \( V(G) = \emptyset \) as well, we let \( \text{ad}(\Delta) := 0 \) and \( \text{wd}(\Delta) := -1 \). A decomposition \( \Delta \) is over a class \( \mathcal{A} \) of graphs if \( \tau(t) \in \mathcal{A} \) for all \( t \in V(\Delta) \).

**Definition 4.2.2.** A treelike decomposition of a graph \( G \) is a decomposition \( \Delta = (D, \sigma, \alpha) \) of \( G \) that satisfies the following axioms.
4.2. Treelike Decompositions

Figure 4.2. A tree $T$ and the digraph of the treelike decomposition $\Delta(T)$ defined in Example 4.2.3

(TL.1) $D$ is acyclic.

(TL.2) For all $t \in V(D)$ it holds that $\alpha(t) \cap \sigma(t) = \emptyset$ and $N^G(\alpha(t)) \subseteq \sigma(t)$.

(TL.3) For all $t \in V(D)$ and $u \in N^D_+(t)$ it holds that $\alpha(u) \subseteq \alpha(t)$ and $\gamma(u) \subseteq \gamma(t)$.

(TL.4) For all $t \in V(D)$ and $u_1, u_2 \in N^D_+(t)$, either $u_1 \parallel u_2$ or $u_1 \perp u_2$, that is, $u_1$ and $u_2$ are either parallel or orthogonal.

(TL.5) For every connected component $A$ of $G$ there is a $t \in V(D)$ with $\sigma(t) = \emptyset$ and $\alpha(t) = V(A)$.

Example 4.2.3. For every forest $F$, we define a decomposition $\Delta(F) = (D, \sigma, \alpha)$ as follows:

\[
V(D) := \{ (v_1, v_2) \in V(F)^2 \mid v_1 = v_2 \text{ or } v_1 v_2 \in E(F) \},
\]

\[
E(D) := \{ (v_1, v_2)(v'_1, v'_2) \in V(D)^2 \mid v'_1 = v_2 \text{ and } v'_2 \neq v_1 \text{ and } v'_1 v'_2 \in E(F) \},
\]

\[
\sigma(v_1, v_2) := \begin{cases} 
\emptyset & \text{if } v_1 = v_2, \\
v_1 & \text{otherwise},
\end{cases}
\]

\[
\alpha(v_1, v_2) := V(A), \text{ where } A \text{ is the connected component of } F \setminus \sigma(v_1, v_2) \text{ that contains } v_2.
\]
Figure 4.2 illustrates the definition.

Claim 1. \( \Delta(F) \) is a treelike decomposition of \( F \).

Proof. \( \Delta(F) \) satisfies \([\text{TL.2}]\) because for every node \((v_1, v_2) \in V(D)\) the set \( \alpha(v_1, v_2) \) is the vertex set of a connected component of \( F \setminus \sigma(v_1, v_2) \).

To verify \([\text{TL.3}]\) let \((v_1, v_2)(v'_1, v'_2) \in E(D)\). Then \( v'_1, v'_2 \in \alpha(v_1, v_2) \), because there are paths from \( v_2 \) to both \( v'_1 = v_2 \) and \( v'_2 \) in \( F \setminus \{ v_1 \} \). Thus \( \sigma(v'_1, v'_2) = \{ v'_1 \} \subseteq \alpha(v_1, v_2) \). Moreover, \( \alpha(v'_1, v'_2) \subseteq \alpha(v_1, v_2) \), because every path from \( v'_2 \) to a vertex \( w \) in \( F \setminus \{ v'_1 \} \) is also a path in \( F \setminus \{ v_1 \} \). Thus \( \gamma(v'_1, v'_2) = \sigma(v'_1, v'_2) \cup \alpha(v'_1, v'_2) \subseteq \alpha(v_1, v_2) \subseteq \gamma(v_1, v_2) \).

Note that for all \((v_1, v_2)(v'_1, v'_2) \in E(D)\) it also holds that \( \sigma(v'_1, v'_2) \cap \alpha(v_1, v_2) = \emptyset \) and thus \( \alpha(v'_1, v'_2) \subset \alpha(v_1, v_2) \). This implies \([\text{TL.1}]\).

To see that \( \Delta \) satisfies \([\text{TL.4}]\) let \((v_1, v_2) \in V(D) \) and \((w_{11}, w_{12}),(w_{21}, w_{22}) \in N^D_+(v_1, v_2) \). Then \( w_{11} = w_{21} = v_2 \) and thus \( \sigma(w_{11}, w_{12}) = \sigma(w_{21}, w_{22}) = \{ v_2 \} \). Moreover, both \( \alpha(w_{11}, w_{12}) \) and \( \alpha(w_{21}, w_{22}) \) are connected components of \( F \setminus v_2 \). It follows that either \( \alpha(w_{11}, w_{12}) = \alpha(w_{21}, w_{22}) \) and thus \( (w_{11}, w_{12}) \parallel (w_{21}, w_{22}) \) or \( \alpha(w_{11}, w_{12}) \cap \alpha(w_{21}, w_{22}) = \emptyset \) and \( \alpha(w_{11}, w_{12}) \cap \alpha(w_{21}, w_{22}) = \emptyset \) and thus \( (w_{11}, w_{12}) \perp (w_{21}, w_{22}) \).

To verify \([\text{TL.5}]\) let \( A \) be a connected component of \( F \). Then for every \( v \in V(A) \) we have \((v, v) \in V(D) \) with \( \sigma(v, v) = \emptyset \) and \( \alpha(v, v) = V(A) \).

Claim 2. For all \((v_1, v_2) \in E(F)\) it holds that \( \tau(v_1, v_2) = K[\{v_1, v_2\}] \).

Proof. Let \((v_1, v_2) \in V(D) \).

Case 1: \( v_1 = v_2 \).

Let \( T \) be the connected component of \( F \) that contains \( v_1 \). If \( V(T) = \{ v_1 \} \) then \( N^D_+(v_1, v_2) = \emptyset \) and thus \( \beta(v_1, v_2) = \gamma(v_1, v_2) = \{ v_1 \} \). Otherwise, let \( T_1, \ldots, T_m \) be the connected components of \( T \setminus \{ v_1 \} \). As \( T \) is a tree, for all \( i \in [m] \) there is a unique \( w_i \in V(T_i) \) such that \( v_1 w_i \in E(F) \). Then \((v_1, w_i) \in V(D) \) and \((v_1, v_2)(v_1, w_i) \in E(D) \) and \( \alpha(v_1, w_i) = V(T_i) \). Moreover, we have \( N^D_+(v_1, v_2) = \{ (v_1, w_i) \mid i \in [m] \} \). Hence

\[
\beta(v_1, v_2) = \gamma(v_1, v_2) \setminus \bigcup_{i=1}^{m} \alpha(v_1, w_i) = V(T) \setminus \bigcup_{i=1}^{m} V(T_i) = \{ v_1 \} = \{ v_1, v_2 \}.
\]

This immediately implies that \( \tau(v_1, v_2) = (\{ v_1 \}, \emptyset) = K[\{v_1, v_2\}] \).

Case 2: \( v_1 v_2 \in E(F) \).

By a similar argument as in Case 1 we see that \( \beta(v_1, v_2) = \{ v_1, v_2 \} \). (We leave the details to the reader.) As \( v_1 v_2 \in E(F) \), this implies that \( \tau(v_1, v_2) = F[\{v_1, v_2\}] = K[\{v_1, v_2\}] \).

Claim 2 implies that the decomposition \( \Delta(F) \) is over the class \( \mathcal{K}_2 \) of all complete graphs of order at most 2. Furthermore, it implies that \( \text{wd}(\Delta(F)) \leq 1 \). Clearly, if \( E(F) \neq \emptyset \) we have \( \text{wd}(\Delta(F)) = 1 \). For the sake of completeness, let us observe that \( \text{ad}(\Delta(F)) \leq 1 \), and if \( E(F) \neq \emptyset \) then \( \text{ad}(\Delta(F)) = 1 \).

Example 4.2.4. For every cycle \( C \), we define a decomposition \( \Delta(C) = (D, \sigma, \alpha) \) as follows:

\[
V(D) := \{ (v_1, v_2, v_3) \in V(C)^3 \mid v_1 \neq v_2, v_3 \}.
\]

M. Grohe, Definable Graph Structure Theory
We leave it to the reader to verify that for every cycle we have

\( \Delta(C) \) is a treelike decomposition of \( C \). Furthermore, for all \( \overline{v} \in V(D) \) it holds that \( \beta(t) = \overline{v} \) and \( \tau(t) = K[\overline{v}] \). Hence \( \Delta(C) \) is a decomposition over the class \( \mathcal{K}_3 \) of complete graphs of order at most 3.
Example 4.2.5. A decomposition \((D, \sigma, \alpha)\) of a graph \(G\) is trivial if \(E(D) = \emptyset\) and \(\sigma(t) := \emptyset, \alpha(t) = V(G)\) for all \(t \in V(D)\). In particular, for every \(\ell \in \mathbb{N}\) the trivial \(\ell\)-dimensional decomposition of \(G\) is the trivial decomposition \((D, \sigma, \alpha)\) of \(G\) with \(V(D) := V(G)^\ell\).

Observe that every trivial decomposition of a connected graph is treelike. \(\square\)

The main motivation we gave for introducing treelike decompositions was that we can make them invariant under isomorphisms. The following definition introduces a weak notion of invariance that only refers to the decomposition of a single graph.

Definition 4.2.6. A decomposition \((D, \sigma, \alpha)\) of a graph \(G\) is automorphism-invariant if for every automorphism \(f\) of \(G\) there is an automorphism \(g\) of \(D\) such that for all \(t \in V(D)\) we have \(\sigma(g(t)) = f(\sigma(t))\) and \(\alpha(g(t)) = f(\alpha(t))\).

In Chapter 5 we will discuss a more general notion of invariance. All definable decompositions (in the sense introduced in Chapter 5) will trivially be invariant. As we are mainly concerned with definable decompositions, except in the following examples we will rarely mention the automorphism-invariance of our decompositions.

Example 4.2.7. The treelike decompositions of forests and cycles constructed in Examples 4.2.3 and 4.2.4 are automorphism-invariant.

Example 4.2.8. The treelike decomposition of the cycle \(C_5\) shown in Figure 1.5 in the introduction is automorphism-invariant. This decomposition exemplifies a general construction of automorphism-invariant decompositions. Let \(G\) be a connected graph, and let \((T, \beta)\) be a tree decomposition of \(G\). Define \(\sigma\) and \(\alpha\) as in (4.1.2) and (4.1.3). Then \((T, \sigma, \alpha)\) is a treelike decomposition of \(G\) (by Lemma 4.6.1 proved later in this chapter). For every automorphism \(f\) of \(G\), let \(T^f\) be a fresh copy of the tree \(T\) (so that for \(f \neq g\) the trees \(T^f\) and \(T^g\) are disjoint), and let \(t \mapsto t^f\) be an isomorphism from \(T\) to \(T^f\). Define \(\sigma^f, \alpha^f: V(T^f) \to 2^{V(G)}\) by \(\sigma^f(t^f) := f(\sigma(t))\) and \(\alpha^f(t^f) := f(\alpha(t))\). Now let \(D^f\) be the union of all the trees \(T^f\), so that \(V(D^f) = \{t^f \mid t \in V(T), f \in \text{Aut}(G)\}\). Let \(\sigma^f, \alpha^f\) be the unions of the \(\sigma^f, \alpha^f\), respectively. It is easy to verify that \((D^f', \sigma^f', \alpha^f')\) is a treelike decomposition of \(G\). Moreover, the decomposition is automorphism-invariant. To see this, let \(h\) be an automorphism of \(G\). Let \(g: V(D^f) \to V(D')\) be defined by \(g(t^f) := t^f h\), for all \(t \in V(T)\) and \(f \in \text{Aut}(f)\). Here we write \(f h\) instead of \(h \circ f\) for the concatenation of \(f\) and \(h\). Then \(g\) is an automorphism of \(D^f\). Moreover, for all \(t \in V(T)\) and \(f \in \text{Aut}(f)\) we have \(\sigma'\)(\(g(t^f)\)) = \(\sigma'(t^f h)\).

Hence \((D', \sigma', \alpha')\) is an automorphism-invariant treelike decomposition of \(G\). The problem with this decomposition is that it can be exponentially large in the graph. In a second step, we can try to shrink the decomposition by identifying nodes in a bottom-up fashion. We repeatedly carry out the following operation: whenever we have nodes with the same separator and the same component and the same children, we identify the two nodes. The resulting decomposition \((D'', \sigma'', \alpha'')\) is still an automorphism-invariant treelike decomposition of \(G\), and it may be substantially smaller than \((D', \sigma', \alpha')\). Note that the treelike decomposition of \(C_5\) shown in Figure 1.5 is obtained by this construction from the tree decomposition of \(C_5\) shown in Figure 1.4.

It is easy to see that the decompositions \((T, \sigma, \alpha)\) and \((D'', \sigma'', \alpha'')\) have the same torsos. More precisely, for every node \(t \in V(T)\) there is a node \(t'' \in V(D'')\) such that \(\tau(t) = \tau''(t'')\) and for every node \(t'' \in V(D'')\) there is a node \(t \in V(T)\) such that \(\tau(t) \cong \tau''(t'')\). Hence for every class \(A\) of graphs, the decomposition \((T, \sigma, \alpha)\) is over \(A\) if and only if \((D'', \sigma'', \alpha'')\) is over \(A\). \(\square\)
The following lemma, which is probably the most frequently used lemma in this book, collects a few basic properties of treelike decompositions.

**Lemma 4.2.9 (β-γ-σ-Lemma).** Let $\Delta = (D, \sigma, \alpha)$ be a decomposition of a graph $G$ that satisfies \([TL.1]\)\([TL.4]\) and let $t, u \in V(\Delta)$. Then:

1. $\gamma(t) = \bigcup_{u \in V(D)} \beta(u)$.
2. $\sigma(t) = \beta(t) \setminus \alpha(t)$ and thus, in particular, $\sigma(t) \subseteq \beta(t)$.
3. If $tu \in E(D)$ then $\beta(t) \cap \beta(u) = \beta(t) \cap \gamma(u) = \sigma(u)$.
4. If $t \prec^D u$ then $\gamma(u) \subseteq \gamma(t)$ and $\beta(t) \cap \gamma(u) \subseteq \sigma(u)$.

**Proof.** Assertion (1) is proved by induction on $t \in V(D)$. If $N^D_+(t) = \emptyset$, then $\gamma(t) = \beta(t)$. Otherwise,

$$\gamma(t) \subseteq \beta(t) \cup \bigcup_{u \in N^D_+(t)} \alpha(u) \subseteq \beta(t) \cup \bigcup_{u \in N^D_+(t)} \gamma(u) \subseteq \gamma(t)$$

where the last inclusion holds by \([TL.3]\). Hence

$$\gamma(t) = \beta(t) \cup \bigcup_{u \in N^D_+(t)} \gamma(u) = \beta(t) \cup \bigcup_{u \in V(D) \text{ with } t \prec^D u} \beta(u)$$

by the induction hypothesis.

To prove (2), we first prove $\sigma(t) \subseteq \beta(t)$. Note that $\sigma(t) \cap \alpha(t) = \emptyset$ by \([TL.2]\) and therefore $\sigma(t) \cap \alpha(u) = \emptyset$ for all $u \in N^D_+(t)$ by \([TL.3]\). Hence $\sigma(t) \subseteq \beta(t)$ follows from the definition of $\beta(t)$ (see (4.2.2)).

Now $\sigma(t) = \beta(t) \setminus \alpha(t)$ follows from $\sigma(t) = \gamma(t) \setminus \alpha(t)$ and $\sigma(t) \subseteq \beta(t) \subseteq \gamma(t)$.

To prove (3), let $tu \in E(D)$. Observe that

$$\beta(t) \cap \beta(u) \subseteq \beta(t) \cap \gamma(u) \subseteq \sigma(u),$$

because $\beta(t) \cap \alpha(u) = \emptyset$ by the definition of $\beta$. For the converse inclusion $\sigma(u) \subseteq \beta(t) \cap \beta(u)$, we only have to prove $\sigma(u) \subseteq \beta(t)$, because $\sigma(u) \subseteq \beta(u)$ by (2). Let $v \in \sigma(u)$. Then $v \in \gamma(u) \subseteq \gamma(t)$ by \([TL.3]\). Moreover, $v \not\in \alpha(u)$ by \([TL.2]\) and $v \not\in \alpha(u')$ for any other $u' \in N^D_+(t)$ by \([TL.4]\). Hence $v \in \beta(t)$.

To prove (4), let $t \prec^D u$, and let $t = u_0, u_1, \ldots, u_m = u$ be a path from $t$ to $u$ in $D$. We prove $\gamma(u_i) \subseteq \gamma(t)$ and $\beta(t) \cap \gamma(u_i) \subseteq \sigma(u_i)$ by induction on $i \in [m]$. For $i = 1$, we have $\gamma(u_1) \subseteq \gamma(u_0) = \gamma(t)$ by \([TL.3]\) and $\beta(t) \cap \gamma(u_1) = \sigma(u_1)$ by (3). For $i \geq 2$, we have $\gamma(u_i) \subseteq \gamma(u_{i-1}) \subseteq \gamma(t)$, where the first inclusion holds by \([TL.3]\) and the second by the induction hypothesis. Furthermore, we have $\beta(t) \cap \gamma(u_i) \subseteq \sigma(u_{i-1}) \cap \gamma(u_i)$, because $\beta(t) \cap \gamma(u_{i-1}) \subseteq \sigma(u_{i-1})$ by the induction hypothesis and $\gamma(u_i) \subseteq \gamma(u_{i-1})$ by \([TL.3]\). Hence

$$\beta(t) \cap \gamma(u_i) \subseteq \sigma(u_{i-1}) \cap \gamma(u_i) \subseteq \beta(u_{i-1}) \cap \gamma(u_i) = \sigma(u_i) \quad \text{(by (2))}$$

$$\gamma(u_i) \subseteq \gamma(u_{i-1}) \quad \text{(by (3))}. \quad \Box$$
We close this section with another simple lemma, which will often be useful when we verify that certain decompositions are treelike. For example, the lemma can be used to simplify the proof that the decomposition of a forest in Example 4.2.3 is treelike.

**Lemma 4.2.10.** Let \( \Delta = (D, \sigma, \alpha) \) be a decomposition of a graph \( G \). Let \( t \in V(D) \) and \( u, u' \in N_D^\perp(t) \). Suppose that \( \sigma(u) = N^G(\alpha(u)) \) and \( \sigma(u') = N^G(\alpha(u')) \) and there is a set \( X \subseteq V(G) \) such that both \( \alpha(u) \) and \( \alpha(u') \) are vertex sets of connected components of \( G \setminus X \). Then either \( u \parallel u' \) or \( u \perp u' \).

**Proof.** We leave the straightforward proof to the reader. \( \square \)

### 4.3 Normalising Treelike Decompositions

In this subsection, we shall establish various “normal forms” for treelike decompositions.

**Definition 4.3.3.** Let \( \Delta = (D, \sigma, \alpha) \) be a treelike decomposition of a graph \( G \).

1. \( \Delta \) is normal if in addition to (TL.1) (TL.5) it satisfies the following two axioms.
   - (TL.6) For every \( \preceq \)-minimal \( t \in V(D) \) it holds that \( \sigma(t) = \emptyset \) and that \( \alpha(t) \) is the vertex set of a connected component of \( G \).
   - (TL.7) For every \( t \in V(D) \) it holds that \( \gamma(t) \neq \emptyset \).

2. \( \Delta \) is strict if it satisfies the following strengthening of (TL.3)

   - (TL.3s) For all \( t \in V(D) \) and \( u \in N_D^\perp(t) \) it holds that \( \alpha(u) \subseteq \alpha(t) \) and \( \gamma(u) \subseteq \gamma(t) \), and one of the two inclusions is strict.

Observe that (TL.3s) implies (TL.1) and (TL.3). Hence to prove that a decomposition is a strict treelike decomposition it suffices to verify (TL.2), (TL.3s), (TL.4), and (TL.5).

**Example 4.3.2.** The treelike decompositions of forests and cycles defined in Examples 4.2.3 and 4.2.4 are strict and normal.

**Definition 4.3.3.** Let \( \Delta, \Delta' \) be two decompositions of the same graph \( G \). Two nodes \( t \in V(\Delta), t' \in V(\Delta') \) are \( \Delta, \Delta' \)-indistinguishable (we write \( t \approx_{\Delta, \Delta'} t' \)) if \( \alpha^\Delta(t) = \alpha^\Delta'(t) \), \( \beta^\Delta(t) = \beta^\Delta'(t) \), \( \gamma^\Delta(t) = \gamma^\Delta'(t) \), \( \delta^\Delta(t) = \delta^\Delta'(t) \), and \( \tau^\Delta(t) = \tau^\Delta'(t) \).

If the decompositions \( \Delta \) and \( \Delta' \) are clear from the context, we just speak of “indistinguishable” and “parallel” nodes and omit the index \( \Delta, \Delta' \). Observe that \( t \approx t' \) implies \( t \parallel t' \), but not vice versa.

**Lemma 4.3.4.** Let \( \Delta, \Delta' \) be two treelike decompositions of the same graph \( G \) and \( t \in V(\Delta), t' \in V(\Delta') \).

1. If \( \beta^\Delta(t) = \beta^\Delta'(t) \) and \( \delta^\Delta(t) = \delta^\Delta'(t) \) then \( \tau^\Delta(t) = \tau^\Delta'(t) \).
2. If \( \beta^\Delta(t) = \beta^\Delta'(t) \) and \( \delta^\Delta(t) = \delta^\Delta'(t) \) and \( \alpha^\Delta(t) = \alpha^\Delta'(t) \), then \( t \approx t' \).

**Proof.** Assertion (1) is trivial. To prove (2), note that by the \( \beta, \gamma, \sigma \)-Lemma 4.2.9(2) \( \beta^\Delta(t) = \beta^\Delta'(t) \) and \( \alpha^\Delta(t) = \alpha^\Delta'(t) \) imply

\[
\sigma^\Delta(t) = \beta^\Delta(t) \setminus \alpha^\Delta(t) = \beta^\Delta'(t) \setminus \alpha^\Delta'(t) = \sigma^\Delta'(t).
\]

M. Grohe, *Definable Graph Structure Theory*
Hence we also have

\[ \gamma^\Delta(t) = \sigma^\Delta(t) \cup \alpha^\Delta(t) = \sigma'^\Delta(t) \cup \alpha'^\Delta(t) = \gamma'^\Delta(t). \]

**Lemma 4.3.5.** Let \( \Delta = (D, \sigma, \alpha) \) be a treelike decomposition of a graph \( G \). Then there is a normal treelike decomposition \( \Delta' = (D', \sigma', \alpha') \) of \( G \) such that the following two conditions are satisfied.

(i) \( D' \) is an induced subgraph of \( D \).

(ii) For all \( t \in V(D') \) it holds that \( t \approx^\Delta'^\Delta t \).

**Proof.** We recursively define a set of bad nodes of \( D \) as follows.

- If \( \alpha(t) \) is not connected and all parents of \( t \) in \( D \) are bad, then \( t \) is bad.
- If \( \sigma(t) \neq \emptyset \) and all parents of \( t \) in \( D \) are bad, then \( t \) is bad.

A node \( t \in V(D) \) is good if it is not bad and if \( \gamma(t) \neq \emptyset \).

We define the decomposition \( \Delta' = (D', \sigma', \alpha') \) by letting \( D' \) be the induced subgraph of \( D \) whose vertex set \( V(D') \) consists of all good nodes and \( \sigma'(t) := \sigma(t) \), \( \alpha'(t) := \alpha(t) \) for all \( t \in V(D') \).

It follows immediately from the definition of the cone that \( \gamma'(t) = \gamma(t) \) for all \( t \in V(D') \), but there is no a priori reason that the same should hold for \( \beta, \delta, \) and \( \tau \).

**Claim 1.** \( \Delta' \) is a normal treelike decomposition of \( G \)

**Proof.** (TL.1) (TL.4) for \( \Delta' \) follow immediately from the respective properties of \( \Delta \). To see that (TL.5) holds, let \( A \) be a connected component of \( G \). Let \( t \in V(D) \) such that \( \sigma(t) = \emptyset \) and \( \alpha(t) = V(A) \). Then \( \alpha(t) \) is connected and \( \sigma(t) = \emptyset \). Hence \( t \) is not bad. Furthermore, \( \gamma(t) = V(A) \neq \emptyset \). Hence \( t \) is good and therefore \( t \in V(D') \).

To prove (TL.6) let \( t \in V(D') \) be \( \preceq^{D'} \)-minimal. Suppose for contradiction that \( \sigma(t) \neq \emptyset \). Then there is an \( s \in V(D) \) such that \( t \in N^D_+(s) \) and \( s \) is not bad, because otherwise \( t \) would be bad and hence not contained in \( V(D') \). Actually, \( s \) is good, because \( \gamma(s) \supseteq \sigma(t) \neq \emptyset \). Thus \( s \in V(D') \), which contradicts the \( \preceq^{D'} \)-minimality of \( t \). Hence \( \sigma(t) = \emptyset \). If \( \alpha(t) = \emptyset \), then \( \gamma(t) = \emptyset \), which contradicts \( t \) being good. Hence \( \alpha(t) \neq \emptyset \). Suppose for contradiction that \( \alpha(t) \) is not connected. Then there is an \( s \in V(D) \) such that \( t \in N^D_+(s) \) and \( s \) is not bad, because otherwise \( t \) would be bad. Again, \( s \) is good, because \( \gamma(s) \supseteq \gamma(t) \supseteq \alpha(t) \neq \emptyset \). Thus \( s \in V(D') \), which contradicts the \( \preceq^{D'} \)-minimality of \( t \). Thus \( \alpha(t) \) is connected. By (TL.2) and \( \sigma(t) = \emptyset \), it follows that \( \alpha(t) \) is the vertex set of a connected component of \( G \).

Finally, (TL.7) holds trivially, because we defined good nodes to have nonempty cones.

It follows from the “top-down” definition of the bad nodes that for all \( t \in V(D') \) it holds that

\[ N^D_+(t) = N^D_+(t) \cap \{ u \in V(D) \mid \gamma(u) \neq \emptyset \}. \]  \hspace{1cm} (4.3.1)

**Claim 2.** For all \( t \in V(D') \) it holds that \( \beta'(t) = \beta(t) \) and \( \delta'(t) = \delta(t) \).

**Proof.** Let \( t \in V(D') \). We have

\[ \beta'(t) = \gamma'(t) \setminus \bigcup_{u \in N^D_+(t)} \alpha'(u) \]
\[= \gamma(t) \setminus \bigcup_{u \in N_+^D(t) \text{ with } \gamma(u) \neq \emptyset} \alpha(u) \quad \text{(by (4.3.1))}\]
\[= \gamma(t) \setminus \bigcup_{u \in N_+^D(t)} \alpha(u) \quad \text{(because } \alpha(u) \subseteq \gamma(u) \text{ for all } u \in V(D))\]
\[= \beta(t).\]

All separators of \(\Delta'\) at \(t\) are also separators of \(\Delta\) at \(t\), because \(N_+^{D'}(t) \subseteq N_+^D(t)\). Conversely, by (4.3.1) all nonempty separators of \(\Delta\) at \(t\) are also separators of \(\Delta'\) at \(t\). This implies that \(\Delta'\) and \(\Delta\) have the same nonempty maximal separators at \(t\). Thus \(\delta'(t) = \delta(t).\) 

For all \(t \in V(D')\) we have \(\alpha'(t) = \alpha(t)\). Then by Claim 2 and Lemma 4.3.6(2) it follows that \(t \approx_{\Delta',\Delta} t\).

**Lemma 4.3.6.** Let \(\Delta = (D,\sigma,\alpha)\) be a treelike decomposition of a graph \(G\). Then there is a strict treelike decomposition \(\Delta' = (D',\sigma',\alpha')\) of \(G\) that satisfies the following conditions.

(i) \(V(D') \subseteq V(D)\).

(ii) For all \(t,u \in V(D')\) it holds that \(t \preceq_{D'} u \iff t \preceq_D u\).

(iii) For all \(t \in V(D')\) it holds that \(t \approx_{\Delta',\Delta} t\).

**Proof.** Let \(\Delta = (D,\sigma,\alpha)\). We call a node \(t \in V(D)\) bad if there is a \(u \in N_+^D(t)\) such that \(\gamma(u) = \gamma(t)\) and \(\alpha(u) = \alpha(t)\), and we call \(t\) good otherwise.

**Claim 1.** Let \(t \in V(D)\) be bad. Then for all \(u \in N_+^D(t)\), either \(u \parallel t\) or \(\sigma(u) \subseteq \sigma(t)\) and \(\alpha(u) = \emptyset\).

**Proof.** Let \(u_0 \in N_+^D(t)\) such that \(\gamma(u_0) = \gamma(t)\) and \(\alpha(u_0) = \alpha(t)\). Observe that
\[\sigma(t) = \gamma(t) \setminus \alpha(t) = \gamma(u_0) \setminus \alpha(u_0) = \sigma(u_0).\]
Thus \(u_0 \parallel t\). Now let \(u \in N_+^D(t)\) be arbitrary. Then by (TL.4) either \(u \parallel u_0 \parallel t\), or \(\gamma(u) \cap \alpha(u_0) = \emptyset\). If \(\gamma(u) \cap \alpha(u_0) = \emptyset\), then \(\alpha(u) \cap \alpha(u_0) = \alpha(u) \cap \alpha(t) = \emptyset\), and as \(\alpha(u) \subseteq \alpha(t)\) by (TL.3) this implies \(\alpha(u) = \emptyset\). Furthermore, \(\sigma(u) = \gamma(u) \subseteq \gamma(t) \setminus \alpha(u_0) = \gamma(t) \setminus \alpha(t) = \sigma(t)\).

We now define a sequence of directed graphs \(D_i\) for \(i \in \mathbb{N}^+\) as follows:

- We let \(D^1 := D\).

- If there are no bad nodes in \(D^i\), we let \(D^{i+1} := D^i\). Here a node \(t \in V(D^i)\) is bad in \(D^i\) if there is a \(u \in N_+^{D^i}(t)\) such that \(\gamma(u) = \gamma(t)\) and \(\alpha(u) = \alpha(t)\).

- Otherwise, to define \(D^{i+1}\) from \(D^i\) we delete all \(\preceq_{D^i}\)-minimal bad nodes \(t \in V(D^i)\), and for each bad node \(t\) that we delete, we add edges from all \(D^i\)-parents \(s\) of \(t\) to all \(D^i\)-children \(u\) of \(t\).

Observe that for all \(i \in \mathbb{N}^+\) and all \(t,u \in V(D^{i+1})\) it holds that
\[t \preceq_{D^i} u \iff t \preceq_{D^{i+1}} u.\]

**Claim 2.** Let \(i \in \mathbb{N}^+\).
1. The decomposition $\Delta^i := (D^i, \sigma|_{V(D^i)}, \alpha|_{V(D^i)})$ is treelike.

2. If $t \in V(D^i)$ is a bad node of $D^i$, then for all $u \in N^D_+(t)$ either $u \parallel t$ or $\sigma(u) \subseteq \sigma(t)$ and $\alpha(u) = \emptyset$.

Furthermore,

**Proof.** We prove the claim by induction on $i$. For the base step $i = 1$, (1) holds because $\Delta^1 = \Delta$ is treelike, and (2) holds by Claim 1.

For the inductive step $i \mapsto i + 1$, observe that

- (TL.1) for $\Delta^{i+1}$ follows from (TL.1) for $\Delta^i$ and (4.3.2);
- (TL.2) for $\Delta^{i+1}$ follows from (TL.2) for $\Delta^i$;
- (TL.3) for $\Delta^{i+1}$ follows from (TL.3) for $\Delta^i$ and (4.3.2).

To prove (TL.4) for $\Delta^{i+1}$, let $t \in V(D^{i+1})$ and $u_1, u_2 \in N^D_{+}^{i+1}(t)$. For $j = 1, 2$ we define a node $x_j$ as follows: If $u_j \in N^D_{+}^{i}(t)$ we let $x_j := u_j$. Otherwise, we let $x_j \in N^D_+(t)$ such that $u_j \in N^D_+(x_j)$. In both cases we have either $u_j \parallel x_j$ or $\sigma(u_j) \subseteq \sigma(x_j)$ and $\alpha(u_j) = \emptyset$. This is trivial if $x_j = u_j$ and follows from the induction hypothesis (2) otherwise.

**Case 1:** $u_1 \parallel x_1$ and $u_2 \parallel x_2$.

Then $u_1 \parallel u_2$ or $u_1 \perp u_2$ by (TL.4) for $\Delta^i$.

**Case 2:** $u_1 \parallel x_1$ and $\sigma(u_2) \subseteq \sigma(x_2)$ and $\alpha(u_2) = \emptyset$.

Then $\gamma(u_1) \cap \alpha(u_2) = \emptyset$. Furthermore, $\alpha(u_1) \cap \gamma(u_2) = \alpha(u_1) \cap \sigma(u_2) \subseteq \alpha(x_1) \cap \sigma(x_2) = \emptyset$ by (TL.4) for $\Delta^i$. Hence $u_1 \perp u_2$.

**Case 3:** $\sigma(u_1) \subseteq \sigma(x_1)$ and $\alpha(u_1) = \emptyset$ and $u_2 \parallel x_2$.

Symmetric to Case 2.

**Case 4:** $\sigma(u_1) \subseteq \sigma(x_1)$ and $\alpha(u_1) = \emptyset$ and $\sigma(u_2) \subseteq \sigma(x_2)$ and $\alpha(u_2) = \emptyset$.

Then $\gamma(u_j) = \sigma(u_j)$ for $j = 1, 2$ and thus $u_1 \perp u_2$.

To prove (TL.5) for $\Delta^{i+1}$, let $A$ be a connected component of $G$ and $t \in V(D^i)$ such that $\alpha(t) = V(A)$ and $\sigma(t) = \emptyset$. Such a node $t$ exists by (TL.5) for $\Delta^i$. If $t \in V(D^{i+1})$, then there is nothing to prove. Otherwise, there is a $u \in N^D_+(t)$ such that $\alpha(u) = \alpha(t) = V(A)$ and $\gamma(u) = \gamma(t)$ and thus $\sigma(u) = \sigma(t) = \emptyset$. As we only delete minimal bad nodes from $D^i$ to define $D^{i+1}$, we have $u \in V(D^{i+1})$.

To prove (2), let $t \in V(D^{i+1})$ be a bad node in $D^{i+1}$, and let $u \in N^D_{+}^{i+1}(t)$ such that $\gamma(u) = \gamma(t)$ and $\alpha(u) = \alpha(t)$. We first prove that $u \in N^D_+(t)$. Suppose not. Then there is a $x \in N^D_+(t)$ with $u \in N^D_+(x)$ such that $x$ is a minimal bad node in $D^i$. As $\Delta^i$ is treelike, we have $\gamma(t) = \gamma(u) \subseteq \gamma(x) \subseteq \gamma(t)$ and hence $\gamma(x) = \gamma(t)$, and similarly $\alpha(x) = \alpha(t)$. Hence $t$ is bad in $D^i$, which contradicts the minimality of $x$. Thus this proves that $u \in N^D_+(t)$. Thus $t$ is bad in $D^i$. But this implies $N^D_{+}^{i+1}(t) = N^D_+(t)$, because no child of $t$ is a minimal bad node in $D^i$. Now assertion (2) for $t$ in $D^{i+1}$ follows from the induction hypothesis.

As the graph shrinks in each step, the sequence $(D^i)_{i\geq 0}$ reaches a fixed-point. Let $n \in \mathbb{N}^+$ such that $D^i = D^n$ for all $i \geq n$. Then $D^n$ has no bad nodes. Thus $\Delta^n$ is a strict treelike decomposition. It remains to prove that $t \approx t$ for all $t \in V(D^i)$. 

**Preliminary Version**
For all $i \in \mathbb{N}^+$ and $t \in V(D^i)$, we have $\sigma^{\Delta^i}(t) = \sigma(t)$ and $\alpha^{\Delta^i}(t) = \alpha(t)$ by definition. This implies $\gamma^{\Delta^i}(t) = \gamma(t)$.

Thus by Lemma 4.3.4, the following claim implies that $t \approx^{\Delta^{i+1}} t$.

**Claim 3.** For all $i \in \mathbb{N}^+$ and all $t \in V(D^i)$ we have $\beta^{\Delta^i}(t) = \beta(t)$ and $\delta^{\Delta^i}(t) = \delta(t)$.

**Proof.** The proof is by induction on $i$. The base step $i = 1$ is trivial. For the inductive step $i \mapsto i + 1$, let $t \in V(D^{i+1})$. Let $N^+_+ := N^+_+(t) \cap V(D^{i+1})$ and $N^-_- := N^-_+(t) \setminus V(D^{i+1})$. By the definition of $D^{i+1}$ we have

$$N^+_{i+1}(t) = N^+_+(t) \cup \bigcup_{u \in N^-_-} N^+_+(u).$$

Let $u \in N^-_-$. Then $u$ is bad in $D^i$, and hence there is at least one $x \in N^+_+(u)$ such that $\gamma(x) = \gamma(u)$ and $\alpha(x) = \alpha(u)$. By Claim 2(2), for all $x \in N^+_+(u)$ either $\alpha(x) = \alpha(u)$ or $\alpha(x) = \emptyset$. Hence

$$\bigcup_{x \in N^+_+(u)} \alpha(x) = \alpha(u)$$

and

$$\beta^{\Delta^i}(t) = \gamma(t) \setminus \bigcup_{u \in N^+_+(t)} \alpha(u)$$

$$= \gamma(t) \setminus \left( \bigcup_{u \in N^+_+} \alpha(u) \cup \bigcup_{u \in N^-_+} \bigcup_{x \in N^+_+(u)} \alpha(x) \right)$$

$$= \gamma(t) \setminus \bigcup_{u \in N^+_{i+1}(t)} \alpha(u)$$

$$= \beta^{\Delta^{i+1}}(t).$$

It follows from the induction hypothesis that $\beta^{\Delta^{i+1}}(t) = \beta^{\Delta^i}(t) = \beta(t)$.

It remains to prove that $\delta^{\Delta^{i+1}}(t) = \delta^{\Delta^i}(t)$, because then $\delta^{\Delta^{i+1}}(t) = \delta(t)$ by the induction hypothesis. It suffices to prove that for every nonempty separator $S'$ of $\Delta^{i+1}$ at $t$ there is a separator $S$ of $\Delta^i$ at $t$ such that $S' \subseteq S$ and conversely, for every nonempty separator $S$ of $\Delta^i$ at $t$ there is a separator $S'$ of $\Delta^{i+1}$ at $t$ such that $S \subseteq S'$.

So let $S'$ be a separator of $\Delta^{i+1}$ at $t$. If $S' = \sigma(t)$ or $S' = \sigma(u)$ for some $u \in N^+_+$, then $S'$ is also a separator of $\Delta^i$ at $t$. Assume that $S' = \sigma(x)$ for some $x \in N^+_{i+1}(t) \setminus N^+_+(t)$. Then there is an $u \in N^-_-$ such that $x \in N^+_+(u)$. By Claim 2(2) it holds that $S' = \sigma(x) \subseteq \sigma(u) =: S$, and $S$ is a separator of $\Delta^i$ at $t$.

Conversely, let $S$ be a separator of $\Delta^i$ at $t$. If $S = \sigma(t)$ or $S = \sigma(u)$ for some $u \in N^+_+$, then $S$ is also a separator of $\Delta^{i+1}$ at $t$. Assume that $S = \sigma(u)$ for some $u \in N^-_-$. Then $u$ is bad in $D^i$, and hence there is an $x \in N^+_+(u) \subseteq N^+_{i+1}(t)$ such that $S' := \sigma(x) = \sigma(u)$. The set $S'$ is a separator of $\Delta^{i+1}$ at $t$. $\square$

**Lemma 4.3.7 (Normalisation Lemma for Treelike Decompositions).** Let $\Delta = (D, \sigma, \alpha)$ be a treelike decomposition of a graph $G$. Then there is a strict and normal treelike decomposition $\Delta' = (D', \sigma', \alpha')$ of $G$ that satisfies the following conditions:

M. Grohe, Definable Graph Structure Theory
(i) \( V(D') \subseteq V(D) \).

(ii) For all \( t, u \in V(D') \) it holds that \( t \leq D' u \iff t \leq D u \).

(iii) For all \( t \in V(D') \) it holds that \( t \approx_{\Delta'} \Delta t \).

Proof. This follows from Lemma 4.3.5 and Lemma 4.3.6, because if we apply Lemma 4.3.5 to a strict treelike decomposition to turn it into a normal decomposition, the decomposition remains strict.

Lemma 4.3.8. Let \( \Delta = (D, \sigma, \alpha) \) be a strict and normal treelike decomposition of a graph \( G \). Then for all \( t \in V(D) \) it holds that 

\[
\text{t is } \leq D\text{-minimal } \iff \sigma(t) = \emptyset.
\]

Proof. The forward direction follows trivially from (TL.6). For the backward direction, let \( t \in V(D) \) such that \( \sigma(t) = \emptyset \). Let \( s \leq D t \) be \( \leq D \text{-minimal} \). Then by (TL.6) \( \sigma(s) = \emptyset \), and \( \alpha(s) = \gamma(s) \) is the vertex set of a connected component of \( G \). By (TL.3) \( \gamma(t) \subseteq \gamma(s) \). By (TL.2) \( \alpha(t) = \gamma(t) \) is a union of vertex sets of connected components of \( G \), and as \( \gamma(t) \neq \emptyset \) by (TL.7) \( \gamma(s) \supseteq \gamma(t) \) is the vertex set of a connected component, we have \( \gamma(t) = \gamma(s) \). Hence by (TL.3) we have \( s = t \).

Lemma 4.3.9. Let \( \Delta = (D, \sigma, \alpha) \) be a normal treelike decomposition of a graph \( G \). Then for every \( t \in V(D) \) the torso \( \tau(t) \) is connected.

Proof. Let \( t \in V(D) \), and let \( s \) be \( \leq D\text{-minimal} \) such that \( s \leq D t \). Then by (TL.6) \( \alpha(s) = \gamma(s) \), and \( A := G[\gamma(s)] \) is a connected component of \( G \). By (TL.3) we have \( \beta(t) \subseteq \gamma(t) \subseteq V(A) \).

Let \( v, w \in \beta(t) = V(\tau(t)) \). Let \( P \) be a path from \( v \) to \( w \) in \( A \). Then \( V(P) \cap \beta(t) \) induces a connected subgraph of \( \tau(t) \), because if \( P \) leaves \( \beta(t) \), say, in a vertex \( v' \), and later re-enters it, say, in a vertex \( w' \), then either \( v', w' \in \sigma(t) \) or \( v', w' \in \sigma(u) \) for some \( u \in N_+(t) \). This follows from (TL.2). But then there is an edge between \( v' \) and \( w' \) in \( \tau(t) \), because \( \sigma(t) \) and \( \sigma(u) \) for \( u \in N_+(t) \) are cliques in \( \tau(t) \). Hence there is a path from \( v \) to \( w \) in \( \tau(t) \).

4.4 Tight Decompositions

In an ideal treelike decompositions, the “components” \( \alpha(t) \) of the nodes \( t \) should be connected, and the “separators” \( \sigma(t) \) should be minimal separators that separate their component from the rest of the graph. We call decompositions satisfying these two conditions tight. We will see that we can easily transform every treelike decomposition into a tight treelike decomposition, but not into one that is “indistinguishable” from the original decomposition in the strong sense of the Normalisation Lemma 4.3.7.

Definition 4.4.1. A decomposition \( \Delta = (D, \sigma, \alpha) \) of a graph \( G \) is tight if for all \( t \in V(D) \), the set \( \alpha(t) \) is connected in \( G \) and \( \sigma(t) = N(\alpha(t)) = \partial(\gamma(t)) \).

Observe that in a tight treelike decomposition \( \Delta \), it is indeed the case that for every node \( t \in V(\Delta) \) the set \( \sigma(t) \) is a minimal \( (\alpha(t), V(G) \setminus \gamma(t)) \)-separator.

Lemma 4.4.2. Let \( \Delta = (D, \sigma, \alpha) \) be a treelike decomposition of a graph \( G \). Then there exists a tight treelike decomposition \( \Delta' = (D', \sigma', \alpha') \) of \( G \) and a homomorphism \( h \) from \( D' \) to \( D \) such that for every node \( t' \in V(D') \) with \( t := h(t') \) it holds that \( \beta'(t') \subseteq \beta(t) \), \( \gamma'(t') \subseteq \gamma(t) \), \( \sigma'(t') \subseteq \sigma(t) \), and \( \tau'(t') \subseteq \tau(t) \).
Proof. We first define a decomposition $\Delta'' = (D'', \sigma'', \alpha'')$ as follows:

- $V(D'') := \{(t, a) \mid t \in V(D), a \in \alpha(t)\}$.
- For all $(t, a) \in V(D'')$ we let $\alpha''(t, a)$ be the vertex set of the connected component of $G \setminus \sigma(t)$ that contains $a$, and we let $\sigma''(t, a) := N(\alpha''(t, a))$.
- $E(D'') := \{(t_1, a_1)(t_2, a_2) \mid t_1t_2 \in E(D) \text{ and } \alpha''(t_1, a_1) \supseteq \alpha''(t_2, a_2)\}$.

Observe that for every $(t, a) \in V(D'')$, the set $\alpha''(t, a)$ is the vertex set of a connected component of $G[\alpha(t)]$, because $a \in \alpha(t)$ and by (TL.2) $G[\alpha(t)]$ is a union of connected components of $G \setminus \sigma(t)$. In particular, this implies that $\alpha''(t, a) \subseteq \alpha(t)$. Furthermore, as $\alpha''(t, a)$ is the vertex set of a connected component of $G \setminus \sigma(t)$ and $\sigma''(t, a) = N^G(\alpha''(t, a))$, we have $\sigma''(t, a) \subseteq \sigma(t)$.

It follows immediately from the definitions that $\alpha''(t'')$ is connected and $\sigma''(t'') = N^G(\alpha''(t''))$ for all $t'' \in V(D'')$.

Claim 1. $\Delta''$ is a treelike decomposition of $G$.

Proof. $\Delta''$ satisfies [TL.1] because the mapping $h : V(D'') \to V(D)$ defined by $h(t, a) := t$ is a homomorphism from $D''$ to $D$. It follows from the definition of $\sigma''$ that $\Delta''$ satisfies [TL.2] if it follows from the definitions of $E(D'')$ and $\sigma''$ that $\Delta''$ satisfies [TL.3].

To prove [TL.4] let $(t, a) \in V(D'')$ and $(u_1, b_1), (u_2, b_2) \in N^G_+(t, a)$. Then $u_1, u_2 \in N^G_+(t)$. If $u_1 \not\Delta u_2$, then both $\alpha''(u_1, b_1)$ and $\alpha''(u_2, b_2)$ are vertex sets of connected components of $G \setminus \sigma(u_1)$ and thus either equal or disjoint. If they are equal, then $\sigma''(u_1, b_1) = N^G(\alpha''(u_1, b_1)) = N^G(\alpha''(u_2, b_2)) = \sigma''(u_2, b_2)$ and thus $(u_1, b_1) \not\Delta'' (u_2, b_2)$. If they are disjoint, then $(u_1, b_1) \not\Delta'' (u_2, b_2)$. If $u_1 \not\Delta u_2$, then $(u_1, b_1) \not\Delta'' (u_2, b_2)$, because for $i = 1, 2$ we have $\alpha''(u_i, b_i) \subseteq \alpha(u_i)$ and $\sigma''(u_i, b_i) \subseteq \sigma(u_i)$.

Finally, to prove [TL.5] let $A$ be a connected component of $G$ and $a \in V(A)$. Let $t \in V(D)$ such that $\sigma(t) = \emptyset$ and $\alpha(t) = V(A)$. Then $\alpha''(t, a) = V(A)$ and $\sigma''(t, a) = \emptyset$.

We define $h : V(D'') \to V(D)$ by letting $h(t, a) := t$ for all $t \in V(D)$. We have already observed that for all $t'' \in V(D'')$ we have $\sigma''(t'') \subseteq \sigma(h(t''))$ and $\alpha''(t'') \subseteq \alpha(h(t''))$. Thus $\gamma''(t'') \subseteq \gamma(h(t''))$ as well. Suppose for contradiction that for some $t'' = (t, a) \in V(D'')$ there holds $h(t, a) \not\Delta'' t''$. Then $b \not\sigma''(t'') \subseteq \beta(t)$ and $b \not\alpha''(t'') \subseteq \alpha(t)$. As $b \not\beta(t)$, there is an $u \in N^G_+(t)$ such that $b \alpha(u)$. Let $u'' := (u, b)$. Then $u'' \in V(D'')$ and $\alpha''(u'') \subseteq \alpha''(t'')$, because $b \alpha(u'')$. Thus $t'' u'' \in E(D'')$ and $b \alpha''(u'')$, which contradicts $b \not\beta''(t'')$. Hence $b \not\beta''(t'') \subseteq \beta(t)$. This implies $\tau''(t'') \subseteq \tau(t)$, because $\sigma''(t'') \subseteq \sigma(t)$, and for every $u'' = (u, b) \in N^G_+(t'')$ we have $u \in N^D_+(t)$ and $\sigma''(u'') \subseteq \sigma(u)$.

Unfortunately, the decomposition $\Delta''$ is not necessarily tight, because there may be nodes $t$ with $\sigma''(t) \neq \partial(\gamma''(t))$. We define the tight decomposition $\Delta' = (D', \sigma', \alpha')$ as follows.

- We let $D' := D''$.
- For all $t \in V(D')$ we let $\gamma'(t) := \gamma''(t)$ and $\sigma'(t) := \partial^G(\gamma'(t))$ and $\alpha'(t) := \gamma'(t) \setminus \sigma'(t)$.

Note that for all $t \in V(D')$ we have

$$\sigma'(t) \subseteq \sigma''(t) \quad \text{and} \quad \alpha'(t) \supseteq \alpha''(t) \quad \text{and} \quad \gamma'(t) = \gamma''(t).$$

(4.4.1)

To see that the decomposition $\Delta'$ is tight, let $t \in V(D')$. Then $\partial^G(\gamma'(t)) = \sigma'(t)$ by definition. Since $\sigma'(t) \subseteq \sigma''(t) = N^G(\alpha''(t))$, every vertex $v \in \sigma'(t)$ has a neighbour in $\sigma''(t) \subseteq \alpha'(t)$,
which implies that \( \sigma'(t) = N^G(\alpha'(t)) \). Finally, \( \alpha'(t) \) is connected, because \( \alpha''(t) \) is connected and \( \alpha''(t) \subseteq \alpha'(t) \subseteq \alpha''(t) \cup N^G(\alpha''(t)) \).

**Claim 2.** \( \Delta' \) is a treelike decomposition.

**Proof.** Axioms (TL.1) and (TL.5) are obviously inherited from \( \Delta'' \), and (TL.2) follows from the tightness of \( \Delta' \).

To prove (TL.3) let \( t \in V(D') \) and \( u \in N^D_+(t) \). Then \( \gamma'(u) \subseteq \gamma'(t) \) by (4.4.1) and (TL.3) for \( \Delta'' \). This implies \( \sigma'(t) \cap \gamma'(u) = D^G(\gamma'(t)) \cap \gamma'(u) \subseteq D^G(\gamma'(u)) = \sigma'(u) \) and thus

\[
\alpha'(u) = \gamma'(u) \setminus \sigma'(u) \subseteq \gamma'(u) \setminus \sigma'(t) \subseteq \gamma'(t) \setminus \sigma'(t) = \alpha'(t).
\]

To prove (TL.4) let \( t \in V(D') \) and \( u_1, u_2 \in N^D_+(t) \). If \( u_1 \parallel \Delta'' u_2 \) then \( u_1 \parallel \Delta' u_2 \). So suppose that \( u_1 \perp \Delta'' u_2 \). Then

\[
\gamma'(u_1) \cap \gamma'(u_2) = \gamma''(u_1) \cap \gamma''(u_2) = \sigma''(u_1) \cap \sigma''(u_2) \supseteq \sigma'(u_1) \cap \sigma'(u_2)
\]

by (4.4.1). To prove the converse inclusion, let \( v \in \sigma''(u_1) \cap \sigma''(u_2) \). Then \( v \in N^G(\alpha''(u_1)) \), and since \( \alpha''(u_1) \cap \gamma''(u_2) = \emptyset \), this implies \( v \in \partial(\gamma''(u_2)) \). Hence \( v \in \sigma'(u_2) \). Similarly, \( v \in N^G(\alpha''(u_2)) \) and thus \( v \in \sigma'(u_1) \). Altogether, we have \( \gamma'(u_1) \cap \gamma'(u_2) = \sigma'(u_1) \cap \sigma'(u_2) \) and thus \( u_1 \perp \Delta' u_2 \).

It follows from (4.4.1) that for all \( t \in V(D') \) we have \( \beta'(t) \subseteq \beta''(t) \) and \( \tau'(t) \subseteq \tau''(t) \). Hence \( h \) is a homomorphism from \( D' \) to \( D \) with the desired properties.

It is often not good enough to transform a decomposition \( \Delta \) into a tight decomposition \( \Delta' \) whose torsos are just subgraphs of the torsos of \( \Delta \). For example, the torsos of \( \Delta \) may be 3-connected, and we want to preserve this property in the transformation. Occasionally, \( \Delta \) meets the stronger assumptions of the following lemma, which allows us to transform the decomposition to a tight decomposition \( \Delta' \) with the same torsos.

**Lemma 4.4.3.** Let \( \Delta = (D, \sigma, \alpha) \) be a treelike decomposition of a graph \( G \) such that for all \( t \in V(D) \) the following conditions are satisfied.

(i) \( \sigma(t) = N(\alpha(t)) = \partial(\gamma(t)) \).

(ii) \( \tau(t) \setminus \sigma(t) \) is either empty or connected.

(iii) If \( \beta(t) \setminus \sigma(t) \neq \emptyset \) then \( N^\tau(t)(\beta(t) \setminus \sigma(t)) = \sigma(t) \). That is, every vertex \( v \in \sigma(t) \) has a neighbour \( w \in \beta(t) \setminus \sigma(t) \) in the graph \( \tau(t) \).

Then there exists a tight treelike decomposition \( \Delta' = (D', \sigma', \alpha') \) of \( G \) with \( V(D') \subseteq V(D) \) and \( \beta'(t) = \beta(t), \delta'(t) = \delta(t), \sigma'(t) = \sigma(t), \tau'(t) = \tau(t), \alpha'(t) \subseteq \alpha(t), \gamma'(t) \subseteq \gamma(t) \) for all \( t \in V(D') \).

**Proof.** To understand the proof, it is important to understand why \( \Delta \) is not tight in the first place. Let \( t \) be a \( \partial^D \)-maximal node such that \( \alpha(t) \) is not connected in \( D \). Then it follows from (i) and (ii) that either \( \alpha(t) = \sigma(t) = \emptyset \) or there is some \( u \in N^D_+(t) \) such that \( \sigma(u) \subseteq \sigma(t) \). If we assume, without loss of generality, that \( \Delta \) is normal, then we can rule out the case \( \alpha(t) = \sigma(t) = \emptyset \).

We define the decomposition \( \Delta' := (D', \sigma', \alpha') \) as follows:
For all $t \in V(D)$ we let $\sigma'(t) := \sigma(t)$ and
\[
\alpha'(t) := \alpha(t) \setminus \bigcup_{\sigma(u) \subseteq \sigma(t), u \in t^{D}D} \alpha(u).
\]

We let $V(D') := \{ t \in V(D) \mid \alpha'(t) \neq \emptyset \}$.

We let $E(D')$ be the set of all pairs $tx \in V(D')^2$ such that $t \leq D x$ and $\sigma(x) \not\subseteq \sigma(t)$ and $\sigma(x) \subseteq \sigma(u)$ for all $u \in V(D)$ with $t \leq D u \leq D x$.

We shall prove that $\Delta'$ is a treelike decomposition of $G$ with the desired properties. Obviously, for all $t \in V(D')$ we have $\alpha'(t) \subseteq \alpha(t)$ and $\gamma'(t) \subseteq \gamma(t)$.

**Claim 1.** For all $t \in V(D)$ and $v, w \in \beta(t)$ there is a path from $v$ to $w$ with all internal vertices in $\alpha(t)$.

**Proof.** We prove the claim by induction on $D$. Let $t \in V(D)$. If $t$ is a leaf then the claim follows immediately from (ii) and (iii). Otherwise, by (ii) and (iii) we have a path $P \subseteq \tau(t)$ from $v$ to $w$ in with all internal vertices in $\beta(t) \setminus \sigma(t)$. If $e = vw \in E(P) \setminus E(G)$, then there is an $u \in N^D_P(t)$ with $v, w \in \sigma(u)$. By the induction hypothesis, there is a path $P_e$ from $v$ to $w$ with all internal vertices in $\alpha(u)$, and we can replace the edge $e$ on $P$ by the path $P_e$. If we do this for all edges $e \in E(P) \setminus E(G)$, we get a connected subgraph $P' \subseteq G$ with $v, w \in V(P')$ and $V(P') \setminus \{v, w\} \subseteq \alpha(t)$.

**Claim 2.** For all $t \in V(D')$ and $u \in N^D_P(t)$ we have $\alpha'(u) \subseteq \alpha'(t)$.

**Proof.** Let $t \in V(D')$ and $u \in N^D_P(t)$. Suppose for contradiction that $\alpha'(u) \not\subseteq \alpha'(t)$. Let $v \in \alpha'(u) \setminus \alpha'(t)$. Then $v \in \alpha(u) \subseteq \alpha(t)$, and thus there is an $x' \succ t D$ such that $\sigma(x') \subseteq \sigma(t)$ and $u \in \sigma(x')$. Let $u' \in N^D_P(t)$ such that $u' \preceq D x'$. Then $v \in \alpha(u) \cap \alpha(u')$, and thus by Lemma 4.2.9 we have $u \parallel D u'$. Let $A$ be the connected component of $G \setminus \sigma(x')$ that contains $v$ and $S := N^G(A)$. Then $V(A) \subseteq \sigma(x') \subseteq \sigma(u') = \alpha(u)$ and thus $V(A) \cap \sigma(u) = \emptyset$. Moreover, $\sigma(x') \subseteq \gamma(x') \subseteq \gamma(u') = \gamma(u)$ and $\sigma(x') \subseteq \sigma(t) \subseteq \beta(t)$, and thus
\[
S \subseteq \sigma(x') \subseteq \sigma(u)
\]
by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9. Since $N^G(A) = S$, actually $A$ is a connected component of $G \setminus \sigma(u)$.

Let $x \succ D u$ such that $V(A) \subseteq \sigma(x)$ and $V(A) \not\subseteq \sigma(y)$ for any $y \succ D x$. Then $\sigma(x) \not\subseteq \sigma(u)$, because $\sigma(x) \subseteq \sigma(u)$ would imply $\alpha(x) \cap \alpha'(u) = \emptyset$, contradicting $v \in \alpha(x) \cap \alpha'(u)$. Let $w \in \sigma(x) \setminus \sigma(u)$. We have $w \not\in V(A) \subseteq \sigma(x)$, and thus $S \subseteq \sigma(u)$ separates $V(A)$ from $w$. As $A$ is connected and $V(A) \not\subseteq \sigma(y)$ for any $y \in N^D_P(x)$, we have $\beta(x) \cap V(A) \neq \emptyset$. Let $v' \in \beta(x) \cap V(A)$. Then by Claim 1, there is a path $P$ from $v'$ to $w$ with all internal vertices in $\alpha(x)$. As $S$ separates $v' \in V(A)$ from $w$, we have $V(P) \cap S \neq \emptyset$ and hence $\alpha(x) \cap S \neq \emptyset$. However, since $S = N^G(A) \subseteq \gamma(x)$ by Lemma 4.2.9 and $S \subseteq \sigma(u) \subseteq \beta(u)$, we have $S \subseteq \sigma(x)$ by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9. This is a contradiction, because $\sigma(x) \cap \alpha(x) = \emptyset$.

**Claim 3.** For all $t \in V(D')$ it holds that $\emptyset \neq \beta(t) \setminus \sigma(t) \subseteq \alpha'(t)$.

**Proof.** Let $t \in V(D')$. We have $\beta(t) \setminus \sigma(t) \subseteq \alpha(t)$, and for all $u \succ D t$ it holds that $\alpha(u) \cap \beta(t) = \emptyset$. Thus $\beta(t) \setminus \sigma(t) \subseteq \alpha'(t)$.
4.4. Tight Decompositions

Let \( v \in \alpha'(t) \subseteq \alpha(t) \). Then either \( v \in \beta(t) \setminus \sigma(t) \) and thus \( \beta(t) \setminus \sigma(t) \neq \emptyset \), or \( v \in \alpha(u) \) for some \( u \in N^D_+(t) \) with \( \alpha(u) \not\subseteq \sigma(t) \). In the latter case we have \( \beta(t) \setminus \sigma(t) \supseteq \sigma(u) \setminus \sigma(t) \neq \emptyset \).

Claim 4. For all \( t \in V(D') \), the set \( \alpha'(t) \) is connected in \( G \) and \( \sigma'(t) = N^G(\alpha'(t)) \).

Proof. We prove the claim by induction on \( D \). Let \( t \in V(D') \) and suppose that the claim is proved for all \( u \in V(D') \) with \( u \searrow D t \). By Claim 3, we have \( \emptyset \neq \beta(t) \setminus \sigma(t) \subseteq \alpha'(t) \).

We need three auxiliary claims:

(A) Let \( u \in N^D_+(t) \) and \( v, w \in \sigma(u) \setminus \sigma(t) \). Then there is a path \( P \subseteq G \) from \( v \) to \( w \) with all internal vertices in \( \alpha'(t) \).

By Claim 1 there is a path \( P \) from \( v \) to \( w \) with all internal vertices in \( \alpha(u) \subseteq \alpha(t) \). \( P \) has an empty intersection with \( \alpha(x) \) for all \( x \searrow D t \) with \( \sigma(x) \subseteq \sigma(t) \), because \( P \cap \sigma(t) \subseteq \{v\} \). Thus all internal vertices of \( P \) are in \( \alpha'(t) \). This proves (A).

(B) Let \( u \in N^D_+(t) \) with \( \sigma(u) \not\subseteq \sigma(t) \) and \( v \in \alpha'(t) \cap \alpha(u) \). Then there is a path \( P \subseteq G[\alpha'(t)] \) from \( v \) to a vertex in \( \sigma(u) \setminus \sigma(t) \).

Suppose for contradiction that (B) is false, and let \( u \in N^D_+(t) \) be a counterexample such that \( \text{dist}^D(t, u) \) is maximal. Let \( v \in \alpha'(t) \cap \alpha(u) \).

Case 1: \( v \in \alpha'(u) \).

Let \( w \in \sigma(u) \setminus \sigma(t) \). By the induction hypothesis (of the overall induction proving Claim 4), \( \alpha'(u) \) is connected and \( N^G(\alpha'(u)) = \sigma'(u) = \sigma(u) \). Hence there exists a path \( P \) from \( v \) to \( w \) with all internal vertices in \( \alpha'(u) \). By Claim 2, we have \( P \subseteq G[\alpha'(t)] \).

This contradicts our assumption that \( u \) is a counterexample to (B).

Case 2: \( v \notin \alpha'(u) \).

Then \( v \in \alpha(x) \) for some \( x \searrow D u \) with \( \sigma(x) \subseteq \sigma(u) \). Since \( v \in \alpha'(t) \), we have \( \sigma(x) \not\subseteq \sigma(t) \). Then \( x \in N^D_+(t) \). To see this, note that for \( u' \in V(D) \) with \( t \searrow D u' \searrow D u \) we have \( \sigma(x) \subseteq \sigma(u) \subseteq \sigma(u') \), because \( u \in N^D_+(t) \). For \( u' \in V(D) \) with \( u \searrow D u' \leq D x \) we have \( \sigma(x) \subseteq \gamma(u') \setminus \beta(u) \subseteq \sigma(u') \) by the \( \beta-\gamma-\sigma \)-Lemma 4.2.9.

As \( \text{dist}^D(t, x) > \text{dist}^D(t, u) \), by the maximality of \( u \) there is a path \( P \subseteq G[\alpha'(t)] \) from \( v \) to a vertex in \( \sigma(x) \setminus \sigma(t) \subseteq \sigma(u) \setminus \sigma(t) \). Again, this contradicts our assumption that \( u \) is a counterexample to (B).

(B) immediately implies the following claim (C), because if \( u \in N^D_+(t) \) with \( \sigma(u) \not\subseteq \sigma(t) \) then \( u \in N^D_+(t) \).

(C) Let \( u \in N^D_+(t) \) with \( \sigma(u) \not\subseteq \sigma(t) \) and \( v \in \alpha'(t) \cap \alpha(u) \). Then there is a path \( P \subseteq G[\alpha'(t)] \) from \( v \) to a vertex in \( \beta(t) \setminus \sigma(t) \).

Now we are ready to prove that \( \alpha'(t) \) is connected and that \( \sigma'(t) = \sigma(t) = N^G(\alpha'(t)) \). To prove the former, let \( v, w \in \alpha'(t) \). By (C), we may assume without loss of generality that \( v, w \in \beta(t) \setminus \sigma(t) \). By (ii), there is path \( P \subseteq \tau(t) \setminus \sigma(t) \) from \( v \) to \( w \). If \( e = vw \in E(P) \setminus E(G) \), then there is a \( u \in N^D_+(t) \) with \( v, w \in \sigma(u) \). By (A), there is a path \( P_e \subseteq G[\alpha'(t)] \) from \( v \) to \( w \).

We can replace the edge \( e \) on \( P \) by the path \( P_e \). If we do this for all edges \( e \in E(P) \setminus E(G) \), we get a connected subgraph \( P' \subseteq G[\alpha'(t)] \) with \( v, w \in V(P') \).
Finally, to prove that \( \sigma'(t) = N^G(\alpha'(t)) \), let \( v \in \sigma'(t) = \sigma(t) \). By (iii) there is a \( w \in \beta(t) \setminus \sigma(t) \) such that \( vw \in E(\sigma(t)) \). If \( vw \in E(G) \), then \( v \in N^G(w) \) and thus \( v \in N^G(\alpha'(t)) \). Otherwise, there is a \( u \in N^D_+(t) \) such that \( v, w \in \sigma(u) \). By (A), there is a path \( P \subseteq G \) from \( v \) to \( w \) with all internal vertices in \( \alpha'(t) \). Let \( w' \) be the neighbour of \( v \) on \( P \). Then \( v \in N^G(w') \) and thus \( v \in N^G(\alpha'(t)) \). This proves \( \sigma'(t) = N^G(\alpha'(t)) \). \( \qed \)

Claim 5. \( \Delta' \) is a treelike decomposition of \( G \).

Proof. The digraph \( D' \) is acyclic, because \( tu \in E(D') \) implies \( t \leq_D u \). Thus \( \Delta' \) satisfies (TL.1).

To prove (TL.2) let \( t \in V(D') \). Note that \( \sigma'(t) \cap \alpha'(t) \subseteq \sigma(t) \cap \alpha(t) = \emptyset \). Suppose for contradiction that there is an edge \( vw \in E(G) \) such that \( \sigma'(t) \subseteq \sigma(t) \) and \( v \in \sigma(t) \) and \( w \in \sigma(t) \). By (TL.2) for \( \Delta \) we have \( w \in \gamma(t) \setminus \gamma'(t) \), and since \( \sigma'(t) = \sigma(t) \), we have \( w \in \alpha(t) \setminus \alpha'(t) \). Thus \( w \in \alpha(u) \) for some \( u \triangleright_D t \) with \( \sigma(u) \subseteq \sigma(t) \). We have \( v \notin \alpha(u) \) and \( v \notin \sigma(t) \). This contradicts \( N^G(\alpha(u)) \subseteq \sigma(u) \).

To prove (TL.3) let \( t \in V(D') \) and \( u_1, u_2 \in N^D_+(t) \). For \( i = 1, 2 \), let \( u_i \in N^D_+(t) \) such that \( u_i \leq_D u_i \). Then \( \sigma(u_i) \subseteq \sigma(u_i) \subseteq \beta(t) \). By Claim 4, \( \alpha'(u_i) \) is a connected component of \( G \setminus \beta(t) \) and \( \alpha'(u_i) \subseteq \beta(t) \). By Lemma [4.2.10] this implies (TL.4).

Finally, to prove (v), let \( A \) be a connected component of \( G \) and \( t \in V(D) \) such that \( \sigma(t) = \emptyset \) and \( \alpha(t) = V(A) \). We can choose \( t \) in such a way that for all \( u \in N^D_+(t) \) it holds that \( \alpha(u) \subseteq V(A) \). Then for all \( x \triangleright_D t \) with \( \sigma(x) \subseteq \sigma(t) \) we have \( \sigma(x) = \emptyset \) and thus \( \alpha(x) = \emptyset \) by (i). Hence \( \alpha'(t) = \alpha(t) = V(A) \). \( \qed \)

Claim 6. \( \Delta' \) is tight.

Proof. Let \( t \in V(D') \). By Claim 4, \( \alpha'(t) \) is connected, and we have \( \sigma'(t) = N^G(\alpha'(t)) \). This implies \( \sigma'(t) \supseteq \partial^G(\gamma'(t)) \). To prove equality, let \( v \in \sigma'(t) = \sigma(t) = \partial^G(\gamma(t)) \). Then there is a \( w \in V(G) \setminus \gamma(t) \subseteq G \setminus \gamma'(t) \) with \( vw \in E(G) \). Thus \( w \in \partial^G(\gamma'(t)) \). \( \qed \)

Claim 7. For all \( t \in V(D) \) it holds that \( \beta'(t) = \beta(t) \).

Proof. To prove that \( \beta'(t) \subseteq \beta(t) \), let \( v \in \beta'(t) \). If \( v \in \sigma'(t) = \sigma(t) \), then \( v \in \beta(t) \). Let us assume that

\[
v \in \alpha'(t) \setminus \bigcup_{u \in N^D_+(t)} \alpha'(u).
\]

As \( \alpha'(t) \subseteq \alpha(t) \), we have \( v \in \alpha(t) \). Suppose for contradiction that \( v \in \alpha(u) \) for a \( u \in N^D_+(t) \). Then \( \sigma(u) \not\subseteq \sigma(t) \), because otherwise \( \alpha(u) \cap \alpha'(t) = \emptyset \), which contradicts \( v \in \alpha'(t) \). Hence \( u \in N^D_+(t) \). By (4.4.2) we have \( v \notin \alpha'(u) \). Thus there is an \( x \triangleright_D u \) such that \( \sigma(x) \subseteq \sigma(u) \) and \( v \in \alpha(x) \). As \( v \in \alpha(x) \) \( \cap \alpha'(t) \), we have \( \sigma(x) \not\subseteq \sigma(t) \). This implies \( x \in N^D_+(t) \) (by a similar argument as in the proof of (B) in Claim 4). By (4.4.2) we have \( v \notin \alpha'(x) \). Iterating the argument above, we find a sequence \( u \triangleleft_D x \triangleleft_D x^1 \triangleleft_D x^2 \triangleleft_D \ldots \) of vertices \( x^i \in N^D_+(t) \) with \( v \in \alpha(x^i) \setminus \alpha'(x^i) \). This leads to a contradiction, because \( D \) is a finite graph. This proves \( \beta'(t) \subseteq \beta(t) \).

To prove the converse inclusion \( \beta(t) \subseteq \beta'(t) \), let \( v \in \beta(t) \). If \( v \in \sigma(t) = \sigma'(t) \), then \( v \in \beta(t) \). Suppose that \( v \in \beta(t) \setminus \sigma(t) \). Then \( v \in \alpha(t) \). Furthermore, \( v \notin \alpha(u) \) for any \( u \in V(D) \) with \( t \triangleleft_D u \). Thus \( v \in \alpha'(t) \) and \( v \notin \alpha'(u) \subseteq \alpha(u) \) for any \( u \in N^D_+(t) \). This implies \( v \in \beta'(t) \). \( \qed \)
Claim 8. For all $t \in V(D')$ it holds that $\delta'(t) = \delta(t)$.

Proof. Let $t \in V(D')$. We shall prove the following two assertions, which imply the claim.

\begin{itemize}
\item[(D)] For all $x' \in \{t\} \cup N^D_+(t)$ there is an $x \in \{t\} \cup N^D_+(t)$ such that $\sigma(x') \subseteq \sigma(x)$.
\item[(E)] For all $x \in \{t\} \cup N^D_+(t)$ there is an $x' \in \{t\} \cup N^D_+(t)$ such that $\sigma(x) \subseteq \sigma(x')$.
\end{itemize}

To prove (D), let $x' \in \{t\} \cup N^D_+(t)$. If $x' = t$ we let $x = t$. Suppose that $x' \in N^D_+(t)$. Then $t \sim^D x'$. Let $x \in N^D_+(t)$ such that $x \sim^D x'$. Then by the definition of $E(D')$ we have $\sigma(x') \subseteq \sigma(x)$.

To prove (E), let $x \in \{t\} \cup N^D_+(t)$. If $x = t$ we let $x' = t$. Suppose that $x \in N^D_+(t)$. If $\sigma(x) \subseteq \sigma(t)$ we let $x' := t$. Otherwise, $x \in N^D_+(t)$, and we let $x' := x$.

\section*{4.5 Isomorphisms, Homomorphisms, and Bisimulations}

In this section, we introduce different ways of comparing decompositions. Recall that a \textit{homomorphism} from a digraph $D$ to a digraph $D'$ is a mapping $h : V(D) \to V(D')$ that preserves edges, that is, for all $tu \in E(D)$ it holds that $h(t)h(u) \in E(D')$. A homomorphism $h$ is a \textit{strong homomorphism} if it also preserves non-edges, that is, for all $tu \in V(D)^2 \setminus E(D)$ it holds that $h(t)h(u) \notin E(D')$. An \textit{isomorphism} is a bijective strong homomorphism.

\begin{definition}
Let $\Delta, \Delta'$ be two decompositions of the same graph $G$.

\begin{enumerate}
\item A \textit{(strong) homomorphism} from $\Delta$ to $\Delta'$ is a (strong) homomorphism $f$ from $D^\Delta$ to $D^{\Delta'}$ such that for all $t \in V(\Delta)$ it holds that $t \parallel f(t)$.
\item An \textit{isomorphism} from $\Delta$ to $\Delta'$ is an isomorphism $f$ from $D^\Delta$ to $D^{\Delta'}$ such that for all $t \in V(\Delta)$ it holds that $t \parallel f(t)$. $\Delta$ and $\Delta'$ are \textit{isomorphic} (we write $\Delta \cong \Delta'$) if there is an isomorphism from $\Delta$ to $\Delta'$.
\end{enumerate}

Clearly, isomorphic decompositions of a graph behave in the same way. However, there is a weaker notion of “behavioural equivalence” that is sufficient for our purposes.

\begin{definition}
Let $\Delta = (D, \sigma, \alpha), \Delta' = (D', \sigma', \alpha')$ be two decompositions of the same graph $G$.

\begin{enumerate}
\item A \textit{bisimulation} between $\Delta$ and $\Delta'$ is a binary relation $R \subseteq V(D) \times V(D')$ such that for all $(t, t') \in R$ the following three conditions are satisfied:

\begin{enumerate}
\item $(t, t')$.
\item For all $u \in N^D_+(t)$ there is a $u' \in N^{D'}_+(t')$ such that $(u, u') \in R$.
\item For all $u' \in N^{D'}_+(t')$ there is a $u \in N^D_+(t)$ such that $(u, u') \in R$.
\end{enumerate}
\item Two nodes $t \in V(D)$ and $t' \in V(D')$ are \textit{bisimilar} (we write: $t \sim^{\Delta, \Delta'} t'$) if there is a bisimulation $R$ between $\Delta$ and $\Delta'$ such that $(t, t') \in R$.
\item $\Delta'$ \textit{simulates} $\Delta$ if for all $t \in V(D)$ there is a $t' \in V(D')$ such that $t$ and $t'$ are bisimilar.
\item $\Delta$ and $\Delta'$ are \textit{bisimilar} if $\Delta$ simulates $\Delta'$ and $\Delta'$ simulates $\Delta$.
\end{enumerate}
\end{definition}
As usual, we omit the index $\Delta_0, \Delta'_0$ if the decompositions are clear from the context. Observe that the bisimilarity relation $\sim \subseteq V(D) \times V(D')$ is a bisimulation. This implies that $\Delta'$ simulates $\Delta$ if and only if there is a bisimulation $R$ between $\Delta$ and $\Delta'$ that is total in the first component, that is, for all $t \in V(D)$ there is a $t' \in V(D')$ such that $(t, t') \in R$. Furthermore, $\Delta$ and $\Delta'$ are bisimilar if and only if there is a bisimulation $R$ between $\Delta$ and $\Delta'$ that is total in both components.

**Lemma 4.5.3.** Let $\Delta, \Delta'$ be decompositions of the same graph $G$, and let $R$ be a bisimulation between $\Delta$ and $\Delta'$. Then for all $(t, t') \in R$ we have $t \approx t'$.

**Proof.** Suppose that $\Delta = (D, \sigma, \alpha)$ and $\Delta' = (D', \sigma', \alpha')$. Let $(t, t') \in R$. Then $\sigma(t) = \sigma'(t')$ and $\alpha(t) = \alpha'(t')$ by Definition 4.5.2 and thus $\gamma(t) = \gamma'(t)$. It follows from (ii) and (iii) that $\bigcup_{u \in N_+^D(t)} \alpha(u) = \bigcup_{u' \in N_+^{D'}(t')} \alpha'(u')$. This implies $\beta(t) = \beta'(t)$. It also follows from (ii) and (iii) that

$$\{|\sigma(u) | u \in N_+^D(t)\} = \{|\sigma'(u') | u' \in N_+^{D'}(t')\}.$$ 

This implies $\delta(t) = \delta'(t')$ and $\tau(t) = \tau'(t')$.

**Corollary 4.5.4.** For all decompositions $\Delta, \Delta'$ and all $t \in V(\Delta), t' \in V(\Delta')$ we have

$$t \approx t' \implies t \approx t' \implies t \parallel t'.$$

**Lemma 4.5.5.** Let $\Delta = (D, \sigma, \alpha), \Delta' = (D', \sigma', \alpha')$ be decompositions of the same graph $G$, and let $h$ be a homomorphism from $\Delta$ to $\Delta'$ such that for all $t \in V(D)$ it holds that $h(N_+^D(t)) = N_+^{D'}(h(t))$. Then $\Delta'$ simulates $\Delta$.

**Proof.** Let

$$R := \{ (t, t') \in V(D) \times V(D') \mid t' = h(t) \}.$$ 

Then $R$ is a bisimulation between $\Delta$ and $\Delta'$. To see this, note that conditions (i) and (ii) of Definition 4.5.2 are satisfied because $h$ is a homomorphism. Condition (iii) is satisfied because $h(N_+^D(t)) = N_+^{D'}(h(t))$.

**Corollary 4.5.6.** Let $\Delta, \Delta'$ be decompositions of the same graph $G$, and let $h$ be a surjective strong homomorphism from $\Delta$ onto $\Delta'$. Then $\Delta$ and $\Delta'$ are bisimilar.

### 4.6 The Relation Between Tree Decompositions and Treelike Decompositions

In this section, we study the relationship between tree decompositions and treelike decompositions. We prove two theorems. The first characterises the treelike decompositions that “are” tree decompositions. The second shows that for all treelike decompositions there is a tree decomposition with the same torsos.

**Lemma 4.6.1.** Let $\Delta = (T, \beta)$ be a tree decomposition of a graph $G$, and let $\sigma := \sigma^\Delta$, and $\alpha := \alpha^\Delta$ (defined as in (4.1.2) and (4.1.3)). Then $\Delta' := (T, \sigma, \alpha)$ satisfies (TL.1), (TL.4), and for every node $t \in V(T)$ it holds that $t \approx^{\Delta', \Delta'} t$.

Furthermore, if $G$ is connected then $\Delta'$ is treelike.
4.6. The Relation Between Tree Decompositions and Treelike Decompositions

Proof. We have $\alpha^\Delta = \alpha^{\Delta'} = \alpha$ and $\sigma^\Delta = \sigma^{\Delta'} = \sigma$ by definition. This immediately implies $\gamma^\Delta = \gamma^{\Delta'} = \gamma$. Then it follows from (4.1.6) and (4.2.2) that $\beta = \beta^\Delta = \beta^{\Delta'}$. This implies $\delta^\Delta = \delta^{\Delta'}$ and $\tau^\Delta = \tau^{\Delta'}$. Thus all nodes of $T$ are indistinguishable in the two decompositions.

We still have to prove that $\Delta'$ satisfies (TL.1), (TL.4), and (TL.5) if $G$ is connected.

To establish (TL.3), let $t \in V(T)$ and $u \in N^T(t)$. It follows immediately from the definition (4.1.1) of $\gamma$ that $\gamma(u) \subseteq \gamma(t)$. By Fact 4.1.3(1) we have

$$\gamma(u) \cap \beta(t) \subseteq \sigma(u) \subseteq \beta(u). \quad (4.6.1)$$

Hence

$$\alpha(u) = \gamma(u) \setminus (\beta(t) \cap \beta(u)) \quad \text{(by (4.1.2) and (4.1.3))}$$
$$\subseteq \gamma(u) \setminus \beta(t) \quad \text{(because $\gamma(u) \subseteq \gamma(t)$ and $\beta(t) \supseteq \sigma(t)$)}$$
$$\subseteq \gamma(t) \setminus \sigma(t)$$
$$= \alpha(t).$$

To prove that $\Delta'$ satisfies (TL.4), let $t \in V(T)$ and $u_1, u_2 \in N^T(t)$ with $u_1 \neq u_2$. It follows from (T.1) that $\gamma(u_1) \cap \gamma(u_2) \subseteq \beta(t)$. As $\gamma(u_i) \cap \beta(t) = \sigma(u_i)$ for $i = 1, 2$, this implies

$$\gamma(u_1) \cap \gamma(u_2) = \sigma(u_1) \cap \sigma(u_2).$$

To establish (TL.5) suppose that $G$ is connected. Let $r$ be the root of $T$. Then $\sigma(r) = \emptyset$ and $\gamma(r) = V(G)$. As $G$ is connected, this proves (TL.5) \qed

Corollary 4.6.2. Let $A$ be a class of graphs and $G$ a connected graph that has a tree decomposition over $A$. Then $G$ has a treelike decomposition over $A$.

Remark 4.6.3. Lemma 4.6.1 and Corollary 4.6.2 do not extend to disconnected graphs. Consider the graph $G := \{v_1, v_2\}, \emptyset$ with two distinct vertices $v_1, v_2$ and no edges. Let $T = \{\{1\}, \emptyset\}$ and define $\beta : V(T) \to 2^{V(G)}$ be $\beta(t) := \{v_1, v_2\}$. Then $(T, \beta)$ is a tree decomposition of $G$. If we define $\sigma$ and $\alpha$ according to (4.1.2) and (4.1.3), then we have $\sigma(t) = \emptyset$ and $\alpha(t) = \{v_1, v_2\}$. Obviously, the decomposition $(T, \sigma, \alpha)$ does not satisfy (TL.5).

Moreover, for every treelike decomposition $\Delta$ of $G$ there are nodes $t_1, t_2$ such that for $i = 1, 2$ it holds that $\beta(t_i) = \{v_i\}$. To see this, let $s_i \in V(\Delta)$ such that $\gamma(s_i) = \alpha(s_i) = \{v_i\}$. Such an $s_i$ exists by (TL.5). As $D^\Delta$ is acyclic by (TL.1) there is a $t_i \in V(\Delta)$ such that $s_i \preceq^\Delta t_i$ and $\gamma(t_i) = \alpha(t_i) = \{v_i\}$ and for all $u \in N^\Delta(t_i)$ either $\gamma(u) \neq \gamma(t_i)$ or $\alpha(u) \neq \alpha(t_i)$. Then it follows from (TL.3) and $\alpha(u) \subseteq \gamma(u)$ that $\alpha(u) = \emptyset$ for all $u \in N^\Delta(t_i)$. Hence $\beta(t_i) = \gamma(t_i) = \{v_i\}$. This means that $G$ has a tree decomposition over the class of all graphs of order at least 2, but no treelike decomposition over this class. Hence Corollary 4.6.2 does not extend to disconnected graphs. \qed

Theorem 4.6.4. Let $\Delta$ be a decomposition of a graph $G$ and $\Delta' := (D^\Delta, \beta^\Delta)$. Then $\Delta'$ is a tree decomposition if and only if $\Delta$ satisfies conditions (TL.2), (TL.3) and the following three conditions.

(TD.1) $D^\Delta$ is a directed tree.

(TD.4) For all $t \in V(\Delta)$ and $u_1, u_2 \in N(t)$, if $u_1 \neq u_2$ then $u_1 \perp u_2$.

(TD.5) Let $r$ be the root of $D^\Delta$. Then $\sigma^\Delta(r) = \emptyset$ and $\alpha^\Delta(r) = V(G)$. 

Preliminary Version
Furthermore, if $\Delta'$ is a tree decomposition then for every node $t \in V(\Delta)$ it holds that $t \equiv_{\Delta,\Delta'} t$.

To simplify the notation, we let $(\text{TD.2}) := (\text{TL.2})$ and $(\text{TD.3}) := (\text{TL.3})$. In the following, we use the term tree decomposition both for decompositions $(D, \sigma, \alpha)$ satisfying $(\text{TD.1})$, $(\text{TD.5})$ and for pairs $(T, \beta)$ satisfying $(\text{T.1})$ and $(\text{T.2})$ Observe that $(\text{TD.1})$ implies $(\text{TL.1})$ and $(\text{TD.4})$ implies $(\text{TL.4})$. Furthermore, if $G$ is a connected graph then $(\text{TD.5})$ implies $(\text{TL.5})$. Hence every tree decomposition of a connected graph is a treelike decomposition.

**Proof of Theorem 4.6.4.** The forward direction follows from Lemma 4.6.1 (to be precise, $(\text{TD.4})$ only follows from the proof of Lemma 4.6.1).

For the backward direction, suppose that $\Delta = (T, \sigma, \alpha)$ is a decomposition that satisfies $(\text{TD.1})$, $(\text{TD.3})$ Then $\Delta$ satisfies $(\text{TL.1})$, $(\text{TL.4})$ and thus we can apply the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9. Let $\tau$ be the root of $T$.

**Claim 1.** $(T, \beta)$ satisfies $(\text{T.1})$

**Proof.** Let $v \in V(G)$ and $U := \{ t \in V(T) \mid v \in \beta(t) \}$. To prove that $U$ is nonempty, let $t \in V(T)$ such that $v \in \gamma(t)$ and $v \not\in \gamma(u)$ for all $u \in N^+_T(t)$. Such a $t$ exists because $v \in \gamma(r) = V(G)$ by $(\text{TD.5})$ and $T$ is a tree. Then

$$v \in \gamma(t) \setminus \bigcup_{u \in N^+_T(t)} \gamma(u) \subseteq \gamma(t) \setminus \bigcup_{u \in N^+_T(t)} \alpha(u) = \beta(t)$$

and thus $t \in U$.

To prove that $U$ is connected, we first note that if $s, u \in U$ with $s \preceq_T u$ then $t \in U$ for all $t \in V(T)$ with $s \preceq_T t \preceq_T u$. Indeed, by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9(4) we have $v \in \beta(s) \cap \gamma(u) \subseteq \sigma(t) \subseteq \beta(t)$. It remains to prove that $U$ has a unique $\preceq_T$-minimal element. Let $u_1, u_2 \in U$ with $u_1 \neq u_2$. Let $s \in V(T)$ be the greatest common ancestor of $u_1, u_2$, and let $t_1, t_2 \in N^+_T(s)$ such that $t_1 \neq t_2$ and $t_1 \preceq_T u_1$ and $t_2 \preceq_T u_2$. Then by $(\text{TL.3})$ for $i = 1, 2$ it holds that $v \in \beta(u_i) \subseteq \gamma(u_i) \subseteq \gamma(t_i)$. By $(\text{TD.4})$ it follows that $v \in \gamma(t_1) \cap \gamma(t_2) = \sigma(t_1) \cap \sigma(t_2) \subseteq \beta(s)$. $\square$

**Claim 2.** $(T, \beta)$ satisfies $(\text{T.2})$

**Proof.** Let $e := \{v, w\} \in E(G)$. Let $t \in V(T)$ such that $e \subseteq \gamma(t)$ and $e \not\subseteq \gamma(u)$ for all $u \in N^+_T(t)$. Such a $t$ exists by $(\text{TD.1})$ and $(\text{TD.5})$. It follows from $(\text{TD.2})$ that $e \cap \alpha(u) = \emptyset$ for all $u \in N^+_T(u)$. Hence

$$e \subseteq \gamma(t) \setminus \bigcup_{u \in N^+_T(t)} \alpha(u) = \beta(t).$$

Hence $\Delta' := (T, \beta)$ is a tree decomposition (in the original sense). It remains to prove that for all $t \in V(T)$ we have $t \parallel_{\Delta,\Delta'} t$. Let $\alpha' := \alpha^{\Delta'}$, \ldots, $\tau' := \tau^{\Delta'}$.

We have $\beta = \beta'$ by definition. To see that $\sigma = \sigma'$, note that for the root $r$ we have $\sigma(r) = \sigma'(r) = \emptyset$ by $(\text{TD.5})$ and the definition (4.1.2) of $\sigma'$, and for every node $t \in V(T)$ with parent $s$ we have $\sigma(t) = \beta(s) \cap \beta(t) = \beta'(s) \cap \beta'(t) = \sigma'(t)$ by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9(3) and (4.1.2). Then it follows from the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9(1) and the definition (4.1.1) of $\gamma'$ that $\gamma = \gamma'$ and thus $\alpha = \alpha'$. Since $D^\Delta = D^{\Delta'} = T$, it also follows that $\delta = \delta'$ and thus $\tau = \tau'$.

We will prove next that every treelike decomposition, and not only those satisfying $(\text{TD.1})$, $(\text{TD.5})$ can be turned into a closely related tree decomposition.

M. Grohe, Definable Graph Structure Theory
Lemma 4.6.5. Let $\Delta = (D, \sigma, \alpha)$ be a treelike decomposition of a graph $G$. Then there is a treelike decomposition $\Delta_U = (D_U, \sigma_U, \alpha_U)$ of $G$ such that:

(i) $D_U$ is a forest.

(ii) There is a strong homomorphism from $\Delta_U$ onto $\Delta$.

Note that, by Corollary 4.5.6, (ii) implies that $\Delta_U$ and $\Delta$ are bisimilar.

Proof of Lemma 4.6.5. We start by defining a decomposition $\Delta_U = (D_U, \sigma_U, \alpha_U)$, the forest unfolding of $\Delta$, and a mapping $h_U : V(D') \to V(D)$ as follows:

- For every root $r$ of $D$ we define a directed tree $T_r$ with vertex set $V(T_r)$ consisting of all paths $P \subseteq D$ with head $r$ and edges $PP'$ between all paths $P, P' \in V(T_r)$ with $P \subseteq P'$ and $|P| + 1 = |P'|$. Then we let $D_U$ be the union of all the trees $T_r$.
- For all $P \in V(D_U)$ we let $h_U(P)$ be the tail of $P$.
- For all $P \in V(D_U)$ we let $\sigma_U(P) := \sigma(h_U(P))$ and $\alpha_U(P) := \alpha(h_U(P))$.

Then $h_U$ is a strong homomorphism from $\Delta_U$ onto $\Delta$. \hfill $\square$

Lemma 4.6.6. Let $\Delta = (D, \sigma, \alpha)$ be a treelike decomposition of a graph $G$. Then there is a treelike decomposition $\Delta' = (D', \sigma', \alpha')$ of $G$ such that:

(i) $D'$ is a forest.

(ii) For all connected components $A$ of $G$ there is exactly one root $r_A$ of $D'$ such that

$$\sigma'(r_A) = \emptyset \quad \text{and} \quad \alpha'(r_A) := V(A).$$

Furthermore, there are no other roots, that is, for each root $r$ of $D'$ there is a connected component $A$ of $G$ such that $r = r_A$.

(iii) $\Delta'$ satisfies [TD.4].

(iv) There is a strong homomorphism $h$ from $\Delta'$ to $\Delta$ such that for all $t \in V(D')$ it holds that $t \approx h(t)$.

Proof. By Lemma 4.6.5, without loss of generality we may assume that $D$ is a forest. We define the digraph $D' \subseteq D$ inductively as follows:

- For every connected component $A$ of $G$ we choose a root $r_A$ of $D$ with $\sigma(r_A) = \emptyset$ and $\alpha(r_A) := V(A)$ and add $r_A$ to $V(D')$.
- For every $t$ already in $V(D')$ such that $N^D(t)$ is not yet defined, we arbitrarily choose $u_1, \ldots, u_m \in N^D(t)$ such that for every $u \in N^D(t)$ there is exactly one $i \in [m]$ such that $u \parallel u_i$. For all $i \in [m]$, we add $u_i$ to $V(D')$, and we add an edge $tu_i$ to $E(D')$.

Then for every $t \in V(D')$ we let $\sigma'(t) := \sigma(t)$ and $\alpha'(t) := \alpha(t)$. It is easy to see that $\Delta' = (D', \sigma', \alpha')$ is a treelike decomposition of $G$ and that for every $t \in V(D')$ it holds $t \approx \Delta' \Delta$. Hence the inclusion map is a strong homomorphism from $\Delta'$ to $\Delta$ that satisfies (iv). As $D' \subseteq D$, (i) is satisfied by our assumption that $D$ be a forest. Assertions (ii) and (iii) follow immediately from the definition of $D'$.

\hfill $\square$
Corollary 4.6.7. Let $\Delta$ be a treelike decomposition of a connected graph $G$. Then there is a tree decomposition $\Delta'$ of $G$ and a strong homomorphism $h$ from $\Delta'$ to $\Delta$ such that for all $t \in V(D')$ it holds that $t \approx h(t)$.

Theorem 4.6.8. Let $\Delta = (D, \sigma, \alpha)$ be a treelike decomposition of a graph $G$. Then there is a tree decomposition $\Delta' = (D', \sigma', \alpha')$ of $G$ such that for all $t' \in V(D')$ there is a $t \in V(D)$ with $\beta(t) = \beta'(t')$ and $\delta(t) = \delta'(t')$ and thus $\tau(t) = \tau'(t')$.

Proof. Let $\Delta'' = (D'', \sigma'', \alpha'')$ be the decomposition obtained from $\Delta$ by applying Lemma 4.6.6. Let $r_1, \ldots, r_m$ be the roots of $D''$. We let $D'$ be the tree obtained from the forest $D''$ by adding the edges $r_ir_i$ for all $i \in [2, m]$. We let $\sigma' := \sigma''$ and $\alpha' := \alpha''$. It is easy to verify that the decomposition $\Delta'$ has the desired properties. $\square$

Corollary 4.6.9. Let $A$ be a class of graphs and $G$ a graph that has a treelike decomposition over $A$. Then $G$ has a tree decomposition over $A$.

Corollary 4.6.10. Let $A$ be a class of graphs and $G$ a connected graph. Then $G$ has a treelike decomposition over $A$ if and only if $G$ has a tree decomposition over $A$. 

M. Grohe, Definable Graph Structure Theory
Chapter 5

Definable Decompositions

The main advantage treelike decompositions have over tree decompositions is that they can be made invariant under automorphisms of the underlying graph. This makes them logically definable, at least in principle. To actually define them, we introduce decomposition schemes, which may be viewed as a form of transductions. We exclusively study definability in the logic IFP, though most of the time we use little of the power of IFP.

Our first results are “definable versions” of some of the results of the previous chapter, in particular the normal forms. We then turn to issues that are more specific to definability. A simple, but useful result is the Definability Lifting Lemma 5.4.3 essentially saying that the class of all graphs that have a definable treelike decomposition over a definable class of graphs is definable as well. Throughout this book, we will frequently be in situations where we have to define a decomposition of some graph $H$ within some other graph $G$. Typically $H$ is a subgraph of $G$ defined by some subgraph transduction, or $H$ is a torso of some definable decomposition of $G$. Section 5.5 introduces the framework for dealing with this.

The main result of this chapter is the Transitivity Lemma 5.6.1, stating that if a class $C$ of graphs admits definable treelike decompositions over a class $B$, and $B$ admits definable treelike decompositions over a class $A$, then $C$ admits definable treelike decompositions over $A$.

5.1 Decomposition Schemes

Definition 5.1.1. Let $L$ be a logic and $\tau$ a vocabulary. An $L[\tau]$-decomposition scheme is a tuple

$$\Lambda = (\lambda_V(\overline{x}), \lambda_E(\overline{x}, \overline{x}'), \lambda_\sigma(\overline{x}, y), \lambda_\alpha(\overline{x}, y))$$

of formulae in $L[\tau]$, where $\overline{x}, \overline{x}'$ are tuples of individual variables of the same type and $y$ is a vertex variable.

The dimension of $\Lambda$ is the length of the tuple $\overline{x}$. 

Throughout this book, we will only consider $\text{IFP}[\{E\}]$-decomposition schemes. For this reason, we will no longer mention the logic or vocabulary and just speak of decomposition schemes or d-schemes, for short. Furthermore, we will only apply d-schemes to graphs. Most of our results extend to arbitrary relational structures. In particular, this is the case for the basic theory of definable treelike decompositions and definable ordered treelike decompositions developed in this chapter and Chapter 7. Many of the results also apply to other logics than IFP.
IFP, at least if the logics are at least as expressive as IFP. For weaker logics, some care needs to be taken, because our definitions occasionally involve inductions over tuples (the tuples $\bar{x}$ in the d-scheme).

**Definition 5.1.2.** Let $\Lambda = (\lambda_V(\bar{x}), \lambda_E(\bar{x}, \bar{x}'), \lambda_\sigma(\bar{x}, y), \lambda_\alpha(\bar{x}, y))$ a d-scheme.

1. For every graph $G$, the *decomposition defined by $\Lambda$ on $G$* is the decomposition

   $$\Lambda[G] := (D^{\Lambda[G]}, \sigma^{\Lambda[G]}, \alpha^{\Lambda[G]}).$$

   with

   $$V(D^{\Lambda[G]}) := \lambda_V[G, \bar{x}],$$

   $$E(D^{\Lambda[G]}) := \lambda_E[G, \bar{x}, \bar{x}'] \cap V(\Lambda[G])^2,$$

   and for every $\bar{v} \in V(\Lambda[G])$

   $$\sigma^{\Lambda[G]}(\bar{v}) := \lambda_\sigma[G, \bar{v}, y],$$

   $$\alpha^{\Lambda[G]}(\bar{v}) := \lambda_\alpha[G, \bar{v}, y].$$

2. $T_\Lambda$ is the class of all graphs $G$ such that $\Lambda[G]$ is a treelike decomposition.

   If $G \in T_\Lambda$, we say occasionally say that $\Lambda$ *defines a treelike decomposition on $G$*.

**Definition 5.1.3.** Let $\Lambda$ be a d-scheme, $G$ a graph, and $\mathcal{A}, \mathcal{C}$ classes of graphs.

1. $T_\Lambda(\mathcal{A})$ denotes the class of all graphs $G$ such that $\Lambda[G]$ is a treelike decomposition of $G$ over $\mathcal{A}$.

2. $\mathcal{C}$ *admits IFP-definable treelike decompositions over $\mathcal{A}$* if there is a d-scheme $\Lambda$ such that $\mathcal{C} \subseteq T_\Lambda(\mathcal{A})$.

   If $G \in T_\Lambda(\mathcal{A})$, we occasionally say that $\Lambda$ *defines a treelike decomposition over $\mathcal{A}$ on $G$*.

**Example 5.1.4.** Recall Example 4.2.3. Let $\Lambda$ be the 2-dimensional d-scheme defined as follows:

- $\lambda_V(x_1, x_2) := (x_1 = x_2) \lor E(x_1, x_2),$
- $\lambda_E(x_1, x_2, x'_1, x'_2) := (x'_1 = x_2) \land (x'_2 \neq x_1) \land E(x'_1, x'_2),$
- $\lambda_\sigma(x_1, x_2, y) := (x_1 \neq x_2 \land y = x_1),$
- $\lambda_\alpha(x_1, x_2, y) := (x_1 = x_2 \land \text{path}(x, y)) \lor (x_1 \neq x_2 \land \text{path}'(x_1, x_2, y_2)),$

where $\text{path}(x, y)$ is the IFP-formula from Example 2.3.3 saying that there is a path from $x$ to $y$, and $\text{path}'(x_1, x_2, y_2)$ is a modification such that for every graph $G$ and all $v_1, v_2, w \in V(G)$ it holds that

$$G \models \text{path}'[v_1, v_2, w] \iff \text{there is a path from $v_2$ to $w$ in the graph $G \setminus \{v_1\}$.}$$

Cf. Example 2.4.10 for the definition of $\text{path}'(x_1, x_2, y_2)$.

M. Grohe, *Definable Graph Structure Theory*
Then for every graph $G$, the decomposition $\Lambda[G]$ is precisely the decomposition $\Delta(G)$ of Example 4.2.3. Hence for all forests $F$, the d-scheme $\Lambda$ defines a treelike decomposition on $F$ over $K_2$. We may also write $T_1 \subseteq T_\Lambda(K_2)$, because the class of forests is precisely the class of graphs of tree width 1. As a matter of fact, we have $T_1 = T_\Lambda(K_2)$, because every graph in $T_\Lambda(K_2)$ has a treelike decomposition of width 1 and thus by Theorem 4.6.8 a tree decomposition of width 1.

Example 5.1.5. The treelike decomposition of cycles introduced in Example 4.2.4 is definable as well. We leave it to the reader to define a d-scheme defining the decomposition.

Example 5.1.6. Remember the definition of trivial decompositions from Example 4.2.5. For every $\ell \in \mathbb{N}$ there is an $\ell$-dimensional d-scheme $\Lambda_\ell$ such that $T_{\Lambda_\ell}$ is the class of all connected graphs, and $\Lambda_\ell[G]$ is the trivial $\ell$-dimensional decomposition of $G$, for every connected graph $G$.

Example 5.1.7. As a pathological example, let us decompose the empty graph $\emptyset$. Let $\Delta_\emptyset$ be the decomposition in which $D^\Delta_\emptyset$ is the empty digraph and $\sigma^\Delta_\emptyset, \alpha^\Delta_\emptyset$ are the empty mappings. Then $\Delta_\emptyset$ is a treelike decomposition of $\emptyset$. Furthermore, for every class $A$ of graphs, $\Delta_\emptyset$ is a decomposition over $A$.

The decomposition $\Delta_\emptyset$ is IFP-definable. Indeed, for every d-scheme $\Lambda$ that is at least unary it holds that $\Lambda[\emptyset] = \Delta_\emptyset$.

From now on, we shall ignore the empty graph when discussing decompositions.

The following lemma shows that the decomposition of a graph into its connected components is definable.

Lemma 5.1.8 (CC Decomposition Lemma). There is a d-scheme $\Lambda_{cc}$ such that for all graphs $G$ the decomposition $\Lambda_{cc}[G]$ is a treelike decomposition of adhesion 0, and its torsos are precisely the connected components of $G$.

Proof. Let $\Lambda_{cc} = (\lambda_V(x), \lambda_E(x, x'), \lambda_\sigma(x, y), \lambda_\alpha(x, y))$ be the d-scheme with $\lambda_V(x) := \text{true}$, $\lambda_E(x, x') := \text{false}$, $\lambda_\sigma(x, y) := \text{false}$, and $\lambda_\alpha(x, y) := \text{path}(x, y)$, where $\text{path}(x, y)$ is an IFP-formula stating that there is a path from $x$ to $y$.

Then for every graph $G$ it holds that $V(\Lambda[\emptyset]) = V(G)$ and $E(\Lambda[\emptyset]) = \emptyset$. Furthermore, for every $v \in V(\Lambda[G])$, the set $\sigma(v)$ is empty, and $\alpha(v)$ is the connected component of $v$. It is easy to verify that the decomposition $\Lambda[G]$ is treelike.

5.2 Normalising Definable Decompositions

In this section, we prove “definable versions” of the normalisation lemmas proved in Section 4.3.

Lemma 5.2.1. Let $\Lambda$ be a d-scheme. Then there is a d-scheme $\Lambda'$ such that for every graph $G \in T_\Lambda$ the decomposition $\Lambda'[G]$ is a normal treelike decomposition of $G$ that satisfies the following conditions:

(i) $D^{\Lambda'[G]} \subseteq D^{\Lambda[G]}$.

(ii) For all $t \in V(\Lambda'[G])$ it holds that $t \approx^{\Lambda'[G], \Lambda[G]} t$.
Chapter 5. Definable Decompositions

Proof. We shall prove that the decomposition $\Delta'$ defined in the proof of Lemma 4.3.5 is IFP-definable, provided the decomposition $\Delta$ is IFP-definable. To explain our IFP-definition, it will convenient to fix a graph $G \in T_{\Lambda}$.

Using the IFP-sentence $\text{conn}$ of Example 2.3.3 and the Transduction Lemma (Fact 2.4.6 also see Example 2.4.11), we can construct a formula $\text{conn}_\alpha(\bar{x})$ such that for all $\bar{v} \in V(\Lambda[G])$ it holds that

$$G \models \text{conn}_\alpha(\bar{v}) \iff G[\alpha^\Lambda(\bar{v})] \text{ is connected.}$$

The following IFP-formula $\text{bad}(\bar{x})$ defines the set of all bad nodes of $\Lambda[G]$:

$$\text{bad}(\bar{x}) := \lambda_V(\bar{x}) \land \text{ifp} \left( X \bar{x} \leftarrow (\neg \text{conn}_\alpha(\bar{x}) \lor \exists y \lambda_\sigma(\bar{x}, y)) \land \forall \bar{x}' \left( \lambda_V(\bar{x}') \land \lambda_E(\bar{x}', \bar{x}) \rightarrow X\bar{x}' \right) \right).$$

Then the following formula defines the set of good nodes:

$$\text{good}(\bar{x}) := \neg \text{bad}(\bar{x}) \land \exists y (\lambda_\sigma(\bar{x}, y) \lor \lambda_\alpha(\bar{x}, y)).$$

We let $\Lambda'$ be the $d$-scheme with $\lambda'_V(\bar{x}) := \text{good}(\bar{x})$ and $\lambda'_E(\bar{x}, \bar{x}') := \lambda_E(\bar{x}, \bar{x}')$, $\lambda'_\sigma(\bar{x}, y) := \lambda_\sigma(\bar{x}, y)$, $\lambda'_\alpha(\bar{x}, y) := \lambda_\alpha(\bar{x}, y)$.

Lemma 5.2.2. Let $\Lambda$ be a $d$-scheme. Then there is a $d$-scheme $\Lambda'$ such that for every graph $G \in T_{\Lambda}$ the decomposition $\Lambda'[G]$ is a strict treelike decomposition of $G$ that satisfies the following conditions:

(i) $V(D^{\Lambda'[G]}) \subseteq V(D^{\Lambda[G]})$.

(ii) For all $t, u \in V(D^{\Lambda'[G]})$ it holds that $t \preceq^{\Lambda'[G]} u \iff t \preceq^{\Lambda[G]} u$.

(iii) For all $t \in V(\Lambda'[G])$ it holds that $t \approx^{\Lambda'[G], \Lambda[G]} t$.

Proof. We shall prove that the decomposition $\Delta'$ defined in the proof of Lemma 4.3.6 is IFP-definable, provided the decomposition $\Delta$ is IFP-definable. Let $G \in T_{\Lambda}$ and $\Delta := \Lambda[G]$. We will use the notation of the proof of Lemma 4.3.6.

Suppose that $\Lambda$ is $\ell$-dimensional, and let $X$ be an $\ell$-ary relation variable. We shall define a deflationary fixed-point formula that defines $X$ in such a way that for all $i \in \mathbb{N}^+$ the $i$th stage $X^i[G]$ is $V(D^i)$. We need a few auxiliary formulæ. For $\ell$-tuples $\bar{x}, \bar{x}'$ of variables we write $\bar{x} \parallel \bar{x}'$ to abbreviate the formula $\forall y \left( (\lambda_\sigma(\bar{x}, y) \leftrightarrow \lambda_\sigma(\bar{x}', y)) \land (\lambda_\alpha(\bar{x}, y) \leftrightarrow \lambda_\alpha(\bar{x}', y)) \right)$.

Let

$$\text{bad}(X, Y, \bar{x}) := X\bar{x} \land \exists \bar{x}' \left( Y\bar{x}' \land \bar{x} \parallel \bar{x}' \right).$$

Then for every $i \in \mathbb{N}^+$, the set $\text{bad}[G, V(D^i), E(D^i), \bar{x}]$ consists of all bad nodes of $D^i$. We let

$$\text{leq}(X, Y, \bar{x}, \bar{x}') := \text{ifp} \left( Z\bar{z} \leftarrow X\bar{z} \land (\bar{z} = \bar{x} \lor \exists \bar{z}' (Z\bar{z}' \land Y\bar{z}' \bar{z})) \right) \bar{x}'$$

and

$$\text{min-bad}(X, Y, \bar{x}) := \text{bad}(X, Y, \bar{x}) \land \neg \exists \bar{x}' \left( \text{leq}(X, Y, \bar{x}, \bar{x}') \land \bar{x}' \neq \bar{x} \land \text{bad}(X, Y, \bar{x}') \right).$$

Then for every $i \in \mathbb{N}^+$, the set $\text{min-bad}[G, V(D^i), E(D^i), \bar{x}]$ consists of all $\preceq$-minimal bad nodes of $D^i$.

M. Grohe, Definable Graph Structure Theory
Recall that for every $i \in \mathbb{N}^+$ and all $\overline{v}, \overline{v}' \in V(D^{i+1})$ it holds that

$$(\overline{v}, \overline{v}') \in E(D^{i+1}) \iff (\overline{v}, \overline{v}') \in E(D^i) \text{ or there is a } \overline{v}'' \in V(D^i) \text{ such that } (\overline{v}, \overline{v}''), (\overline{v}, \overline{v}') \in E(D^i).$$

Now a straightforward induction proves that for all $i \in \mathbb{N}^+$ and all $\overline{v}, \overline{v}' \in V(D^i)$ it holds that

$$(\overline{v}, \overline{v}') \in E(D^i) \iff \text{there are an } n \in \mathbb{N} \text{ and } \overline{v}^0, \overline{v}^1, \ldots, \overline{v}^{n+1} \in V(D) \text{ such that } \overline{v}^0 = \overline{v}, \overline{v}^{n+1} = \overline{v}' \text{ and } \overline{v}^1, \ldots, \overline{v}^n \in V(D) \setminus V(D^i) \text{ and } (\overline{v}^i, \overline{v}^{i+1}) \in E(D) \text{ for all } i \in [0, n].$$

Let

$$\text{edge}(X, \overline{v}, \overline{v}') := X\overline{v} \land X\overline{v}' \land \text{dfp} \left( Y \overline{v}' \leftarrow \lambda_E(Y, \overline{v}) \lor \exists \overline{v}'' \left( \neg X\overline{v}' \land \lambda_V(Y, \overline{v}'') \land Y \overline{v} \land Y \overline{v}'' \land \overline{v}' \right) \right) \overline{v}' .$$

Then for all $i \in \mathbb{N}^+$ we have $\text{edge}[G, V(D^i), \overline{v}, \overline{v}'] = E(D^i)$. Let $\text{min-bad'}(X, \overline{v})$ be the formula obtained from $\text{min-bad}(X, Y, \overline{v})$ by replacing each atomic subformula of the form $Y \overline{v} \overline{v}'$ by $\text{edge}(X, \overline{v}, \overline{v}')$. Then for all $i \in \mathbb{N}^+$ the set $\text{min-bad'}[G, V(D^i), \overline{v}]$ consists of all $\leq$-minimal bad nodes of $D^i$. Now we let

$$\lambda'_{V}(\overline{v}) := \text{dfp} \left( X\overline{v} \leftarrow \lambda_V(\overline{v}) \land \neg \text{min-bad'}(X, \overline{v}) \right) \overline{v} .$$

Then $V(D^i) = \lambda'_{V}[G, \overline{v}]$. Let $\lambda'_{E}(\overline{v}, \overline{v}')$ be the formula obtained from the formula $\text{edge}(X, \overline{v}, \overline{v}')$ by replacing each subformula of the form $X\overline{v}$ by $\lambda'_{V}(\overline{v})$. We let

$$\Lambda' := (\lambda'_{V}(\overline{v}), \lambda'_{E}(\overline{v}, \overline{v}'), \lambda_{\sigma}(\overline{v}, y), \lambda_{\alpha}(\overline{v}, y)) .$$

It is easy to verify that $\Delta' = \Lambda'[G]$.

Lemma 5.2.3 (Normalisation Lemma for Definable Treelike Decompositions). Let $\Lambda$ be a $d$-scheme. Then there is a $d$-scheme $\Lambda'$ such that for every graph $G \in \mathcal{T}_\Lambda$ the decomposition $\Lambda'[G]$ is a strict and normal treelike decomposition on $G$ that satisfies the following conditions:

(i) $V(\Lambda'[G]) \subseteq V(\Lambda[G])$.

(ii) For all $t, u \in V(\Lambda'[G])$ it holds that $t \leq \Lambda'[G] u \iff t \leq \Lambda[G] u$.

(iii) For all $t \in V(\Lambda'[G])$ it holds that $t \approx \Lambda'[G], \Lambda[G] t$.

Proof. This follows immediately from the previous two lemmas. 

5.3 Definable Tight Decompositions

In this section, we give “definable versions” of the two lemmas in Section 4.4.

Lemma 5.3.1. Let $\Lambda$ be a $d$-scheme. Then there exists a $d$-scheme $\Lambda'$ such that for every graph $G \in \mathcal{T}_\Lambda$ the decomposition $\Lambda'[G]$ is a tight treelike decomposition, and there is a strong homomorphism $h$ from $D^{\Lambda'[G]}$ to $D^{\Lambda[G]}$ such that for every node $t' \in V(\Lambda')$ with $t := h(t')$ it holds that $\beta^{\Lambda'[G]}(t') \subseteq \beta^{\Lambda[G]}(t)$, $\gamma^{\Lambda'[G]}(t') \subseteq \gamma^{\Lambda[G]}(t)$, $\sigma^{\Lambda'[G]}(t') \subseteq \sigma^{\Lambda[G]}(t)$, and $\tau^{\Lambda'[G]}(t') \subseteq \tau^{\Lambda[G]}(t)$.
Chapter 5. Definable Decompositions

Proof. It is easy to see that the tight decomposition constructed in the proof of Lemma 4.4.2 is IFP-definable.

Lemma 5.3.2. Let \( \Lambda \) be a d-scheme. Then there exists a d-scheme \( \Lambda' \) such that the following holds for every graph \( G \in T_\Lambda \). Suppose that \( \Lambda[G] = (D, \sigma, \alpha) \) satisfies the following conditions for all \( t \) in \( V(D) \).

1. \( \sigma(t) = N(\alpha(t)) = \partial(\gamma(t)) \).
2. \( \tau(t) \setminus \sigma(t) \) is either empty or connected.
3. If \( \beta(t) \setminus \sigma(t) \neq \emptyset \) then \( N^{\tau(t)}(\beta(t) \setminus \sigma(t)) = \sigma(t) \).

Then \( \Lambda'[G] = (D', \sigma', \alpha') \) is a tight treelike decomposition of \( G \) with \( V(D') \subseteq V(D) \) and \( \beta'(t) = \beta(t), \delta'(t) = \delta(t), \sigma'(t) = \sigma(t), \tau'(t) = \tau(t) \) and \( \alpha'(t) \subseteq \alpha(t), \gamma'(t) \subseteq \gamma(t) \) for all \( t \) in \( V(D') \).

Proof. The tight decomposition constructed in the proof of Lemma 4.4.3 is IFP-definable.

5.4 Lifting Definability

Throughout this book, we will prove many of lemmas about definable treelike decomposition. Roughly, these lemmas can be classified into the following three categories. Lifting lemmas allow us to “lift” definability results from the torsos of a definable decomposition of a graph to the whole graph. Extension lemmas allow us to “extend” definable treelike decompositions form certain graphs \( H, H', \ldots \) to other graphs \( G \); typically \( H, H', \ldots \) will be definable subgraphs or minors of \( G \). Completion lemmas allow us to “complete” so-called pre-decompositions (to be introduced in Chapter [12]) to full treelike decompositions. Extension lemmas will be the most frequent. Our first extension lemma will be Lemma 5.5.7. Completion lemmas will only appear in Part II of the book. In this section, we shall prove our first lifting lemma, the Definability Lifting Lemma 5.4.3. It says that if a class of graphs admits IFP-definable treelike decompositions over an IFP-definable class of graphs then the class itself is IFP-definable.

Lemma 5.4.1. Let \( \Lambda \) be a d-scheme.

1. There are IFP-formulae \( \lambda_\gamma(\overline{x}, \overline{y}) \) and \( \lambda_\beta(\overline{x}, \overline{y}) \) such that for all graphs \( G \) and all \( \overline{v} \in V(\Lambda[G]) \) we have
   \[ \gamma(\overline{v}) = \lambda_\gamma[G, \overline{v}, \overline{y}] \quad \text{and} \quad \beta(\overline{v}) = \lambda_\beta[G, \overline{v}, \overline{y}] \]

2. There is an IFP-formula \( \lambda_\tau(\overline{x}, y_1, y_2) \) such that for all graphs \( G \) and all tuples \( \overline{v} \in V(\Lambda[G]) \) we have
   \[ E(\tau(\overline{v})) = \lambda_\tau[G, \overline{v}, y_1, y_2] \]

3. There is an IFP-sentence \( \lambda_T \) that defines the class \( T_\Lambda \).

Proof. It is straightforward to formalise the definitions in IFP.

Corollary 5.4.2. Let \( \Lambda \) be a d-scheme. Then there is a simple 1-dimensional IFP-graph transduction \( \Theta(\overline{x}) \) such that for all graphs \( G \) and all tuples \( \overline{v} \in V(\Lambda[G]) \) we have \( \tau(\overline{v}) = \Theta[G, \overline{v}] \).
Lemma 5.4.3 (Definability Lifting Lemma). Let $\Lambda$ be a d-scheme, and let $A$ be an IFP-definable class of graphs. Then $T_\Lambda(A)$ is IFP-definable.

Proof. Let $\varphi$ be an IFP-sentence that defines $A$, and let $\Theta(\pi)$ be the IFP-graph transduction obtained by Corollary 5.4.2. Applying the Transduction Lemma (Corollary 2.4.7) to $\Theta(\pi)$ and $\varphi$, we obtain an IFP-formula $\varphi^{-\Theta(\pi)}$ such that for all graphs $G$ and all $\pi \in V(\Lambda[G])$ we have
\[
G \models \varphi^{-\Theta(\pi)} \iff \tau(\pi) \in A.
\]
Then the formula
\[
\lambda_T \land \forall \pi (\lambda_V(\pi) \rightarrow \varphi^{-\Theta(\pi)})
\]
defines the class $T_\Lambda(A)$.

Recall that for every class $A$ of graphs, $T(A)$ denotes the class of all graphs that have a tree decomposition over $A$.

Corollary 5.4.4. Let $A$ be an IFP-definable class of graphs such that $T(A)$ admits IFP-definable treelike decompositions over $A$. Then $T(A)$ is IFP-definable.

Proof. Let $\Lambda$ be a d-scheme such that $T(A) \subseteq T_\Lambda(A)$. Then it follows from Corollary 4.6.9 that $T_\Lambda(A) = T(A)$. Thus $T(A)$ is IFP-definable by the Definability Lifting Lemma 5.4.3.

Lemma 5.4.5 (Union Lemma for Definable Decompositions). Let $A, B, C$ be classes of graphs such that $A$ is IFP-definable and $B, C$ admit IFP-definable treelike decompositions over $A$. Then $B \cup C$ admits IFP-definable treelike decompositions over $A$.

Proof. Let $\Lambda^B$ and $\Lambda^C$ be d-schemes such that $B \subseteq T_{\Lambda^B}(A)$ and $C \subseteq T_{\Lambda^C}(A)$. Without loss of generality we may assume that both $\Lambda^B$ and $\Lambda^C$ are $\ell$-dimensional, because we can always increase the dimension of a d-scheme artificially. Let $\varphi^B$ be an IFP-formula that defines the class $T_{\Lambda^B}(A)$. Let $\Lambda$ be the d-scheme defined as follows:
\[
\begin{align*}
\lambda_V(\pi) &:= (\varphi_B \land \lambda^B_V(\pi)) \lor (\neg \varphi_B \land \lambda^C_V(\pi)); \\
\lambda_E(\pi, \pi') &:= (\varphi_B \land \lambda^B_E(\pi, \pi')) \lor (\neg \varphi_B \land \lambda^C_E(\pi, \pi')); \\
\lambda_\sigma(\pi, y) &:= (\varphi_B \land \lambda^B_\sigma(\pi, y)) \lor (\neg \varphi_B \land \lambda^C_\sigma(\pi, y)); \\
\lambda_\alpha(\pi, y) &:= (\varphi_B \land \lambda^B_\alpha(\pi, y)) \lor (\neg \varphi_B \land \lambda^C_\alpha(\pi, y));
\end{align*}
\]
It is easy to see that $B \cup C \subseteq T_\Lambda(A)$.

5.5 Parametrised Decomposition Schemes

Occasionally, it will be necessary to consider definable treelike decompositions where the definition involves parameters. In particular, we shall use parametrised decompositions in combination with transductions to define decompositions of graphs within other graphs.

Recall that a signature is a tuple of variables and that a graph interpretation for a signature consists of a graph and an assignment for the variables in the signature. We adapt the definitions as follows:
Definition 5.5.1. (1) A parametrised d-scheme is tuple

\[ \Lambda(Z) = \left( \lambda_V(Z, \bar{x}), \lambda_E(Z, \bar{x}, \bar{x}'), \lambda_\sigma(Z, \bar{x}, y), \lambda_\alpha(Z, \bar{x}, y) \right) \]

of \( \text{IFP}\{\{E\}\} \)-formulae, where \( Z \) is a tuple of variables, which are called the parameters of the d-scheme, and \( \bar{x}, \bar{x}' \) are tuples of vertex variables of the same length, and \( y \) is a vertex variable.

(2) Let \( \Lambda(Z) \) be a parametrised d-scheme and \((G, \bar{P})\) a graph interpretation for \( Z \). The decomposition defined by \( \Lambda \) on \((G, \bar{P})\) is the decomposition

\[ \Lambda[G, \bar{P}] := (D^{\Lambda[G, \bar{P}]}, \sigma^{\Lambda[G, \bar{P}]}, \alpha^{\Lambda[G, \bar{P}]}) \]

with

\[ V(D^{\Lambda[G, \bar{P}]}) := \lambda_V[G, \bar{P}, \bar{x}], \]

\[ E(D^{\Lambda[G, \bar{P}]}) := \lambda_E[G, \bar{P}, \bar{x}, \bar{x}'] \cap V(D^{\Lambda[G, \bar{P}]})^2, \]

and for every \( \bar{v} \in V(\Lambda[G, \bar{P}]) \)

\[ \sigma^{\Lambda[G, \bar{P}]}(\bar{v}) := \lambda_\sigma[G, \bar{P}, \bar{v}, y], \]

\[ \alpha^{\Lambda[G, \bar{P}]}(\bar{v}) := \lambda_\alpha[G, \bar{P}, \bar{v}, y]. \]

(3) \( T_{\Lambda(Z)} \) is the class of all graph interpretations \((G, \bar{P})\) for \( Z \) such that \( \Lambda[G, \bar{P}] \) is a treelike decomposition of \( G \) over \( A \).

(4) For every class \( \mathcal{A} \) of graphs, \( T_{\Lambda(Z)}(\mathcal{A}) \) is the class of all graph interpretations \((G, \bar{P})\) for \( Z \) such that \( \Lambda[G, \bar{P}] \) is a treelike decomposition of \( G \) over \( \mathcal{A} \).

It is straightforward to extend our previous results on definable treelike decompositions to the parametrised setting. We state two examples without proof:

Lemma 5.5.2 (Parametrised Normalisation Lemma). Let \( \Lambda(Z) \) be a parametrised d-scheme. Then there is a parametrised d-scheme \( \Lambda'(Z) \) such that for every graph interpretation \((G, \bar{P}) \in T_{\Lambda(Z)} \) the decompositions \( \Lambda'[G, \bar{P}] \) is a strict and normal treelike decomposition of \( G \) that satisfies the following conditions:

(i) \( V(\Lambda'[G, \bar{P}]) \subseteq V(\Lambda[G, \bar{P}]) \).

(ii) For all \( t, u \in V(\Lambda'[G, \bar{P}]) \) it holds that \( t \preceq^{\Lambda'[G, \bar{P}]} u \iff t \preceq^{\Lambda[G, \bar{P}]} u \).

(iii) For all \( t \in V(\Lambda'[G, \bar{P}]) \) it holds that \( t \approx^{\Lambda'[G, \bar{P}], \Lambda[G, \bar{P}]} t \).

Lemma 5.5.3 (Parametrised Definability Lifting Lemma). Let \( \Lambda(Z) \) be a parametrised d-scheme. Then there is an \( \text{IFP} \)-formula \( \lambda_T(Z) \) that defines the class \( T_{\Lambda(Z)} \).

Furthermore, for every \( \text{IFP} \)-definable class \( \mathcal{A} \) of graphs there is an \( \text{IFP} \)-formula \( \lambda_T(\mathcal{A})(Z) \) that defines the class \( T_{\Lambda(Z)}(\mathcal{A}) \).

M. Grohe, Definable Graph Structure Theory
Parameters are less important in definable treelike decompositions than they are in transductions and canonisations, because they often can be eliminated. The simple proof of this fact nicely illustrates the flexibility of treelike decompositions. For the proof we introduce disjoint unions of treelike decompositions. Let \( \Delta_1 = (D_1, \sigma_1, \alpha_1), \ldots, \Delta_m = (D_m, \sigma_m, \alpha_m) \) be decompositions of a graph \( G \). Then the disjoint union of \( \Delta_1, \ldots, \Delta_m \) is the decomposition \( \Delta \) defined by

\[
V(\Delta) := \bigcup_{i=1}^m V(D_i) \times \{i\},
\]

\[
E(\Delta) := \{(t, i)(u, i) \mid i \in [m] \text{ and } tu \in E(D_i)\},
\]

\[
\sigma^\Delta(t, i) := \sigma^i(t) \quad \text{for all } i \in [m] \text{ and } t \in V(D_i),
\]

\[
\alpha^\Delta(t, i) := \alpha^i(t) \quad \text{for all } i \in [m] \text{ and } t \in V(D_i).
\]

Obviously, if the decompositions \( \Delta_1, \ldots, \Delta_m \) are treelike then their disjoint union \( \Delta \) is treelike as well, and if \( \Delta_1, \ldots, \Delta_m \) are decompositions over some class \( A \) then \( \Delta \) is a decomposition over \( A \) as well. The next lemma shows that disjoint unions of definable decompositions are definable as well. Parametrised decomposition enable us to state this in an elegant way.

**Lemma 5.5.4.** Let \( \Lambda(\overline{z}) \) be a parametrised d-scheme, where \( \overline{z} \) is a tuple of individual variables, and let \( A \) be an \( \text{IFP} \)-definable class of graphs. Then there is a d-scheme \( \Lambda' \) such that for all graphs \( G \) the decomposition \( \Lambda'[G] \) is isomorphic to the disjoint union of the decompositions \( \Lambda[G, \overline{p}] \) for all \( \overline{p} \in G^\overline{z} \) such that \( (G, \overline{p}) \in T_{\Lambda(\overline{z})}(A) \).

**Proof.** Let \( \lambda_{T(A)}(\overline{z}) \) be the formula from the Parametrised Definability Lifting Lemma 5.4.3 that defines the class \( T_{\Lambda(\overline{z})}(A) \). Let \( k \) be the dimension of \( \Lambda \) and \( \ell \) the length of the parameter tuple \( \overline{z} \). We let \( \Lambda' \) be the \((k + \ell)\)-dimensional decomposition defined by

\[
\Lambda'_V(\overline{xz}) := \lambda_V(\overline{z}, \overline{x}) \land \lambda_{T(A)}(\overline{z});
\]

\[
\Lambda'_E(\overline{xz}, \overline{xyz'}) := \lambda_E(\overline{z}, \overline{x}, \overline{z'}) \land \overline{z} = \overline{z'};
\]

\[
\Lambda'_\sigma(\overline{zx}, y) := \lambda_\sigma(\overline{z}, \overline{x}, y);
\]

\[
\Lambda'_\alpha(\overline{zx}, y) := \lambda_\alpha(\overline{z}, \overline{x}, y).
\]

It is easy to see that \( \Lambda' \) indeed defines the disjoint union of the \( \Lambda[G, \overline{p}] \) for \( \overline{p} \in G^\overline{z} \) with \( (G, \overline{p}) \in T_{\Lambda(\overline{z})}(A) \). \( \square \)

**Corollary 5.5.5.** Let \( \Lambda(\overline{z}) \) be a parametrised d-scheme, where \( \overline{z} \) is a tuple of individual variables, and let \( A \) be an \( \text{IFP} \)-definable class of graphs. Then there is a d-scheme \( \Lambda' \) such that for all graphs \( G \) it holds that

\[
G \in T_{\Lambda'}(A) \iff \text{there is a tuple } \overline{p} \in G^\overline{z} \text{ such that } (G, \overline{p}) \in T_{\Lambda(\overline{z})}(A).
\]

**Proof.** We let \( \Lambda' \) define the disjoint union of the decompositions \( \Lambda[G, \overline{p}] \) for all \( \overline{p} \in G^\overline{z} \) such that \( (G, \overline{p}) \in T_{\Lambda(\overline{z})}(A) \). \( \square \)
5.5.1 Decomposing Graphs within Other Graphs

Suppose that $G, H$ are graphs with $V(H) \subseteq V(G)$. Further suppose that $\Delta = (D, \sigma, \alpha)$ is a decomposition of $H$. Note that $\Delta$ may also be viewed as a decomposition of $G$, though if $V(H) \neq V(G)$, then $\Delta$ cannot be a treelike decomposition of both $H$ and $G$. But we want to view $\Delta$ as a decomposition of $H$. Now suppose that $\Lambda$ is a d-scheme such that $\Lambda \mid [G] = \Delta$. Then we say that $\Lambda$ defines the decomposition $\Delta$ of $H$ within $G$. Typically, the graph $H$ will be defined by a transduction. Then we can use the Transduction Lemma to define $\Lambda \mid [H]$ within $G$. The following lemma makes this precise.

**Lemma 5.5.6 (Transduction Lemma for Definable Decompositions).** Let $\Theta(Z)$ be a simple 1-dimensional IFP-graph transduction and $\Lambda(Z')$ a parametrised d-scheme. Then there is a parametrised d-scheme $\Lambda'(Z, Z')$ such that for all graph interpretations $(G, \overline{P}, \overline{P}')$ for $Z, Z'$ with $(G, \overline{P}) \in D_{\Theta(Z)}$ and $\overline{P}' \in \Theta[G, \overline{P}]^{Z'}$ the d-scheme $\Lambda'(Z, Z')$ defines the decomposition $\Lambda \mid [\Theta[G, \overline{P}], \overline{P}']$ of $\Theta[G, \overline{P}]$ within $(G, \overline{P}, \overline{P}')$.

**Proof.** This follows immediately from the Transduction Lemma (Fact 2.4.6). □

A consequence of the Transduction Lemma for Definable Decompositions is the following useful lemma.

**Lemma 5.5.7.** Let $\mathcal{B}, \mathcal{C}$ be a classes of graphs such that the class of all connected components of the graphs in $\mathcal{C}$ admits IFP-definable treelike decompositions over $\mathcal{B}$. Then the class $\mathcal{C}$ admits IFP-definable treelike decompositions over $\mathcal{B}$.

**Proof.** Let $\mathcal{A}$ be the class of all connected components of the graphs in $\mathcal{C}$, and let $\Lambda$ be a d-scheme such that $\mathcal{A} \subseteq T_\Lambda(\mathcal{B})$. Let $\Theta(x_0)$ be an IFP-graph transduction such that for all graphs $G$ and all $v_0 \in V(G)$ the graph $\Theta[G, v_0]$ is the connected component of $G$ that contains $v_0$. Then by the Transduction Lemma for Definable Decompositions 5.5.6 there is a parametrised d-scheme $\Lambda'(x_0)$ such that for all graphs $G$, all connected components $A$ of $G$, and all $v_0 \in V(A)$ we have $\Lambda'[G, v_0] = \Lambda[A]$.

Now we let $\Lambda''$ be the d-scheme defined as follows:

$$
\begin{align*}
\lambda''_V(x_0, \overline{x}) &:= \lambda_V(x_0, \overline{x}), \\
\lambda''_E(x_0, \overline{x}, x_0', \overline{x}') &:= (x_0 = x_0') \land \lambda''_E(x_0, \overline{x}, \overline{x}') \\
\lambda''_a(x_0, \overline{x}, y) &:= \lambda'_a(x_0, \overline{x}, y), \\
\alpha''(x_0, \overline{x}, y) &:= \alpha'(x_0, \overline{x}, y).
\end{align*}
$$

Then for every graph $G$ the decomposition $\Lambda''[G]$ is the disjoint union of the decompositions $\Lambda'[G, v_0]$ for all $v_0 \in V(G)$. If $G \in \mathcal{C}$, this is a treelike decomposition over $\mathcal{B}$. □

**Remark 5.5.8.** The simple proof of Lemma 5.5.7 illustrates a typical application of the Transduction Lemma. We will be less explicit in similar applications of the Transduction Lemma in the remainder of this book.

5.6 The Transitivity Lemma

The goal of this section is to prove the following lemma.
Lemma 5.6.1 (Transitivity Lemma). Let \( A, B, C \) be classes of graphs such that \( C \) admits \( \text{IFP} \)-definable treelike decompositions over \( B \) and \( B \) admits \( \text{IFP} \)-definable treelike decompositions over \( A \). Then \( C \) admits \( \text{IFP} \)-definable treelike decompositions over \( A \).

Before we start to prove the lemma, let me remark that in the Transitivity Lemma, definability is the real issue. It is straightforward to prove that \( B \subseteq \mathcal{T}(A) \) and \( C \subseteq \mathcal{T}(B) \) imply \( C \subseteq \mathcal{T}(A) \); the proof uses Fact 4.1.3(3). Then, via Corollary 4.6.10 one can prove that if every graph in \( B \) has a treelike decomposition over \( A \) and every graph in \( C \) has a treelike decomposition over \( B \), then every graph in \( C \) has a treelike decomposition over \( A \). However, this route through tree decompositions is not possible if we care about definable decompositions, and the proof of gets much harder. The proof strategy is straightforward, though. We use a d-scheme defining decompositions on \( B \) and let \( \Lambda \). Let \( \Lambda \) be a parametrised d-scheme, where \( |\bar{x}_1| = \ell_1 \). Then, via Corollary 4.6.10, one can prove that if every graph in \( B \) has a treelike decomposition over \( A \) and every graph in \( C \) has a treelike decomposition over \( B \), then every graph in \( C \) has a treelike decomposition over \( A \). The decompositions can be merged together because intersections between torsos of the first decomposition are cliques of the torsos and cannot be decomposed any further. Hence they will appear in bags of the second decomposition as a whole.

The Transitivity Lemma follows easily from the following lifting lemma, which is important in its own right. To be as general as required later, we state the lemma in a very technical form. Corollary 5.6.3 is a simplified version that is more accessible and sufficient for most purposes, in particular, for proving the Transitivity Lemma 5.6.1.

Lemma 5.6.2 (Decomposition Lifting Lemma). Let \( \Lambda^1 \) be an \( \ell_1 \)-dimensional d-scheme, and let \( \Lambda^2(\bar{x}_1) \) be a parametrised d-scheme, where \( |\bar{x}_1| = \ell_1 \). Then there exists a d-scheme \( \Lambda \) such that for every graph \( G \):

(i) If \( G \in \mathcal{T}_{\Lambda^1} \) and for every \( \bar{v}_1 \in V(\Lambda^1[G]) \) the scheme \( \Lambda^2(\bar{x}_1) \) defines a treelike decomposition of \( \tau^\Lambda[G](\bar{v}_1) \) within \( (G, \bar{v}_1) \), then \( G \in \mathcal{T}_\Lambda \).

Furthermore, for every graph \( G \in \mathcal{T}_\Lambda \):

(ii) \( V(\Lambda[G]) \subseteq \{ \bar{v}_1 \bar{v}_2 \mid \bar{v}_1 \in V(\Lambda^1[G]), \bar{v}_2 \in V(\Lambda^2[G, \bar{v}_1]) \} \).

(iii) For all \( \bar{v} = \bar{v}_1 \bar{v}_2 \in V(\Lambda[G]) \), we have

\[ \tau^\Lambda[G](\bar{v}) = \tau^\Lambda^2[G, \bar{v}_1](\bar{v}_2). \]

(iv) For all \( \bar{v} = \bar{v}_1 \bar{v}_2 \in V(\Lambda[G]) \),

- either \( \sigma^\Lambda[G](\bar{v}) = \sigma^\Lambda^2[G, \bar{v}_1](\bar{v}_2) \) and \( \alpha^\Lambda[G](\bar{v}) \cap \beta^\Lambda[G](\bar{v}_1) = \alpha^\Lambda^2[G, \bar{v}_1](\bar{v}_2) \)
- or there is a \( \bar{w}_2 \in N^\Lambda^2[G, \bar{v}_1](\bar{v}_2) \) such that \( \sigma^\Lambda[G](\bar{v}) = \sigma^\Lambda^2[G, \bar{v}_1](\bar{w}_2) \) and \( \alpha^\Lambda[G](\bar{v}) \cap \beta^\Lambda[G](\bar{v}_1) = (\beta^\Lambda[G](\bar{v}_1) \setminus \gamma^\Lambda^2[G, \bar{v}_1](\bar{w}_2)) \)
- or \( \sigma^\Lambda[G](\bar{v}) = \sigma^\Lambda^1[G](\bar{v}_1) \) and \( \alpha^\Lambda[G](\bar{v}) = \alpha^\Lambda^1[G](\bar{v}_1) \).

Corollary 5.6.3. Let \( \Lambda^1, \Lambda^2 \) be d-schemes. Then there exists a d-scheme \( \Lambda \) such that for every graph \( G \):

(i) If \( G \in \mathcal{T}_{\Lambda^1} \) with \( \tau^\Lambda^1[G](t_1) \in \mathcal{T}_{\Lambda^2} \) for all \( t_1 \in V(\Lambda^1[G]) \), then \( G \in \mathcal{T}_\Lambda \).
Furthermore, for every graph $G \in T_{\Lambda}$:

(i) $V(\Lambda[G]) \subseteq \{ t_1t_2 \mid t_1 \in V(\Lambda^1[G]), t_2 \in V(\Lambda^2[H]) \}$, where $H := \tau^{\Lambda^1[G]}(t_1)$.

(ii) For $t = t_1t_2 \in V(\Lambda[G])$, we have

$$\tau^{\Lambda[G]}(t) = \tau^{\Lambda^2[H]}(t_2),$$

where $H := \tau^{\Lambda^1[G]}(t_1)$.

The Transitivity Lemma [5.6.1] is an easy consequence of the corollary.

Proof of the Transitivity Lemma [5.6.1] Let $\Lambda^1$ be a d-scheme with $C \subseteq T_{\Lambda^1}(B)$. Let $\Lambda^2$ be a d-scheme $B \subseteq T_{\Lambda^1}(A)$. Let $\Lambda$ be the d-scheme obtained from Corollary 5.6.3. Then for every graph $G \in C$, the scheme $\Lambda$ defines a treelike decompositions of $G$ over $A$. 

It remains to prove the Decomposition Lifting Lemma. The proof is long and tedious, but in its style typical for many proofs of lifting or extension lemmas later in this book (mainly in Part II).

### 5.6.1 Proof of the Decomposition Lifting Lemma

**Lemma 5.6.4.** Let $\Delta = (D, \sigma, \alpha)$ be a treelike decomposition of a graph $G$, and let $P \subseteq V(G)$ be a clique in $G$.

1. For all $t \in V(D)$, if $P \cap \alpha(t) \neq \emptyset$ then $P \subseteq \gamma(t)$.

2. For every $t \in V(D)$ with $P \cap \alpha(t) \neq \emptyset$ there is a $u \supseteq_D t$ such that $P \subseteq \beta(u)$ and $P \cap \alpha(u) \neq \emptyset$.

   In particular, there is a $u \in V(D)$ with $P \subseteq \beta(u)$.

Note that the condition $P \cap \alpha(u) \neq \emptyset$ in (2) implies $P \not\subseteq \beta(t')$ for all $t' \prec_D u$.

**Proof of Lemma 5.6.4** (1) follows from (TL.2).

To prove (2), let $t \in V(D)$ with $P \cap \alpha(t) \neq \emptyset$. Let $u \supseteq_D t$ such that $P \cap \alpha(u) \neq \emptyset$ and $P \cap \alpha(x) = \emptyset$ for all $x \in N^D(u)$. Then $P \subseteq \gamma(u)$ by (1) and thus

$$P \subseteq \gamma(u) \setminus \bigcup_{x \in N^D(u)} \alpha(x) = \beta(u).$$

**Lemma 5.6.5.** For every d-scheme $\Lambda$ there is a parametrised d-scheme $\Lambda'(Z)$ such that the following holds. Let $G \in T_{\Lambda}$ be a connected graph and $P \subseteq V(G)$ a clique in $G$. Then

(i) $(G, P) \in T_{\Lambda'(Z)}$;

(ii) $V(\Lambda'[G, P]) \subseteq V(\Lambda[G])$;

(iii) for all $t \in V(\Lambda'[G, P])$ it holds that $\beta^{\Lambda'[G, P]}(t) = \beta^{\Lambda[G]}(t)$ and $\delta^{\Lambda'[G, P]}(t) = \delta^{\Lambda[G]}(t)$;

(iv) for all $t \in V(\Lambda'[G, P])$,

- either $\sigma^{\Lambda'[G, P]}(t) = \sigma^{\Lambda[G]}(t)$ and $\alpha^{\Lambda'[G, P]}(t) = \alpha^{\Lambda[G]}(t)$,
5.6. The Transitivity Lemma

- or there is a \( u \in N^A_G(t) \) such that \( \sigma^A_{t\neg D}(t) = \sigma^A_G(u) \) and \( \alpha^A_{t\neg D}(t) = V(G) \setminus \gamma^A_G(u) \),
- or \( \sigma^A_{t\neg D} = \emptyset \) and \( \alpha^A_{t\neg D} = V(G) \).

(v) for all \( \leq^A_{t\neg D} \)-minimal nodes \( s \) it holds that \( P \subseteq \beta^A_{t\neg D}(s) \).

Proof. The idea of the proof is simple. We take the minimal nodes of a decomposition whose bags contain the clique \( P \) and make them the new roots. To do this, we have to reverse edges in the obvious way. We will explain the transformation for the decomposition \( \Lambda[G] \) of a fixed graph \( G \), ignoring definability issues for most of the proof and just argue on a graph theoretic level. It will be clear that the whole transformation is definable in \( \text{IFP} \). Example 5.6.6 after the proof illustrates the construction.

Let \( G \in \mathcal{T}_\Lambda \) be a connected graph and \( P \subseteq V(G) \) a clique. Let \( \Delta := (D, \sigma, \alpha) := \Lambda[G] \). By the Normalisation Lemma for Definable Decompositions 5.2.3 we may assume that the decomposition \( \Delta \) is strict and normal. Let

\[
U_0 := \{ t \in V(D) \mid P \subseteq \beta(t), P \not\subseteq \beta(t') \text{ for any } t' \triangleleft_D t \}, \\
U_\downarrow := \{ t \in V(D) \mid t \triangleleft_D u \text{ for some } u \in U_0 \}, \\
U_\uparrow := V(D) \setminus U_\downarrow.
\]

Note that \( U_0 \subseteq U_\downarrow \).

Claim 1. For all \( t \in V(D) \) we have

\[
P \cap \alpha(t) = \emptyset \iff t \in U_\uparrow.
\]

Proof. If \( P \cap \alpha(t) = \emptyset \), then either \( P \not\subseteq \gamma(t) \) or \( P \subseteq \sigma(t) \). If \( P \not\subseteq \gamma(t) \), then by [(TL.3)] there is no \( u \triangleleft_D t \) with \( P \subseteq \beta(u) \subseteq \gamma(u) \). Hence \( t \not\in U_\uparrow \). If \( P \subseteq \sigma(t) \), then \( \sigma(t) \neq \emptyset \) and thus \( t \) is not \( \leq_D \)-minimal by [(TL.6)]. Let \( s \) be a parent of \( t \) in \( D \). Then by the \( \beta-\gamma-\sigma \)-Lemma 4.2.9 we have \( P \subseteq \sigma(t) \subseteq \beta(s) \). Suppose for contradiction that \( t \in U_\downarrow \). Let \( u \triangleright_D t \) such that \( u \in U_0 \). Then \( s \triangleleft_D u \), and since \( P \subseteq \beta(s) \), this contradicts the minimality of \( u \).

The converse direction follows from Lemma 5.6.6.

We define a directed graph \( D' \) as follows:

- \( V(D') := V(D) \);
- for all \( t, u \in V(D') \) we let \( tu \in E(D') \) if one of the following two conditions is satisfied:
  - \( tu \in E(D) \) and \( u \in U_\uparrow \);
  - \( ut \in E(D) \) and \( t \in U_\downarrow \).

Note that at most one of the two conditions is satisfied because \( D \) is acyclic.

Claim 2. The relation \( E(D') \) is antisymmetric, and it holds that

\[
\{ \{ t, u \} \mid tu \in E(D') \} = \{ \{ t, u \} \mid tu \in E(D) \}.
\]

(5.6.1)
Proof. Suppose for contradiction that $tu \in E(D')$ and $ut \in E(D')$. Then either $tu \in E(D)$ or $ut \in E(D)$, but not both because $D$ is acyclic. Without loss of generality we assume that $ut \in E(D)$ and $tu \not\in E(D)$. From $tu \in E(D')$, it follows that $t \in U\uparrow$, and from $ut \in E(D')$, it follows that $t \in U\downarrow$. Since $U\downarrow \cap U\uparrow = \emptyset$, this is a contradiction.

To prove Claim 6. observe that the inclusion $\subseteq$ is trivial. For the converse inclusion, suppose that $tu \in E(D)$. If $u \in U\uparrow$, then $tu \in E(D')$. Otherwise, $u \in U\downarrow$. Together with $tu \in E(D)$, this implies that $ut \in E(D')$.

Claim 3. $D'$ is acyclic.

Proof. Suppose for contradiction that $C \subseteq D'$ is a cycle. If $V(C) \subseteq U\uparrow$ then $E(C) \subseteq E(D)$ and thus $C \subseteq D$, which is a contradiction. Thus $V(C) \cap U\downarrow \neq \emptyset$. If $V(C) \subseteq U\downarrow$, then for all $tu \in E(C)$ it holds that $ut \in E(D)$. Hence $C^{-1} := \{tu \mid ut \in E(C)\}$ is a cycle in $D$, which is a contradiction again. Thus $V(C) \cap U\downarrow \neq \emptyset$ and $V(C) \cap U\uparrow \neq \emptyset$. Then there is an edge $tu \in E(C) \subseteq E(D')$ such that $t \in U\uparrow$ and $u \in U\downarrow$. But this contradicts the definition of $E(D')$.

Claim 4. Let $u \in V(D')$. Then either for all $t \in V(D')$ with $tu \in E(D')$ it holds that $tu \in E(D)$ or for all $t \in V(D')$ with $tu \in E(D')$ it holds that $ut \in E(D)$.

Proof. Suppose that $tu \not\in E(D)$ for some $t \in V(D')$ with $tu \in E(D')$. Then by the definition of $E(D')$ it holds that $ut \in E(D)$ and $t \in U\downarrow$. This implies $u \in U\downarrow$. Now let $t'u \in E(D')$. By the definition of $E(D')$, either $t'u \in E(D)$ and $u \in U\uparrow$ or $ut' \in E(D)$ and $t' \in U\downarrow$. The former contradicts $u \in U\downarrow$, hence the latter holds. This proves the claim.

Claim 5. Let $t_1, t_2, u \in V(D')$ such that for $j = 1,2$ it holds that $t_j u \in E(D')$ and $ut_j \in E(D)$. Then $t_1 \parallel t_2$.

Proof. By the definition of $E(D')$, we have $t_1, t_2 \in U\downarrow$. Hence there exist $s_1, s_2 \in U_0$ such that $t_j \preceq_s t_j$ for $j = 1,2$. Then $u \prec s_j$, and by the definition of $U_0$ we have $P \setminus \beta(u) \neq \emptyset$. As $P \subseteq \gamma(s_j) \subseteq \gamma(t_j)$ and $\sigma(t_j) \subseteq \beta(u)$, it follows that $P \setminus \beta(u) \subseteq \alpha(t_j)$ for $j = 1,2$. Now the claim follows from [TL.4].

Claim 6. Let $t \in V(D')$. Then $t$ is $\preceq_{D'}$-minimal if and only if $t \in U_0$.

Proof. For the forward direction, suppose for contradiction that $t \in V(D') \setminus U_0$ is $\preceq_{D'}$-minimal.

Case 1: $t \in U\downarrow$.

Then there is a $u \in U_0$ such that $t \prec_D u$. We have $u \prec_D t$, which contradicts the minimality of $t$.

Case 2: $t \in U\uparrow$ and $P \not\subseteq \gamma(t)$.

Then $\gamma(t) \neq V(G)$, and as $G$ is connected and the decomposition $\Delta$ is normal, $t$ is not $\preceq_{U\downarrow}$-minimal. Let $s \in V(D)$ such that $st \in E(D)$. Then $st \in E(D')$, because $t \in U\uparrow$. Thus $t$ is not $\preceq_{D'}$-minimal. This is a contradiction.

Case 3: $t \in U\uparrow$ and $P \subseteq \gamma(t)$.

If $P \cap \alpha(t) \neq \emptyset$, then by Lemma 5.6.4(2) there is a $u \triangleright_D t$ such that $u \in U_0$. Then $t \in U\downarrow$, a contradiction. Hence $P \cap \alpha(t) = \emptyset$ and thus $P \subseteq \sigma(t) \subseteq \beta(t)$. Let $s \in U_0$ such that $s \preceq_D t$. Then $s \preceq_D t$, because $t \in U\uparrow$. This implies $s \prec_D t$, which contradicts the $\preceq_{D'}$-minimality of $t$.

M. Grohe, Definable Graph Structure Theory
This completes the proof of the forward direction.

To prove the backward direction, let $t \in U_0$. Suppose for contradiction that $t$ is not $\leq D'$-minimal, and let $s \in V(D')$ such that $st \in E(D')$. As $t \in U_0$ and thus $t \notin U \uparrow$, it follows from the definition of $E(D')$ that $ts \in E(D)$ and $s \in U \downarrow$. Let $s' \in U_0$ such that $s \leq D s'$. Then $t \leq D s'$ and $t, s' \in U_0$. This is a contradiction.

Let $t \in V(D')$. If $t$ is $\leq D'$-minimal, we let $\sigma'(t) := \emptyset$ and $\alpha'(t) := V(G)$. If $st \in E(D')$ for some $s \in V(D')$, we let

$$\sigma'(t) := \begin{cases} 
\sigma(t) & \text{if } st \in E(D), \\
\sigma(s) & \text{if } ts \in E(D),
\end{cases}$$

$$\alpha'(t) := \begin{cases} 
\alpha(t) & \text{if } st \in E(D), \\
V(G) \setminus \gamma(s) & \text{if } ts \in E(D).
\end{cases}$$

Then $\sigma'(t)$ and $\alpha'(t)$ are well-defined by Claims 4 and 5. We let $\Delta' := (D', \sigma', \alpha')$. Our next goal, which will achieved in Claim 7, is to prove that $\Delta'$ is a treelike decomposition of $G$.

**Claim 7.** Let $t \in V(D')$.

1. If $t \in U \uparrow$ then $\sigma'(t) = \sigma(t)$ and $\alpha'(t) = \alpha(t)$ and $\gamma'(t) = \gamma(t)$.
2. If $t \in U \downarrow \setminus U_0$, then $N^D_+(t) \cap U \downarrow \neq \emptyset$, and for all $u \in N^D_+(t) \cap U \downarrow$ we have $\sigma'(t) = \sigma(u)$ and $\alpha'(t) = V(G) \setminus \gamma(u)$. And $\gamma'(t) = V(G) \setminus \alpha(u)$.
3. If $t \in U_0$ then $\sigma'(t) = \emptyset$, and $\alpha'(t) = \gamma'(t) = V(G)$.

**Proof.** All assertions follow easily from the definitions and Claims 4 and 5.

**Claim 8.** The decomposition $\Delta' = (D', \sigma', \alpha')$ is treelike.

**Proof.** We have already proved Claim 3.

To prove (TL.2), let $t \in V(D')$. If $t \in U \uparrow$, then $\sigma'(t) = \sigma(t)$ and $\alpha'(t) = \alpha(t)$, and hence (TL.2) for $\Delta'$ follows from (TL.2) for $\Delta$. If $t \in U \downarrow \setminus U_0$, let $u \in N^D_+(t) \cap U \downarrow$. Then $\sigma'(t) = \sigma(u)$, and $\alpha'(t) = V(G) \setminus \gamma(u)$. Thus $N^G(\alpha'(t)) = \partial^G(\gamma(u)) \subseteq \sigma(u)$ by (TL.2) for $\Delta$. Finally, if $t \in U_0$ then $\sigma'(t) = \emptyset$ and $\alpha'(t) = V(G)$, and thus (TL.2) is trivially satisfied for $t$.

To prove (TL.3), let $t \in V(D')$ and $u \in N^D_+(t)$. We make a case distinction:

**Case 1:** $t \in U \uparrow$.

Then $\gamma'(t) = \gamma(t)$, $\alpha'(t) = \alpha(t)$. Furthermore, $u \in N^A_+(t)$ and $u \in U \uparrow$, and thus $\gamma'(u) = \gamma(u)$, $\alpha'(u) = \alpha(u)$. Hence (TL.3) for $\Delta'$ follows from (TL.3) for $\Delta$.

**Case 2:** $t \in U \downarrow \setminus U_0$ and $u \in U \uparrow$.

Then $u \in N^D_+(t)$. Let $x \in N^A_+(t) \cap U \downarrow$. Moreover, $u \parallel^A x$, because $P \cap \alpha(x) \neq \emptyset$ and $P \cap \alpha(u) = \emptyset$ by Claim 1. Hence $\gamma(u) \cap \alpha(x) = \emptyset$ by (TL.4) for $\Delta$ and thus, by (TL.3) for $\Delta$ and Claim 7,

$$\gamma'(u) = \gamma(u) \subseteq \gamma(t) \setminus \alpha(x) \subseteq V(G) \setminus \alpha(x) = \gamma'(t),$$

and similarly $\alpha'(u) \subseteq \alpha'(t)$. 

Preliminary Version
Case 3: \( t \in U \downarrow \setminus U_0 \) and \( u \in U \downarrow \).

Then \( t \in N^D_+(u) \), and hence by Claim 7, \( \gamma'(u) = V(G) \setminus \alpha(t) \) and \( \alpha'(u) = V(G) \setminus \gamma(t) \). Let \( x \in N^A(t) \cap U \downarrow \). Then by Claim 7, we have \( \gamma'(t) = V(G) \setminus \alpha(x) \) and \( \alpha'(t) = V(G) \setminus \gamma(x) \). Hence \( |\text{TL.3}| \) for \( \Delta' \) at the edge \( tu \) follows from \( |\text{TL.3}| \) for \( \Delta \) at the edge \( tx \).

Case 4: \( t \in U_0 \).

Then \( \gamma'(t) = \alpha'(t) = V(G) \), and \( |\text{TL.3}| \) is trivial.

To prove \( |\text{TL.4}| \) let \( t \in V(D') \) and \( u_1, u_2 \in N^D_+(t) \). If \( u_1, u_2 \in U \uparrow \), then \( u_1, u_2 \in N^D_+(t) \) and \( \sigma'(t) = \sigma(u_1), \alpha'(u_1) = \alpha(u_1) \) for \( i = 1, 2 \). In this case, \( |\text{TL.4}| \) for \( \Delta' \) follows from \( |\text{TL.4}| \) for \( \Delta \) (even if \( t \in U \downarrow \)). Hence without loss of generality we may assume that \( u_1 \in U \downarrow \).

Then \( t \in N^D_+(u_1) \cap U \downarrow \), and therefore \( \sigma'(u_1) = \sigma(t) \) and \( \alpha'(u_1) = V(G) \setminus \gamma(t) \) and \( \gamma'(u_1) = V(G) \setminus \alpha(t) \).

Case 1: \( u_2 \in U \downarrow \).

Then \( t \in N^D_+(u_2) \cap U \downarrow \), and therefore \( \sigma'(u_2) = \sigma(t) = \sigma(u_1) \) and \( \alpha'(u_2) = V(G) \setminus \gamma(t) = \alpha(u_1) \). Hence \( u_1 \parallel^\Delta u_2 \).

Case 2: \( u_2 \in U \uparrow \).

Then \( u_2 \in N^D_+(t) \) and \( \gamma'(u_2) = \gamma(u_2) \subseteq \gamma(t), \alpha'(u_2) = \alpha(u_2) \subseteq \alpha(t) \). Hence \( \gamma'(u_1) \cap \alpha'(u_2) = \emptyset \) and \( \alpha'(u_1) \cap \gamma'(u_2) = \emptyset \), which implies \( u_1 \perp^\Delta u_2 \).

Finally, to prove \( |\text{TL.5}| \) let \( t \in U_0 \). Then \( \sigma'(t) = \emptyset \) and \( \alpha'(t) = V(G) \) by Claim 7(3). (Recall that we assumed \( G \) to be connected.)

Claim 9. For all \( t \in V(D') \) it holds that \( \beta'(t) = \beta(t) \) and \( \delta'(t) = \delta(t) \).

Proof. For \( t \in U \uparrow \), the claim follows from Claim 7(1). Let \( t \in U \downarrow \). Let \( s_1, \ldots, s_n \) be the parents of \( t \) in \( D \). Let \( N^D_+(t) \cap U \downarrow = \{ u_1, \ldots, u_m \} \) and \( N^D_+(t) \cap U \uparrow = \{ t_1, \ldots, t_\ell \} \). Then

\[
N^D_\uparrow(t) = \{ s_1, \ldots, s_n, t_1, \ldots, t_\ell \}.
\]

Note that for every \( i \in [n] \) we have \( \sigma'(s_i) = \sigma(t) \) and \( \alpha'(s_i) = V(G) \setminus \gamma(t) \) and \( \gamma'(s_i) = V(G) \setminus \alpha(t) \).

Case 1: \( t \in U \downarrow \setminus U_0 \).

Then \( m \geq 1 \), and for every \( j \in [m] \) we have \( \sigma'(t) = \sigma(u_j) \) and \( \alpha'(t) = V(G) \setminus \gamma(u_j) \) and \( \gamma'(t) = V(G) \setminus \alpha(u_j) \). Hence

\[
\beta'(t) = \gamma'(t) \setminus \left( \bigcup_{i=1}^n \alpha'(s_i) \cup \bigcup_{i=1}^\ell \alpha'(t_i) \right)
= (V(G) \setminus \alpha(u_1)) \setminus \left( (V(G) \setminus \gamma(t)) \cup \bigcup_{i=1}^\ell \alpha(t_i) \right)
= \gamma(t) \setminus \left( \alpha(u_1) \cup \bigcup_{i=1}^\ell \alpha(t_i) \right)
= \gamma(t) \setminus \left( \bigcup_{j=1}^m \alpha(u_j) \cup \bigcup_{i=1}^\ell \alpha(t_i) \right)
\]

M. Grohe, Definable Graph Structure Theory
5.6. The Transitivity Lemma

\[ = \beta(t). \]

It remains to prove that \( \delta'(t) = \delta(t) \). The set \( \delta(t) \) consists of all nonempty maximal sets in

\[ \{ \sigma(t) \} \cup \{ \sigma(u_j) \mid j \in [m] \} \cup \{ \sigma(t_i) \mid i \in [\ell] \}. \tag{5.6.2} \]

The set \( \delta'(t) \) consists of all nonempty maximal sets in

\[ \{ \sigma'(t) \} \cup \{ \sigma'(s_i) \mid i \in [n] \} \cup \{ \sigma'(t_i) \mid i \in [\ell] \}. \tag{5.6.3} \]

As \( \sigma'(s_i) = \sigma(t) \) for all \( i \in [n] \), \( \sigma'(u_j) = \sigma'(t) \) for all \( j \in [m] \), and \( \sigma'(t_i) = \sigma'(t_i) \) for all \( i \in [\ell] \), the two sets in (5.6.2) and (5.6.3) are equal. Thus \( \delta(t) = \delta'(t) \).

**Case 2:** \( t \in U_0 \).

Then \( m = 0 \) and \( \sigma'(t) = \emptyset \) and \( \alpha'(t) = V(G) \). Hence

\[
\beta'(t) = \gamma'(t) \setminus \left( \bigcup_{i=1}^{n} \alpha'(s_i) \cup \bigcup_{i=1}^{\ell} \alpha'(t_i) \right)
\]

\[
= V(G) \setminus \left( (V(G) \setminus \gamma(t)) \cup \bigcup_{i=1}^{\ell} \alpha(t_i) \right)
\]

\[
= \gamma(t) \setminus \bigcup_{i=1}^{\ell} \alpha(t_i)
\]

\[
= \beta(t).
\]

To prove that \( \delta'(t) = \delta(t) \), note that \( \delta(t) \) consists of all nonempty maximal sets in

\[ \{ \sigma(t) \} \cup \{ \sigma(t_i) \mid i \in [\ell] \} \tag{5.6.4} \]

and \( \delta'(t) \) consists of all nonempty maximal sets in

\[ \{ \sigma'(s_i) \mid i \in [n] \} \cup \{ \sigma'(t_i) \mid i \in [\ell] \}. \tag{5.6.5} \]

As \( \sigma'(s_i) = \sigma(t) \) for all \( i \in [m] \) and \( \sigma(t_i) = \sigma'(t_i) \) for all \( i \in [\ell] \), the two sets in (5.6.4) and (5.6.5) are equal. Thus \( \delta(t) = \delta'(t) \).

**Claim 10.** For all \( \preceq \Delta' \)-minimal nodes \( t \in V(D') \) it holds that \( P \subseteq \beta'(t) \).

**Proof.** Let \( t \in V(D') \) be \( \preceq \Delta' \)-minimal. By Claim 6 we have \( t \in U_0 \). Then \( P \subseteq \beta(t) \) by the definition of \( U_0 \) and hence \( P \subseteq \beta'(t) \) by Claim 9. \( \square \)

It is straightforward to define a d-scheme \( \Lambda' \) (not depending on \( G \)) such that \( \Lambda'[G] = \Delta' \). It follows immediately from the construction that (ii) is satisfied. Assertion (i) follows from Claim 8, (iii) follows from Claim 9, (iv) follows from Claim 7, and (v) follows from Claim 10. This completes the proof of the lemma. \( \square \)
To explain the proof, we fix a graph $G$. Let $D := D^2$. Recall that for every node $v \in V(D) \subseteq V(C_4)$ it holds that $\beta(v) = v$. Let $P = \{3, 4\}$. Then $P$ is a clique in $C_4$. Figure 5.1(a) shows the graph $D$ with all $\leq D$-minimal nodes $v \in V(D)$ with $P \subseteq \beta(v)$ in black.

Let us now apply the construction of the proof of Lemma 5.6.5. The black nodes in Figure 5.1(a) are those in the set $U_0$. Figure 5.1(a) shows that graph $D'$ with all nodes in the set $U_0$ in black and all nodes in $U \setminus U_0$ in grey.

**Example 5.6.6.** Consider the decomposition $\Delta$ of the cycle $C_4$ introduced in Example 5.2.4. Let $G \subseteq V(C_4)$. Then $G$ is a clique in $C_4$. Figure 5.1(a) shows the graph $D$ with all $\leq D$-minimal nodes $v \in V(D)$ with $P \subseteq \beta(v)$ in black.

Let us now apply the construction of the proof of Lemma 5.6.5. The black nodes in Figure 5.1(a) are those in the set $U_0$. Figure 5.1(b) shows that graph $D'$ with all nodes in the set $U_0$ in black and all nodes in $U \setminus U_0$ in grey.

**Proof of the Decomposition Lifting Lemma 5.6.2.** To explain the proof, we fix a graph $G \subseteq V(C_4)$ such that for every $v \in V(C_4)$ it holds that $\beta(v) = v$. Let $P = \{3, 4\}$. Then $P$ is a clique in $C_4$. Figure 5.1(a) shows the graph $D$ with all $\leq D$-minimal nodes $v \in V(D)$ with $P \subseteq \beta(v)$ in black.

Let us now apply the construction of the proof of Lemma 5.6.5. The black nodes in Figure 5.1(a) are those in the set $U_0$. Figure 5.1(b) shows that graph $D'$ with all nodes in the set $U_0$ in black and all nodes in $U \setminus U_0$ in grey.

In a first step of the proof, we apply (a parametrised Version of) Lemma 5.6.5 to the d-scheme $\Lambda^2(\overline{\Delta}_1)$ and obtain a parameterised d-scheme $\Lambda^3(\overline{\Delta}_1, Z)$. Then for every $v_1 \in V(D^1)$ and every clique $P$ of $H_{\overline{\Delta}_1}$, the scheme $\Lambda^3(\overline{\Delta}_1, Z)$ defines a tree-like decomposition of $H_{\overline{\Delta}_1}$, which is a clique in $H_{\overline{\Delta}_1}$. In each of the formulae of the d-scheme $\Lambda^3$, we replace each atomic subformula of the form $Z(z)$ by the formula $\Lambda^3(\overline{\Delta}_1, z)$. Let $\Lambda^3(\overline{\Delta}_1)$ be the resulting d-scheme. Then for every $v_1 \in V(D^1)$, the scheme $\Lambda^3(\overline{\Delta}_1)$ defines a tree-like decomposition $\Delta^3_{\overline{\Delta}_1} = (D^3_{\overline{\Delta}_1}, \alpha^3_{\overline{\Delta}_1}, \alpha^3_{\overline{\Delta}_1})$ of the torso $H_{\overline{\Delta}_1}$ within $(G, v_1)$, and by Lemma 5.6.5 this decomposition has the following properties:

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M. Grohe, Definable Graph Structure Theory
(A) \( V(D_{t_1}^1) \subseteq V(D_{t_1}^2) \);

(B) for all \( v_2 \in V(D_{t_1}^2) \) it holds that \( \beta_{t_1}^4(v_2) = \beta_{t_1}^2(v_2) \) and \( \delta_{t_1}^4(v_2) = \delta_{t_1}^2(v_2) \);

(C) for all \( v_2 \in V(D_{t_1}^4) \),

\[
\begin{align*}
\text{either } & \sigma_{t_1}^4(v_2) = \sigma_{t_1}^2(v_2) \text{ and } \alpha_{t_1}^4(v_2) = \alpha_{t_1}^2(v_2), \\
\text{or there is a } & \bar{w}_2 \in N_{t_1}^2(v_2) \text{ such that } \sigma_{t_1}^4(v_2) = \sigma_{t_1}^2(w_2) \text{ and } \alpha_{t_1}^4(v_2) = \beta_1(t_1) \setminus \gamma_{t_1}^2(w_2), \\
\text{or } & \sigma_{t_1}^4(v_2) = \emptyset \text{ and } \alpha_{t_1}^4(v_2) = \beta_1(v_1); \end{align*}
\]

(D) for all \( \leq_{t_1}^4 \)-minimal nodes \( w_2 \) it holds that \( \sigma_1(v_1) \subseteq \beta_{t_1}^4(w_2) \).

By the Normalisation Lemma 5.2.3 we may further assume that the decomposition \( \Delta_{t_1}^4 \) is strict and normal.

To simplify the notation, in the following we denote nodes of the decompositions by \( t, u \) and variants like \( t_1, u_2 \) instead of \( v, \bar{v}, v_1, \bar{w}_2 \).

In the main step of the proof, we take a product of \( \Delta_1 \) with the decompositions \( \Delta_{t_1}^4 \) of the torsos \( H_{t_1} \). We define a new decomposition \( \Delta = (D, \sigma, \alpha) \) as follows.

(E) \( V(D) := \{ t_1 t_2 \mid t_1 \in V(D^1) \text{ and } t_2 \in V(D_{t_1}^4) \} \).

(F) For all \( t_1 t_2, u_1 u_2 \in V(D) \) there is an edge \( (t_1 t_2, u_1 u_2) \in E(D) \) if

\[
\begin{align*}
\text{either } & t_1 = u_1 \text{ and } (t_2, u_2) \in E(D_{t_1}^1), \\
\text{or } (t_1, u_1) \in E(D^1) \text{ and } \sigma_1(u_1) \subseteq \beta_{t_1}^4(t_2) \text{ and } \sigma_1(u_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset \text{ and } u_2 \text{ is} \leq_{t_1}^4-\text{minimal}. 
\end{align*}
\]

(G) For all \( t_1 t_2 \in V(D) \), we let

\[
\sigma(t_1 t_2) := \sigma_{t_1}^4(t_2) \cup (\sigma_1(t_1) \cap \alpha_{t_1}^4(t_2)),
\]

\[
\alpha(t_1 t_2) := (\alpha_{t_1}^4(t_2) \setminus \sigma_1(t_1)) \cup \bigcup_{u_1} \alpha_1(u_1).
\]

where the union ranges over all \( u_1 \in N_{t_1}^1(t_1) \) with \( \sigma_1(u_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset \).

Note that for all \( t_1 t_2 \in V(D) \) we have \( \sigma_{t_1}^4(t_2) \subseteq \sigma(t_1 t_2) \subseteq \gamma_{t_1}^4(t_2) \) and \( \alpha(t_1 t_2) \subseteq \gamma_{t_1}^1(t_2) \). This implies

\[
\gamma(t_1 t_2) = \gamma_{t_1}^4(t_2) \cup \bigcup_{u_1 \in N_{t_1}^1(t_1)} \alpha_1(u_1). \quad (5.6.6)
\]

Claim 1. Let \( t_1 t_2 \in V(D) \). Then:

(1) If \( t_2 \) is \( \leq_{t_1}^4 \)-minimal, then \( \sigma(t_1 t_2) = \sigma_1(t_1) \) and \( \alpha(t_1 t_2) = \alpha_1(t_1) \) and \( \gamma(t_1 t_2) = \gamma_1(t_1) \).

(2) If \( t_2 \) is not \( \leq_{t_1}^4 \)-minimal, then \( \sigma(t_1 t_2) = \sigma_{t_1}^4(t_2) \) and \( \alpha(t_1 t_2) = \alpha_{t_1}^4(t_2) \).
Proof. To prove (1), suppose that $t_2$ is $\leq_{\Delta_1^4}$-minimal. Then $\sigma^4_{\Delta_1^4}(t_2) = \emptyset$, because $\Delta_1^4$ is normal. Thus $\gamma_{\Delta_1^4}(t_2) = \alpha^4_{\Delta_1^4}(t_2) = V(H_{t_1}) = \beta^1(t_1)$, because $H_{t_1}$ is connected. This yields $\sigma(t_1t_2) = \sigma^1(t_1)$ by the definition of $\sigma$. Moreover, for all $u_1 \in N^1_+(t_1)$ we have $\sigma^1(u_1) \neq \emptyset$ by Lemma 4.3.8. Thus $\alpha^1(u_1) \cap \alpha_{\Delta_1^4}^1(t_2) = \sigma^1(u_1) \neq \emptyset$, which implies $\alpha^1(u_1) \subseteq \alpha(t_1t_2)$. Thus $\gamma(t_1t_2) = \beta^1(t_1) \cup \bigcup_{u_1 \in N^1_+(t_1)} \alpha^1(u_1) = \gamma^1(t_1)$.

To prove (2), suppose that $t_2$ is not $\leq_{\Delta_1^4}$-minimal. Let $s_2, s_1 \subseteq t_2$ be $\leq_{\Delta_1^4}$-minimal. Then by (D) we have $\sigma^1(s_1) \subseteq \beta^1_{\Delta_1^4}(s_2)$. By the $\beta$-$\gamma$-Lemma 4.2.9, it follows that $\sigma^1(t_1) \cap \gamma_{\Delta_1^4}(t_2) \subseteq \sigma^1_{\Delta_1^4}(t_2)$, and this implies assertion (2).

Claim 2. The decomposition $\Delta = (D, \sigma, \alpha)$ is treelike.

Proof. Let $t_1t_2 \in V(D)$. It is an immediate consequence of the definitions of $\sigma$ and $\alpha$ that $\sigma(t_1t_2) \cap \alpha(t_1t_2) = \emptyset$. To prove that $N^G(\alpha(t_1t_2)) \subseteq \sigma(t_1t_2)$, let $v \in \alpha(t_1t_2)$, and $w \in N^G(v)$. We shall prove that $w \in \gamma(t_1t_2)$. Suppose first that $v \in \alpha_{\Delta_1^4}(t_2)$. If $w \in V(H_{t_1})$ then by (TL.2) we have $w \in \gamma_{\Delta_1^4}(t_2) \subseteq \gamma(t_1t_2)$. If $w \in \alpha^1(u_1)$ for some $u_1 \in N^1_+(t_1)$, then $v \in \alpha^1(u_1)$ by (TL.2) for $\Delta^4_1$ and thus $\alpha^1(u_1) \cap \alpha_{\Delta_1^4}(t_2) = \emptyset$. This implies $w \in \alpha^1(u_1) \subseteq \alpha(t_1t_2)$. Suppose next that $v \in \alpha^1(u_1)$ for some $u_1 \in N^1_+(t_1)$ with $\sigma(u_1) \cap \alpha_{\Delta_1^4}(t_2) = \emptyset$. As $\sigma(u_1)$ is a clique in the torso $H_{t_1}$ and $\sigma^1(u_1) \cap \alpha_{\Delta_1^4}(t_2) = \emptyset$, by (TL.2) for $\Delta^4_1$ we have $\sigma^1(u_1) \subseteq \gamma_{\Delta_1^4}(t_2) \subseteq \gamma(t_1t_2)$. By (TL.2) for $\Delta^4_1$ we have $N^{G}(v) \subseteq \gamma^1(u_1) \subseteq \gamma(t_1t_2)$.

To prove (TL.3) let $t_1t_2 \in V(D)$ and $u_1u_2 \in N^D_+(t_1t_2)$. If $t_1 = u_1$, then it follows from (TL.3) for $\Delta^4_1$ that $\alpha(u_1u_2) \subseteq \alpha(t_1t_2)$ and $\gamma(u_1u_2) \subseteq \gamma(t_1t_2)$. Suppose that $u_1 \in N^D_+(t_1)$. Then $u_2$ is $\leq_{\Delta_1^4}$-minimal, and thus by Claim 1 we have $\alpha(u_1u_2) = \alpha^1(u_1)$ and $\sigma(u_1u_2) = \alpha^1(u_1)$. Since $u_1u_2 \in N^D_+(t_1t_2)$, we have $\sigma(u_1u_2) = \sigma^1(u_1 \subseteq \beta_{\Delta_1^4}(t_2) \subseteq \gamma(t_1t_2)$, and $\sigma(u_1) \cap \alpha_{\Delta_1^4}(t_2) = \emptyset$, which implies $\alpha(u_1u_2) = \alpha^1(u_1) \subseteq \alpha(t_1t_2)$.

To prove (TL.4) let $t_1t_2 \in V(D)$ and $u_1^1u_2^1, u_1^2u_2^2 \in N^D_+(t_1t_2)$.

Case 1: $t_1 = u_1^1 = u_2^1$.

Then $u_1^1, u_2^1 \in N^4_+(t_1)$ and thus by (TL.4) for $\Delta^4_1$ either $u_2^1 \not\perp_{\Delta^4_1} u_2^2$ or $u_2^1 \perp_{\Delta^4_1} u_2^2$. If $u_2^1 \not\perp_{\Delta^4_1} u_2^2$, then clearly $u_2^1 \perp_{\Delta^4_1} u_2^2$.

So suppose that $u_2^1 \perp_{\Delta^4_1} u_2^2$. Then there is no $x_1 \in N^1_+(t_1)$ such that $\sigma^1(x_1) \cap \alpha_{\Delta_1^4}(u_2^1) = \emptyset$ and $\sigma^1(x_1) \cap \alpha_{\Delta_1^4}(u_2^2) = \emptyset$. To see this, suppose that for some $x_1 \in N^1_+(t_1)$ we have $\sigma^1(x_1) \cap \alpha_{\Delta_1^4}(u_2^1) = \emptyset$. Then $\sigma^1(x_1) \subseteq \gamma_{\Delta_1^4}(u_2^1)$, because $\sigma^1(x_1)$ is a clique in $H_{t_1}$. But $u_2^1 \perp_{\Delta^4_1} u_2^2$ implies $\gamma_{\Delta_1^4}(u_2^1) \cap \alpha_{\Delta_1^4}(u_2^2) = \emptyset$, which implies $\sigma^1(x_1) \cap \alpha_{\Delta_1^4}(u_2^2) = \emptyset$.

Thus for each $x_1 \in N^1_+(t_1)$, either $\alpha(x_1) \cap \gamma(u_2^1u_2^2) = \emptyset$ or $\alpha^1(x_1) \cap \gamma(u_2^1u_2^2) = \emptyset$ and thus $\gamma(u_2^1u_2^2) \cap \gamma(u_2^1u_2^2) = \gamma_{\Delta_1^4}(u_2^1) \cap \gamma_{\Delta_1^4}(u_2^2) = \sigma^4_{\Delta_1^4}(u_2^1) \cap \sigma^4_{\Delta_1^4}(u_2^2) = \emptyset$. As we trivially have $\sigma(u_2^1u_2^2) \subseteq \gamma(u_2^1u_2^2) \cap \gamma(u_2^1u_2^2)$, equality holds.

Case 2: $t_1 = u_1^1 \neq u_2^1$.

Then $u_2^2$ is $\leq_{\Delta_1^4}$-minimal, and thus by Claim 1 we have $\sigma(u_2^1u_2^2) = \sigma^1(u_1^1)$ and $\alpha(u_2^1u_2^2) = \alpha^1(u_1^1)$. Furthermore, $\sigma^1(u_1^1) \subseteq \beta_{\Delta_1^4}(t_2)$ (and thus $\sigma^1(u_1^1) \cap \alpha_{\Delta_1^4}(u_2^2) = \emptyset$). This immediately implies $\sigma(u_2^1u_2^2) \cap \alpha(u_1^1u_2^2) = \emptyset$, and by the definition of $\alpha$ it also implies $\alpha(u_2^1u_2^2) \cap \alpha(u_1^1u_2^2) = \emptyset$. Furthermore, we have $\alpha(u_1^1u_2^2) \cap \sigma(u_1^1u_2^2) \subseteq \alpha(u_1^1u_2^2) \cap \beta_{\Delta_1^4}(t_2) = \emptyset$. Overall, this yields $\gamma(u_1^1u_2^1) \cap \gamma(u_1^1u_2^2) = \sigma(u_1^1u_2^2) \cap \sigma(u_1^1u_2^2)$.

Case 3: $t_1 = u_1^2 \neq u_1^1$.

Symmetric to Case 2.
Case 4: \( t_1 \neq u_1^4 \) and \( t_1 \neq u_2^4 \).

Then \( u_1^4, u_2^4 \in N_{+}^4(t_1) \) and the assertion follows from Claim 1 and \((\text{TL.4})\) for \( \Delta^1 \).

To prove \((\text{TL.5})\), let \( A \) be a connected component of \( G \). Let \( t_1 \in V(D^1) \) such that \( \sigma^1(t_1) = \emptyset \) and \( \alpha(t_1) = V(A) \). Let \( t_2 \in V(D^4_{t_1}) \) be \( \leq t_2^4 \)-minimal. Then by Claim 1 we have \( \sigma(t_1t_2) = \emptyset \) and \( \alpha(t_1t_2) = V(A) \).

Claim 3. For all \( t_1t_2 \in V(D) \),

\[
\bigcup_{u_1u_2 \in N_{+}^4(t_1t_2)} \alpha(u_1u_2) = \bigcup_{u_2 \in N_{+}^4_{t_1}(t_2)} \alpha_{t_1}^4(u_2) \cup \bigcup_{x_1 \in N_{+}^4_{t_1}(t_1)} \alpha^1(x_1). \tag{5.6.7}
\]

Proof. For the \( \subseteq \) inclusion, consider a \( u_1u_2 \in N_{+}^4(t_1t_2) \). If \( u_1 = t_1 \), then \( u_2 \in N_{+}^4_{t_1}(t_2) \) and

\[
\alpha(u_1u_2) \subseteq \alpha_{t_1}^4(u_2) \cup \bigcup_{x_1 \in N_{+}^4_{t_1}(t_1)} \alpha^1(x_1),
\]

which is contained in the set on the left-hand side of \((5.6.7)\), because \( \alpha_{t_1}^4(u_2) \subseteq \alpha_{t_1}^4(t_2) \).

Otherwise, \( u_1 \in N^1(t_1) \) with \( \sigma^1(u_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset \). Then \( \alpha(u_1u_2) = \alpha^1(u_1) \), which is contained in the set on the left-hand side of \((5.6.7)\).

For the converse inclusion \( \supseteq \), note first that for all \( u_2 \in N_{+}^4_{t_1}(t_2) \) we have \( \alpha_{t_1}^4(u_2) \subseteq \alpha(t_1u_2) \) and \( t_1u_2 \in N_{+}^4(t_1t_2) \). Let \( x_1 \in N_{+}^4_{t_1}(t_1) \) such that \( \sigma^1(x_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset \). If \( \sigma^1(x_1) \subseteq \beta_{t_1}^4(t_2) \), let \( x_2 \) be \( \leq t_2^4 \)-minimal. Then \( x_1x_2 \in N_{+}^4(t_1t_2) \) and \( \alpha(x_1x_2) = \alpha^1(x_1) \). Suppose that \( \sigma^1(x_1) \not\subseteq \beta_{t_1}^4(t_2) \). Since \( \sigma^1(x_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset \) and \( \sigma^1(x_1) \) is a clique in \( H_{t_1} \), we have \( \sigma^1(x_1) \subseteq \sigma_{t_1}^4(t_2) \). Thus there is a \( u_2 \in N_{+}^4_{t_1}(t_2) \) such that \( \sigma^1(x_1) \cap \alpha_{t_1}^4(u_2) \neq \emptyset \). Then \( \alpha^1(x_1) \subseteq \alpha(t_1u_2) \) and \( t_1u_2 \in N_{+}^4(t_1t_2) \).

Claim 4. For all \( t_1t_2 \in V(D) \) it holds that \( \tau(t_1t_2) = \tau_{t_1}^4(t_2) \).

Proof. We have

\[
\beta(t_1t_2) = \gamma(t_1t_2) \setminus \bigcup_{u_1u_2 \in N_{+}^4(t_1t_2)} \alpha(u_1u_2)
\]

\[
= \gamma(t_1t_2) \setminus \left( \bigcup_{u_2 \in N_{+}^4_{t_1}(t_2)} \alpha_{t_1}^4(u_2) \cup \bigcup_{x_1 \in N_{+}^4_{t_1}(t_1)} \alpha^1(x_1) \right) \quad \text{by Claim 3}
\]

\[
= \left( \gamma_{t_1}^4(t_2) \cup \bigcup_{x_1 \in N_{+}^4_{t_1}(t_1)} \alpha^1(x_1) \right) \setminus \left( \bigcup_{u_2 \in N_{+}^4_{t_1}(t_2)} \alpha_{t_1}^4(u_2) \cup \bigcup_{x_1 \in N_{+}^4_{t_1}(t_1)} \alpha^1(x_1) \right)
\]

\[
= \gamma_{t_1}^4(t_2) \setminus \bigcup_{u_2 \in N_{+}^4_{t_1}(t_2)} \alpha_{t_1}^4(u_2)
\]

\[
= \beta_{t_1}^4(t_2).
\]

To prove that the torsos \( \tau(t_1t_2) \) and \( \tau_{t_1}^4(t_2) \) are equal, we make a case distinction.
Case 1: $t_2$ is not $\leq_{t_1}^4$-minimal.

Then $\sigma(t_1 t_2) = \sigma_{t_1}^4(t_2)$. Hence

$$\tau(t_1 t_2) = G(\beta(t_1 t_2)) \cup K(\sigma(t_1 t_2)) \cup \bigcup_{u_1 u_2 \in N^4_{t_1}(t_1 t_2)} K(\sigma(u_1 u_2))$$

$$= H_{t_1}[\beta_{t_1}^4(t_2)] \cup K[\sigma_{t_1}^4(t_2)] \cup \bigcup_{u_2 \in N^4_{t_1}(t_2)} K[\sigma_{t_1}^4(u_2)] \quad \text{(proved below)}$$

$$= \tau_{t_1}^4(t_2).$$

To establish the second equality, first recall that $\beta(t_1 t_2) = \beta_{t_1}^4(t_2)$. The inclusion “$\subseteq$” holds, because for $u_1 u_2 \in N^4_{t_1}(t_1 t_2)$ with $u_1 \neq t_1$ the set $\sigma(u_1 u_2) = \sigma^1(u_1)$ is a clique in $H_{t_1}$, and for $u_1 u_2 \in N^4_{t_1}(t_1 t_2)$ with $u_1 = t_1$ we have $u_2 \in N^4_{t_1}(t_2)$ and $\sigma(u_1 u_2) = \sigma_{t_1}^4(u_2)$ by Claim 1. For the converse inclusion “$\supseteq$”, we have to prove that every edge of the graph $H_{t_1}[\beta_{t_1}^4(t_2)]$ that is not an edge of $G[\beta_{t_1}^4(t_2)]$ is an edge of either $K[\sigma(t_1 t_2)]$ or $K[\sigma(u_1 u_2)]$ for some $u_1 u_2 \in N^4_{t_1}(t_1 t_2)$. Suppose that $e \in E(H_{t_1}[\beta_{t_1}^4(t_2)]) \setminus E(G[\beta_{t_1}^4(t_2)])$ and that $e \not\subseteq \sigma(t_1 t_2)$. We shall prove that $e \subseteq \sigma(u_1 u_2)$ for some $u_1 u_2 \in N^4_{t_1}(t_1 t_2)$. Since $e \in E(H_{t_1}[\beta_{t_1}^4(t_2)]) \setminus E(G[\beta_{t_1}^4(t_2)])$, either $e \not\subseteq \sigma^1(t_1)$ or $e \subseteq \sigma^1(u_1)$ for some $u_1 \in N^1_{t_1}(t_1)$. Since $e \not\subseteq \sigma(t_1 t_2)$, we have $e \not\subseteq \sigma^1(t_1)$, because $e \subseteq \beta_{t_1}^4(t_2) \subseteq \gamma_{t_1}^4(t_2)$ and $\sigma^1(t_1) \cap \gamma_{t_1}^4(t_2) \subseteq \sigma(t_1 t_2)$. Let $u_1 \in N^1_{t_1}(t_1)$ such that $e \subseteq \sigma^1(u_1)$. As $e \not\subseteq \sigma^1(t_1)$, we have $e \cap \alpha_{t_1}^4(t_2) \neq \emptyset$ and thus $\sigma^1(u_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset$.

If $\sigma^1(u_1) \cap \alpha_{t_1}^4(u_2) \neq \emptyset$ for some $u_2 \in N^4_{t_1}(t_2)$, then $\sigma^1(u_1) \subseteq \gamma_{t_1}^4(u_2)$, because $\sigma^1(u_1)$ is a clique in $H_{t_1}$. Thus $e \subseteq \beta_{t_1}^4(t_2) \cap \gamma_{t_1}^4(u_2) = \sigma_{t_1}^4(u_2)$ by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9.

Moreover, $t_1 u_2 \in N^4_{t_1}(t_2)$ with $\sigma_{t_1}^4(u_2) = \sigma(t_1 u_2)$.

Otherwise, $\sigma^1(u_1) \cap \alpha_{t_1}^4(u_2) = \emptyset$ for all $u_2 \in N^4_{t_1}(t_2)$. As $\sigma^1(u_1) \cap \alpha_{t_1}^4(t_2) \neq \emptyset$ and $\sigma^1(u_1)$ is a clique, we have $\sigma^1(u_1) \subseteq \gamma_{t_1}^4(t_2)$ and thus $\sigma^1(u_1) \subseteq \beta_{t_1}^4(t_2)$. Let $u_2 \in V(H_{t_1})$ be $\leq_{t_1}^4$-minimal. Then $u_1 u_2 \in N^4_{t_1}(t_1 t_2)$ and $\sigma(u_1 u_2) = \sigma^1(u_1) \supseteq e$ by Claim 1.

Case 2: $t_2$ is $\leq_{t_1}^4$-minimal.

Then $\sigma_{t_1}^4(t_2) = \emptyset$ and $\sigma(t_1 t_2) = \sigma^1(t_1)$. Hence

$$\tau(t_1 t_2) = G(\beta(t_1 t_2)) \cup K(\sigma(t_1 t_2)) \cup \bigcup_{u_1 u_2 \in N^4_{t_1}(t_1 t_2)} K(\sigma(t_1 t_2))$$

$$= H_{t_1}[\beta_{t_1}^4(t_2)] \cup \bigcup_{u_2 \in N^4_{t_1}(t_2)} K[\sigma_{t_1}^4(u_2)] \quad \text{(proved below)}$$

$$= \tau_{t_1}^4(t_2).$$

The proof of the second equality is analogous to the proof in Case 1. For the inclusion “$\subseteq$”, note that $K[\sigma(t_1 t_2)] = K[\sigma^1(t_1)] \subseteq H_{t_1}$.

Claim 5. For all $t_1 t_2 \in V(D_0)$,

- either $\sigma(t_1 t_2) = \sigma_{t_1}^2(t_2)$ and $\alpha(t_1 t_2) \cap \beta^1(t_1) = \alpha_{t_1}^2(t_2)$

- or there is a $u_2 \in N^2_{t_1}(t_2)$ such that $\sigma(t_1 t_2) = \sigma_{t_1}^2(u_2)$ and $\alpha(t_1 t_2) \cap \beta^1(t_1) = (\beta^1(t_1) \setminus \gamma_{t_1}^2(u_2))$.
5.6. The Transitivity Lemma

- or \( \sigma(t) = \sigma^1(t_1) \) and \( \alpha(t) = \alpha^1(t_1) \).

**Proof.** Let \( t_1t_2 \in V(D_0) \). By Claim 1

- either \( t_2 \) is \( \leq_{t_1}^4 \)-minimal and \( \sigma(t_1t_2) = \sigma^1(t_1) \) and \( \alpha(t) = \alpha^1(t_1) \),
- or \( t_2 \) is not \( \leq_{t_1}^4 \)-minimal and \( \sigma(t_1t_2) = \sigma^4_{t_1}(t_2) \) and \( \alpha(t_1t_2) \cap \beta^1(t_1) = \alpha^4_{t_1}(t_2) \).

Note furthermore that if \( t_2 \) is not \( \leq_{t_1}^4 \)-minimal, then \( \sigma^4_{t_1}(t_2) \neq \emptyset \) by Lemma 4.3.8. Now the claim follows from (C).

It is easy to construct a d-scheme \( \Lambda \) not depending on the specific graph \( G \) such that \( \Lambda[G] = \Delta \). Then assertions (i) and (ii) are immediate from the construction. Assertion (iii) follows from Claim 4 and (iv) follows from Claim 5. \( \square \)
Chapter 6

Graphs of Bounded Tree Width

The tree width of a graph may be viewed as a measure of its similarity with a tree. It has gained great importance in algorithmic graph theory, because many algorithmic problems that are \text{NP}-hard in general can be solved efficiently, often in linear time, on graph classes of bounded tree width. Classes of bounded tree width also play an important role in Robertson and Seymour’s graph minor theory (as we shall see later in this book).

In this chapter, we shall prove that graphs of tree width \( k \) admit IFP-definable treelike decompositions of width \( k \). Besides its applications later in this book, this result serves as a first nontrivial example of a “definable structure theorem”. Its proof introduces basic techniques for defining treelike decompositions that will be appear over and over again throughout this book. After proving the theorem in Section 6.1.1, in Section 6.2 we will give a second, quite different proof of a slightly weaker result. The ideas of this second proof will reappear in the proof of our main theorem in Chapter 17.

6.1 Defining Bounded-Width Decompositions

**Theorem 6.1.1 (Definable Structure Theorem for Graphs of Bounded Tree Width).**

For every \( k \in \mathbb{N} \) there is a \( d \)-scheme \( \Lambda \) such that for every graph \( G \) of tree width at most \( k \), the decomposition \( \Lambda[G] \) is a treelike decomposition of \( G \) of width at most \( k \).

Furthermore, for all \( v \in V(\Lambda[G]) \) it holds that \( \beta(v) \subseteq \bar{v} \).

Note that the theorem implies that every graph of tree width \( k \) has a treelike decomposition of width \( k \). (It is also easy to prove this directly; for connected graphs it follows from Corollary 4.6.2.) The converse follows from Corollary 4.6.9: every graph that has a treelike decomposition of width \( k \) also has a tree decomposition of width \( k \), observe that a tree decomposition or treelike decomposition has width \( k \) if and only if it is over the class \( G_{k+1} \).

**Corollary 6.1.2 (Grohe and Mariño [53]).** For every \( k \in \mathbb{N} \), the class of graphs of tree width at most \( k \) is IFP-definable.

**Proof.** This follows from Theorem 6.1.1 and the Definability Lifting Lemma (or rather Corollary 5.4.4).
6.1.1 Proof of the Decomposition Theorem

Let $k \in \mathbb{N}^+$. To explain our decomposition, we fix a graph $G$ of tree width at most $k - 1$. (It is more convenient to work with tree width $k - 1$ instead of $k$ because then we can represent the bags of the decomposition by $k$-tuples.) In the first step of the proof, we shall define a treelike decomposition $\Delta' = (D', \sigma', \alpha')$ of $G$, and in the second step we will delete nodes from $\Delta'$ to turn it into a tree decomposition $\Delta$ of width at most $k - 1$.

**Step 1. Definition of $\Delta'$.**

All nodes of $\Delta'$ will be $2k$-tuples of vertices of $G$. To simplify the notation, for $2k$-tuples $\vec{v} = (v_1, \ldots, v_{2k})$ we let

$$\vec{v}_I := (v_1, \ldots, v_k)$$
$$\vec{v}_{II} := (v_{k+1}, \ldots, v_{2k}).$$

$\Delta'$ will have two kinds of nodes, r-nodes (root nodes) and c-nodes (child nodes). Let $\vec{v} \in V(G)^{2k}$.

(A) $\vec{v}$ is an r-node if $\vec{v}_I = \vec{v}_{II}$ and there is a connected component $A_{\vec{v}}$ of $G$ such that $\vec{v}_{II} \subseteq V(A_{\vec{v}})$.

(B) $\vec{v}$ is a c-node if $\vec{v}_{II} \setminus \vec{v}_I \neq \emptyset$ and there is a connected component $A_{\vec{v}}$ of $G \setminus \vec{v}_I$ such that $\vec{v}_{II} \setminus \vec{v}_I \subseteq V(A_{\vec{v}})$ and $N^G(A_{\vec{v}}) = \vec{v}_I \setminus \vec{v}_{II}$.

We let $V_r$ and $V_c$ be the sets of r-nodes and c-nodes, respectively, and $V(D') := V_r \cup V_c$. For all $\vec{v} \in V(D')$, we let $S_{\vec{v}} := N^G(A_{\vec{v}})$. Note that $S_{\vec{v}} = \emptyset$ if $\vec{v} \in V_r$ and $S_{\vec{v}} = \vec{v}_I \setminus \vec{v}_{II}$ if $\vec{v} \in V_c$.

To define the edge relation $E(D')$, let $\vec{v}, \vec{w} \in V(D')$. Then $\vec{v}\vec{w} \in E(D')$ if

(C) $\vec{w} \notin V_r$ and $\vec{v}_{II} = \vec{w}_I$ and $V(A_{\vec{w}}) \subseteq V(A_{\vec{v}})$.

We define the mappings $\sigma', \alpha' : V(D') \rightarrow 2^{V(G)}$ by

(D) $\sigma' (\vec{v}) := S_{\vec{v}}$ and $\alpha' (\vec{v}) := V(A_{\vec{v}})$

for all $\vec{v} \in V(D')$. This completes the definition of the decomposition $\Delta'$.

**Claim 1.** $\Delta'$ is a strict treelike decomposition of $G$.

**Proof.** It is immediate from the definitions that $\Delta'$ satisfies **[TL.2]**

To prove (TL.3s), let $\vec{v} \in V(D')$ and $\vec{w} \in N^G_+(\vec{v})$. Then $\alpha'(\vec{w}) \subseteq \alpha' (\vec{v})$ follows immediately from **[C]**. As $\vec{w}$ is a c-node,

$$\sigma' (\vec{w}) = \vec{w}_I \setminus \vec{w}_{II} \subseteq \vec{w}_I = \vec{v}_I \subseteq S_{\vec{v}} \cup V(A_{\vec{v}}) = \gamma'(\vec{v})$$

and hence $\gamma'(\vec{w}) \subseteq \gamma'(\vec{v})$. Moreover, $A_{\vec{w}} \subseteq A_{\vec{v}} \setminus \vec{w}_I = A_{\vec{v}} \setminus \vec{v}_{II} \subseteq A_{\vec{v}}$.

**[TL.4]** follows from Lemma **[4.2.10]**

To prove (TL.5), let $\Lambda$ be a connected component of $G$. Then for each $v \in V(A)$, the $2k$-tuple $\vec{v} := (v, \ldots, v)$ is an r-node with $\alpha' (\vec{v}) = V(A_{\vec{v}}) = V(A)$ and $\sigma' (\vec{v}) = \emptyset$. 

**Claim 2.** There is a d-scheme $\Lambda'$ (not depending on $G$) such that $\Lambda'[G] = \Delta'$.

**Proof.** Straightforward.

**Step 2. Pruning the Decomposition.**

For every $\vec{v} \in V(D')$, we let $G_{\vec{v}} := G[\gamma'(\vec{v})]$. We inductively define a sequence of sets $U_h \subseteq V(D')$, for all $h \in \mathbb{N}^+$ as follows.
The **height** of a tree decomposition \((T, \beta)\) of a graph \(H\) is the height of the tree \(T\), that is, the length of the longest path from the root of \(T\) to a leaf.

**Claim 3.** Let \(\pi \in V(D')\) such that \(G_\pi\) has a tree decomposition \((T_\pi, \beta_\pi)\) of height at most \(h\) and width at most \((k-1)\) such that \(\tilde{v}_{II} = \beta_\pi(r_\pi)\) for the root \(r_\pi\) of \(T_\pi\). Then \(\pi \in U_{h+1}\).

**Proof.** We prove the claim by induction on \(h\). For the base step \(h = 0\), suppose that \((T_\pi, \beta_\pi)\) is a tree decomposition of \(G_\pi\) of height 0 and width at most \((k-1)\) such that \(\tilde{v}_{II} = \beta_\pi(r_\pi)\) for the root \(r_\pi\) of \(T_\pi\). Let \(A\) be a connected component of \(G_\pi \setminus \tilde{v}_{II} = \beta_\pi(r_\pi)\). We shall prove that there is a node \(\bar{v} \in N_{D'}(\pi) \cap U_h\) with \(A_{\bar{v}} = A\). This will imply \(\pi \in U_{h+1}\).

Let \(S := N^G(A)\) and \(H := G[V(A) \cup S]\). Let \(t \in V(T_\pi)\) be \(\leq_{T_\pi}\)-maximal with \(V(A) \subseteq \alpha_\pi(t)\), that is, \(V(A) \subseteq \alpha_\pi(t)\) and \(V(A) \not\subseteq \alpha_\pi(u)\) for any \(u \in N_{T_\pi}(t)\). Clearly, such a node \(t\) exists, because \(V(A) \subseteq V(G_\pi) = \alpha_\pi(r_\pi)\). Then
\[
V(A) \cap \beta_\pi(t) \neq \emptyset. \tag{6.1.1}
\]
To see this, suppose for contradiction that \(V(A) \cap \beta_\pi(t) = \emptyset\). As \(V(A) \subseteq \gamma_\pi(t)\), there is at least one child \(u \in N^G(A)\) such that \(V(A) \cap \alpha_\pi(u) \neq \emptyset\). Then it follows from Fact 4.1.3(2) that \(V(A) \subseteq \alpha_\pi(u)\), which is a contradiction. This proves \((6.1.1)\). Note that \((6.1.1)\) implies \(t \neq r_{\pi}\).

Furthermore,
\[
S \subseteq \beta_\pi(t). \tag{6.1.2}
\]
To see this, note that every vertex \(v \in S\) is contained in \(\gamma_\pi(t)\), because it has a neighbour in \(V(A) \subseteq \alpha_\pi(t)\), and the edge \(vu_\pi\) must occur in some bag of \((T_\pi, \beta_\pi)\). Moreover, \(S = N^G(A) \subseteq \beta_\pi(r_\pi)\), because \(A\) is a connected component of \(G_\pi \setminus \tilde{v}_{II} = \beta_\pi(r_\pi)\). Hence \(S \subseteq \gamma_\pi(t) \cap \beta_\pi(t) \subseteq \alpha_\pi(t) \subseteq \beta_\pi(t)\).

Let \(T_A := T_\pi[\{u \in V(T_\pi) \mid t \leq_{T_\pi} u\}]\) be the full subtree of \(T_\pi\) rooted at \(t\), and define \(\beta_A : V(T_A) \rightarrow 2^{V(H)}\) by \(\beta_A(u) := \beta_\pi(u) \cap V(H)\). Then \((T_A, \beta_A)\) is a tree decomposition of \(H\) of height at most \((h-1)\) and width at most \((k-1)\).

Observe that \(\beta_A(t) \cap \beta_\pi(r_\pi) = S\) by \((6.1.2)\) and \(\beta_A(t) \setminus \beta_\pi(r_\pi) \neq \emptyset\) by \((6.1.1)\). Let \(\bar{w} \in V(G)^{2k}\) be a tuple with \(\bar{w}_I = \tilde{v}_{II}\) and \(\bar{w}_{II} = \beta_A(t)\). Then \(\bar{w} \in V(D)\) with \(A_{\bar{v}} = A\) and \(S_{\bar{v}} = S\). By (C) we have \(\bar{w} \in N_{D'}(\tilde{v})\), and by the induction hypothesis we have \(\bar{w} \in U_h\).

We let \(D\) be the induced subgraph of \(D'\) with universe \(V(D) := \bigcup_{h \in N^+} U_h\), and we let \(\sigma, \alpha : V(D) \rightarrow 2^{V(G)}\) be the restrictions of \(\sigma', \alpha'\) to \(V(D)\). We let \(\Delta := (D, \sigma, \alpha)\).

**Claim 4.** \(\Delta\) is a treelike decomposition of \(G\).

**Proof.** \(\Delta\) inherits \((\text{TL.1})\), \((\text{TL.4})\) from \(\Delta\). To see that it satisfies \((\text{TL.5})\), let \(A\) be a connected component of \(G\). Let \((T_A, \beta_A)\) be a tree decomposition of \(A\) of width at most \(k-1\) such that \(\beta(r_A) \neq \emptyset\) for the root \(r_A\) of \(T_A\). Such a decomposition exists, because the tree width of \(A \subseteq G\) is at most \(k-1\). Let \(h\) be the height of \(T_A\). Let \(\bar{b} \in V(G)^k\) such that \(\beta_A(r_A) = \bar{b}\), and
Proof. Let \((\overline{v}, 4k) \subseteq V(G)^2k\). Then \(\overline{v}\) is an r-node with \(\sigma(\overline{v}) = \emptyset\) and \(\alpha(\overline{v}) = A \overline{v} = A\). By Claim 3, we have \(\overline{v} \in U_{h+1} \subseteq V(D)\). Hence \(\Delta\) is treelike.

**Claim 5.** For all \(\overline{v} \in V(D)\) it holds that \(\beta(\overline{v}) \subseteq \overline{v}_{II}\).

**Proof.** Let \(\overline{v} \in V(D)\). Let \(h \in \mathbb{N}^+\) such that \(\overline{v} \in U_h\). Let \(A_1, \ldots, A_m\) be the connected components of \(G_\overline{v} \setminus \overline{v}_{II}\). Then for every \(i \in [m]\) there is a \(\overline{w}_i \in N^D(\overline{v}) \cap U_{h-1} \subseteq N^D(\overline{v})\) such that \(A_{\overline{w}_i} = A_i\). Hence

\[
\beta(\overline{v}) = \gamma(\overline{v}) \setminus \bigcup_{\overline{w} \in N^D(\overline{v})} \alpha(\overline{w}) \subseteq \gamma(\overline{v}) \setminus \bigcup_{i=1}^m \alpha(\overline{w}_i) = V(G_\overline{v}) \setminus \bigcup_{i=1}^m V(A_i) = \overline{v}_{II}.
\]

Claim 5 implies that the width of \(\Delta\) is at most \(k - 1\). It follows from Claim 2 and the inductive definition of \(V(D) = \bigcup_{h \in \mathbb{N}^+} U_h\), which can easily be formalised in \(\text{IFP}\), that there is a d-scheme \(\Lambda\) (not depending on \(G\)) such that \(\Lambda[G] = \Delta\).

### 6.2 Defining Bounded-Width Decompositions Top-Down

In this section we give an alternative proof of the following weaker version of the Decomposition Theorem for Graphs of Bounded Tree Width \[6.1.1\].

**Proposition 6.2.1.** For every \(k \in \mathbb{N}^+\), the class of graphs of tree width at most \(k\) admits \(\text{IFP}\)-definable treelike decompositions of width \(3k + 4\).

The reason for giving this alternative proof is twofold. First, the basic construction underlying this proof will re-appear in the proof of our main theorem, the Definable Structure Theorem \[17.2.1\] and this is a good occasion to see the construction in a simpler context. And second, the decomposition constructed in the proof given here is definable in a much weaker logic than \(\text{IFP}\). This second issue will be further discussed in Remark \[6.2.4\]. Both proofs are inspired by standard algorithms for computing tree decompositions of bounded width: the proof given in Section \[6.1.1\] is based on an algorithm (going back to Arnborg, Corneil, and Proskurowski \[3\]) that computes a tree decomposition of width \(k\) in time \(n^{O(k)}\) if there is one, whereas the proof given here is based on an approximation algorithm (going back to Robertson and Seymour \[109\]) that computes a decomposition of width at most \((4k - 3)\) if there exists a decomposition of width at most \(k\) in time \(2^{O(k)} \cdot n^2\).

**Definition 6.2.2.** Let \(G\) be a graph and \(X \subseteq V(G)\). A set \(Y \subseteq V(G)\) cracks \(X\) if for every connected component \(A\) of \(G \setminus Y\) it holds that \(|(V(A) \cap X) \cup Y| < |X|\).

The following lemma is a variant of well-known lemmas due to Robertson and Seymour \[109\] and Reed \[103\] \[104\].

**Lemma 6.2.3.** Let \(G\) be a graph of tree width at most \(k \in \mathbb{N}\) and \(X \subseteq V(G)\) a set of size \(|X| \geq 2k + 3\). Then there is a set \(Y \subseteq V(G)\) of size \(|Y| \leq k + 1\) that cracks \(X\).

**Proof.** Let \((T, \beta)\) be a tree decomposition of \(G\) of width at most \(k\). Choose a node \(t \in V(T)\) such that

\[
|\gamma(t) \cap X| > k + 1
\]

and for all \(u \in N^T_X(t),\)

\[
|\gamma(u) \cap X| \leq k + 1.
\]
Clearly, such a \( t \) exists, because the root of \( T \) satisfies (6.2.1).

I claim that \( Y := \beta(t) \) cracks \( X \). To see this, let \( A \) be a connected component of \( G \setminus Y \). It follows from Fact 4.1.3(1) that either \( V(A) \subseteq V(G) \setminus \gamma(t) \) or \( V(A) \subseteq \alpha(u) \) for some \( u \in N_r^+(t) \).

Let \( X \subseteq V(G) \setminus \gamma(t) \), then
\[
|V(A) \cap X| \leq |X \setminus \gamma(t)| = |X| - |\gamma(t) \cap X| < |X| - (k + 1) \leq |X| - |Y|,
\]
which implies \(|(V(A) \cap X) \cup Y| < |X|\).

If \( V(A) \subseteq \alpha(u) \) for some \( u \in N_r^+(t) \), then
\[
|(V(A) \cap X) \cup Y| \leq |\gamma(u) \cap X| + |Y| \leq 2k + 2 < |X|.
\]

Proof of Proposition 6.2.1. Let \( k \in \mathbb{N} \). We need to define a \( d \)-scheme \( \Lambda \) such that for every graph \( G \) of tree width at most \( k \) the decomposition \( \Lambda[G] \) is a treelike decomposition of \( G \) of width at most \( 3k + 3 \). To explain our decomposition, we fix a graph \( G \) of tree width at most \( k \).

We define the mappings \( \overline{S} := u_1 \cdots u_{k+1}, \overline{X} := u_{k+2} \cdots u_{4k+6}, \overline{Y} := u_{4k+7} \cdots u_{5k+7} \).

Then \( \overline{u} = u_1 \overline{S} \overline{X} \overline{Y} \). If \( u_1 \in \overline{S} \) we let \( S_{\overline{u}} = \emptyset \), and otherwise we let \( S_{\overline{u}} = \overline{S} \). We let \( A_{\overline{u}} \) be the connected component of \( G \setminus S_{\overline{u}} \) that contains \( u_1 \) and \( H_{\overline{u}} := G[V(A_{\overline{u}}) \cup S_{\overline{u}}] \). Furthermore, we let \( X_{\overline{u}} := \overline{X} \) and \( Y_{\overline{u}} := \overline{Y} \). Note that \( |S_{\overline{u}}| \leq |\overline{S}| = 2k + 2 \) and \( 1 \leq |X_{\overline{u}}| \leq |\overline{X}| = 2k + 3 \).

We let \( U \) be the set of all \( \overline{u} \in V(G)^{k} \) satisfying the following conditions.

(A) \( S_{\overline{u}} = N(A_{\overline{u}}) \);

(B) \( S_{\overline{u}} \subseteq X_{\overline{u}} \subseteq V(H_{\overline{u}}) \) and \( Y_{\overline{u}} \subseteq V(H_{\overline{u}}) \);

(C) either \( X_{\overline{u}} = V(H_{\overline{u}}) \) or \( |X_{\overline{u}}| = 2k + 3 \);

(D) if \( |X_{\overline{u}}| = 2k + 3 \) then \( Y_{\overline{u}} \) cracks \( X_{\overline{u}} \) in \( H_{\overline{u}} \).

Note that, by (A), \( H_{\overline{u}} \) is connected for all \( \overline{u} \in U \). \( \Delta \) will have two kinds of nodes, \( r \)-nodes (“root nodes”) and \( c \)-nodes (“cracked nodes”).

(E) An \( r \)-node is a tuple \( \overline{u} \in U \) with \( u_1 \in \overline{S} \).

(F) A \( c \)-node is a tuple \( \overline{u} \in U \) with \( u_1 \notin \overline{S} \).

Let \( U_r, U_c \) be the sets of all \( r \)-nodes and \( c \)-node, respectively, and \( V(D) := V_r \cup V_c \). Observe that for every \( \overline{u} \in U_r \) the graph \( A_{\overline{u}} = H_{\overline{u}} \) is a connected component of \( G \).

To define the edge relation \( E(D) \), let \( \overline{u}, \overline{v} \in V(D) \). Then \( \overline{uv} \in E(D) \) if

(G) \( \overline{v} \in V_c \) and \( A_{\overline{u}} \) is a connected component of \( H_{\overline{u}} \setminus (X_{\overline{u}} \cup Y_{\overline{u}}) \).

We define the mappings \( \sigma, \alpha : V(D) \to 2^{V(G)} \) as follows:
(H) For all \( \overline{v} \in V(D) \) we let \( \sigma(\overline{v}) := S_{\overline{v}} \) and \( \alpha(\overline{v}) := V(A_{\overline{v}}) \).

This completes the definition of the decomposition \( \Delta \).

Claim 1. \( \Delta \) is a strict treelike decomposition of \( G \).

Proof. It is immediate from the definitions that \( \Delta \) satisfies (TL.2).

To verify (TL.3s), let \( \overline{v} \in V(D) \) and \( \overline{v} \in N_{+}^{D}(\overline{v}) \). Then

\[
\alpha(\overline{v}) = A_{\overline{v}} \subseteq H_{\overline{v}} \setminus X_{\overline{v}} \subseteq H_{\overline{v}} \setminus S_{\overline{v}} = A_{\overline{v}} = \alpha(\overline{v}).
\]

To see that the second inclusion is strict, note that \(|X_{\overline{v}}| = 2k + 3\), because otherwise \(X_{\overline{v}} = V(H_{\overline{v}})\) by (C) and \(A_{\overline{v}}\) cannot be a connected component of \( H_{\overline{v}} \setminus (X_{\overline{v}} \cup Y_{\overline{v}}) = \emptyset \). However, \(|S_{\overline{v}}| \leq 2k + 2\).

Moreover, we have

\[
\sigma(\overline{v}) = N^{G}(A_{\overline{v}}) \subseteq V(A_{\overline{v}}) \cup N^{G}(A_{\overline{v}}) = \gamma(\overline{v})
\]

and thus \( \gamma(\overline{v}) \subseteq \gamma(\overline{v}) \).

To verify (TL.5) let \( A \) be a connected component of \( G \). If \(|A| \leq 2k + 3\), let we can easily find a \( r \)-node \( \overline{v} \) such that \( X_{\overline{v}} = V(A) \). Otherwise, let \( X \subseteq V(A) \) be an arbitrary set of size \(|X| = 2k + 3\), and let \( Y \subseteq V(A) \) be a set of size \(|Y| = k + 1\) that cracks \( X \). Such a set exists by Lemma 6.2.3, because \( \text{tw}(A) \leq \text{tw}(G) \leq k \). Let \( \overline{v} \in V(G) \) such that \( u_{1} \in X \) and \( S_{\overline{v}} = \{ u_{1} \} \), and \( X_{\overline{v}} = X \) and \( Y_{\overline{v}} = Y \). Then \( \overline{v} \) is an \( r \)-node with \( \alpha(\overline{v}) = V(A) \) and \( \sigma(\overline{v}) = \emptyset \).

Claim 2. There is a \( d \)-scheme \( \Lambda \) not depending on \( G \) such that \( \Lambda[G] = \Delta \).

Proof. It is straightforward to formalise the decomposition of \( \Delta \) in \( \text{IFP} \).

Claim 3. For all \( \overline{v} \in V(D) \) we have \( \beta(\overline{v}) \subseteq X_{\overline{v}} \cup Y_{\overline{v}} \).

Proof. Let \( \overline{v} \in V(D) \). As \( \gamma(\overline{v}) = V(H_{\overline{v}}) \), it suffices to prove that for every connected component \( A \) of \( H_{\overline{v}} \setminus (X_{\overline{v}} \cup Y_{\overline{v}}) \) there is a \( \overline{v} \in N_{+}^{D}(\overline{v}) \) such that \( A_{\overline{v}} = A \). This is trivially the case if \( X_{\overline{v}} = V(H_{\overline{v}}) \). Thus by (C), we may assume that \(|X_{\overline{v}}| = 2k + 3\).

Let \( A \) be a connected component of \( H_{\overline{v}} \setminus (X_{\overline{v}} \cup Y_{\overline{v}}) \) and \( S := N^{H_{\overline{v}}}(A) = N^{G}(A) \). Here the equality holds because \( \partial^{G}(H_{\overline{v}}) \subseteq S_{\overline{v}} \subseteq X_{\overline{v}} \). Let \( A' \) be the connected component of \( H_{\overline{v}} \setminus Y_{\overline{v}} \) that contains \( A \). Then \( S \subseteq (X_{\overline{v}} \cap V(A')) \cup Y_{\overline{v}} \). As \( Y_{\overline{v}} \) cracks \( X_{\overline{v}} \), we thus have \(|S| < |X_{\overline{v}}|\).

Let \( H := G[V(A) \cup S] \). If \(|H| < 2k + 3\), let \( X := V(H) \). Then there is a \( c \)-node \( \overline{v} \) such that \( v_{1} \in V(A) \) and \( S_{\overline{v}} = S \) and \( X_{\overline{v}} = X \), and we have \( \overline{v} \in N_{+}^{D}(\overline{v}) \) and \( \alpha(\overline{v}) = V(A_{\overline{v}}) = A \).

So suppose that \(|H| \geq 2k + 3\). Let \( X \subseteq V(H) \) such that \( S \subseteq X \) and \(|X| = 2k + 3\). Let \( Y \subseteq V(H) \) such that \( Y \) cracks \( X \) in \( H \) and \(|Y| \leq k + 1\). Then there is a \( c \)-node \( \overline{v} \) such that \( v_{1} \in V(A) \) and \( S_{\overline{v}} = S \) and \( X_{\overline{v}} = X \) and \( Y_{\overline{v}} = Y \), and we have \( \overline{v} \in N_{+}^{D}(\overline{v}) \) and \( \alpha(\overline{v}) = V(A_{\overline{v}}) = A \).

Claim 3 implies that for all \( \overline{v} \in V(D) \) we have \(|\beta(\overline{v})| \leq 3k + 2\). Hence the width of \( \Delta \) is at most \((3k + 1)\).

Remark 6.2.4. It is easy to see that the treelike decomposition \( \Delta \) constructed in the proof of Proposition 6.2.1 is definable in a much weaker logic than \( \text{IFP} \). Any reasonable extension of first-order logic by an operator that allows it to express graph connectivity will be sufficient (for example, the extension of first-order logic by a unary symmetric transitive closure

M. Grohe, Definable Graph Structure Theory
operator). In particular, it may be interesting to note that the decomposition is definable in monadic second-order logic.

This does not seem to be the case for the decomposition defined in the proof of the Decomposition Theorem for Graphs of Bounded Tree Width 6.1.1, because the inductive definition of the sets $U_h$ of $2k$-tuples seems to require a $2k$-ary fixed-point definition.
Chapter 7

Ordered Treelike Decompositions

While treelike decompositions give a general framework that allows for decompositions over all kinds of interesting structures, our main focus is on decompositions of graphs into torsos that admit definable linear orders. To be able to deal with such decompositions efficiently, in this chapter we introduce ordered treelike decompositions. As for (plain) treelike decompositions, we develop a basic theory of definable ordered treelike decompositions. A key lemma is the Ordered Decomposition Lifting Lemma 7.1.7 stating that from a definable treelike decomposition of a graph into torsos that admit definable ordered treelike decompositions we can obtain a definable ordered treelike decomposition of the whole graph.

The main result of this chapter, proved in Section 7.4, provides the link between our decomposition theory and descriptive complexity theory. It says that if a class of graphs admits \( \text{IFP}\)-definable ordered treelike decompositions, then it admits \( \text{IFP}+\text{C}\)-definable canonisation.

7.1 Definitions and Basic Results

**Definition 7.1.1.** An o-decomposition of a graph \( G \) is a quadruple \( \Delta = (D^\Delta, \sigma^\Delta, \alpha^\Delta, \leq^\Delta) \) such that \( \Delta_0 := (D^\Delta, \sigma^\Delta, \alpha^\Delta) \) is a decomposition of \( G \) and \( \leq^\Delta \) is a mapping that associates a binary relation \( \leq^\Delta_t \subseteq V^2(D^\Delta) \) with every \( t \in V^2(D^\Delta) \).

If \( \Delta = (D, \sigma, \alpha, \leq) \) is an o-decomposition, then we call \( \Delta_0 := (D, \sigma, \alpha) \) the underlying plain decomposition of \( \Delta \). We usually make no notational distinction between \( \Delta \) and \( \Delta_0 \). In particular, we let \( D^\Delta := D^{\Delta_0} \), \( \sigma^\Delta := \sigma^{\Delta_0} \), et cetera. As usual, we omit the index \( \Delta \) if \( \Delta \) is clear from the context.

**Definition 7.1.2.** An ordered treelike decomposition of a graph \( G \) is an o-decomposition \( \Delta \) of \( G \) such that the underlying plain decomposition of \( \Delta \) is treelike and the following axiom is satisfied.

\( (\text{OTL}) \) For every \( t \in V^2(D) \) the relation \( \leq^\Delta_t \) is a linear order of \( \beta^\Delta(t) \).

There is not much more to say about ordered treelike decompositions at this point, and we turn to definability issues right away.

**Definition 7.1.3.** An o-decomposition scheme (for short: od-scheme) is a tuple

\[
\Lambda = (\lambda_V(\vec{x}), \lambda_E(\vec{x}, \vec{x}'), \lambda_\sigma(\vec{x}, y), \lambda_\alpha(\vec{x}, y), \lambda_{\leq}(\vec{x}, y_1, y_2))
\]
of IFP-formulae in the vocabulary of graphs, where $\overline{x}, \overline{x}'$ are tuples of vertex variables of the same length. The dimension of $\Lambda$ is the length of the tuple $\overline{x}$.

For an od-scheme $\Lambda = (\lambda_V(\overline{x}), \lambda_E(\overline{x}, \overline{x}'), \lambda_\sigma(\overline{x}, y), \lambda_\alpha(\overline{x}, y), \lambda_\leq(\overline{x}, y_1, y_2))$, we call $\Lambda_0 = (\lambda_V(\overline{x}), \lambda_E(\overline{x}, \overline{x}'), \lambda_\sigma(\overline{x}, y), \lambda_\alpha(\overline{x}, y))$ the underlying d-scheme of $\Lambda$.

**Definition 7.1.4.** Let $\Lambda = (\lambda_V(\overline{x}), \lambda_E(\overline{x}, \overline{x}'), \lambda_\sigma(\overline{x}, y), \lambda_\alpha(\overline{x}, y), \lambda_\leq(\overline{x}, y_1, y_2))$ an od-scheme, and let $\Lambda_0$ be the underlying d-scheme.

1. For every graph $G$, the ordered decomposition defined by $\Lambda$ on $G$ is the o-decomposition

   $\Lambda[G] := (D^{\Lambda_0[G]}, \sigma^{\Lambda_0[G]}, \alpha^{\Lambda_0[G]}, \leq^{\Lambda[G]})$,

   where for every $\overline{v} \in \lambda_V[G, \overline{x}]$, we let

   $\leq^{\Lambda[G]}_{\overline{v}} := \lambda_\leq[G; \overline{v}, y_1, y_2]$.

2. $\mathcal{OT}_\Lambda$ is the class of all graphs $G$ such that $\Lambda[G]$ is an ordered treelike decomposition of $G$.

3. A class $\mathcal{C}$ of graphs admits **IFP-definable ordered treelike decompositions** if there is an od-scheme $\Lambda$ such that $\mathcal{C} \subseteq \mathcal{OT}_\Lambda$.

We usually make no notational distinction between an od-scheme $\Lambda$ and its underlying d-scheme $\Lambda_0$ and let $D^{\Lambda[G]} := D^{\Lambda_0[G]}, \sigma^{\Lambda[G]} := \sigma^{\Lambda_0[G]}$, et cetera.

**Example 7.1.5.** For every $k \in \mathbb{N}$, the class of graphs of tree width at most $k$ admits IFP-definable ordered treelike decompositions. To see this, recall the Definable Structure Theorem for Graphs of Bounded Tree Width [6.1.1]. Let $\Lambda$ be a d-scheme such that for every graph $G$ of tree width at most $k$, the decomposition $\Lambda[G]$ is a treelike decomposition of $G$ of width at most $k$, and for all $\overline{v} \in V(\Lambda[G])$ it holds that $\beta(\overline{v}) \subseteq \overline{v}$.

Suppose that $\Lambda$ is $\ell$-dimensional. Let $G$ be a graph of tree width at most $k$ and $(D, \sigma, \alpha) := \Lambda[G]$. For every $\overline{v} = (v_1, \ldots, v_\ell) \in V(D)$, we order the vertices in $\beta(\overline{v})$ according to their first occurrence in the tuple $\overline{v}$. That is, we let $v \leq_{\overline{v}} w$ if $v, w \in \beta(\overline{v}) \subseteq \overline{v}$ and $i_0 := \min\{i \mid v = v_i\}$ is less than or equal to $j_0 := \min\{j \mid w = v_j\}$. The following IFP-formula defines this linear order restricted to the bag defined by $\overline{x} = (x_1, \ldots, x_\ell)$

$$
\lambda_{\leq}(\overline{x}, y_1, y_2) := \lambda_\beta(\overline{x}, y_1) \land \lambda_\beta(\overline{x}, y_2) \land \bigvee_{i_0, j_0 \in [\ell]} \left( y_1 = x_{i_0} \land y_2 = x_{j_0} \land \bigwedge_{i = 1}^{i_0-1} y_i \neq x_i \land \bigwedge_{j = 1}^{j_0-1} y_j \neq x_j \right).
$$

Here we use the formula $\lambda_\beta(\overline{x}, y)$ of Lemma [5.4.1].

Then the od-scheme $\left(\lambda_V(\overline{x}), \lambda_E(\overline{x}, \overline{x}'), \lambda_\sigma(\overline{x}, y), \lambda_\alpha(\overline{x}, y), \lambda_\leq(\overline{x}, y_1, y_2)\right)$ defines an ordered treelike decomposition on $G$ and on all other graphs of tree width at most $k$.

The following simple lemma may be viewed as a variant of the Definability Lifting Lemma [5.4.3]. It is particularly useful in combination with the Canonisation Theorem [7.4.1] because it gives us a way to recognise whether a graph belongs to the domain of a canonisation mapping definably in IFP.

**Lemma 7.1.6.** For every od-scheme $\Lambda$ the class $\mathcal{OT}_\Lambda$ is IFP-definable.
Proof. Follows easily from Lemma 5.4.1.

The following lemma is the most important lifting lemma that we prove. It will be used frequently throughout the book.

**Lemma 7.1.7 (Ordered Decomposition Lifting Lemma).** Let $\mathcal{B}, \mathcal{C}$ be classes of graphs such that $\mathcal{C}$ admits IFP-definable treelike decompositions over $\mathcal{B}$ and $\mathcal{B}$ admits IFP-definable ordered treelike decompositions. Then $\mathcal{C}$ admits IFP-definable ordered treelike decompositions.

**Proof.** Let $\Lambda^1$ be a d-scheme such that $\mathcal{C} \subseteq T_{\Lambda^1}(\mathcal{B})$, and let $\Lambda^2$ be an od-scheme such that $\mathcal{B} \in OT_{\Lambda^2}$. Let

$$\Lambda = (\lambda_V(\overline{x}_1 \overline{x}_2), \lambda_E(\overline{x}_1 \overline{x}_2, \overline{x}_1' \overline{x}_2'), \lambda_\sigma(\overline{x}_1 \overline{x}_1, y), \lambda_\alpha(\overline{x}_1 \overline{x}_2, y))$$

be the d-scheme obtained by applying the Decomposition Lifting Lemma (in the simplified version of Corollary 5.6.3) to $\Lambda^1$ and the d-scheme underlying $\Lambda^2$.

Let $G \in \mathcal{C} \subseteq T_{\Lambda^1}$. For every $t_1 \in V(\Lambda^1[G])$, let $H_{t_1} := \tau^{\Lambda^1[G]}(t_1)$. Then $H_{t_1} \in \mathcal{B}$, and therefore $H_{t_1} \in OT_{\Lambda^2}$. Hence $G \in T_{\Lambda}$, and for all $t = t_1 t_2 \in V(\Lambda[G])$ we have $t_1 \in V(\Lambda^1[G])$ and $t_2 \in V(\Lambda^2[H_{t_1}])$ and

$$\tau^{\Lambda[G]}(t) = \tau^{\Lambda^2[H_{t_1}]}(t_2).$$

Hence $\leq_{t_2[H_{t_1}]}$ is a linear order of $\beta^{\Lambda[G]}(t)$.

We apply the Transduction Lemma (Fact 2.4.6) to the simple IFP-graph transduction $\Theta(\overline{x}_1)$ with $\theta_V(\overline{x}_1, y) := \lambda^1_V(\overline{x}_1, y)$ and $\theta_E(\overline{x}_1, y_1, y_2) := \lambda^1_E(\overline{x}_1, y_1, y_2)$ and the formula $\lambda^2_\sigma(\overline{x}_2, y_1, y_2)$ and obtain a formula $\lambda_\leq^2(\overline{x}_1 \overline{x}_2, y_1, y_2) := (\lambda^2_\sigma)^{-\Theta}(\overline{x}_1, \overline{x}_2, y_1, y_2)$ such that for all $\overline{v}_1 \in \Lambda^1[G]^{\overline{x}_1}$ and $\overline{v}_2 \in \Lambda^1[G]^{\overline{x}_2}$ it holds that

$$\lambda_\leq^2[G, \overline{v}_1 \overline{v}_2, y_1, y_2] = \lambda_\leq^2[H_{t_1}, \overline{v}_2, y_1, y_2] = \leq_{t_2[H_{t_1}]}.$$ 

Then the od-scheme

$$(\lambda_V(\overline{x}_1 \overline{x}_2), \lambda_E(\overline{x}_1 \overline{x}_2, \overline{x}_1' \overline{x}_2'), \lambda_\sigma(\overline{x}_1 \overline{x}_1, y), \lambda_\alpha(\overline{x}_1 \overline{x}_2, y), \lambda_\leq(\overline{x}_1 \overline{x}_2, y_1, y_2))$$

defines an ordered treelike decomposition of $G$. 

---

**7.1.1 Definable Ordered Treelike Decompositions and Definable Orders**

**Lemma 7.1.8.** Let $\mathcal{C}$ be a class of graphs that admits IFP-definable orders. Then $\mathcal{C}$ admits IFP-definable ordered treelike decompositions.

**Proof.** Let $\varphi(\overline{x}, y_1, y_2)$ be an IFP-formula that defines a linear order on every graph in $\mathcal{C}$. Suppose that the length of $\overline{x}$ is $k$. Let $\text{check-ord}(\overline{x})$ be an IFP-formula stating that $\varphi(\overline{x}, y_1, y_2)$ defines a linear order, that is, for all graph $G$ and all $\overline{v} \in V(G)^k$ we have $G \models \text{check-ord}[\overline{v}]$ if and only if the binary relation $\varphi[G, \overline{v}, y_1, y_2]$ is a linear order of $V(G)$. We let $\Lambda$ be the $(k + 1)$-dimensional od-scheme defined by

$$\lambda_V(\overline{x}x) := \text{check-ord}(\overline{x}),$$

$$\lambda_E(\overline{x}x, \overline{x}'x') := \text{false},$$

$$\lambda_\sigma(\overline{x}x, y) := \text{false},$$

Preliminary Version
\[
\lambda_\alpha(x,y) := \text{path}(x,y).
\]

Before we complete the definition of \( \Lambda \) by defining \( \lambda_\ell(x,y_1,y_2) \), we observe that for all graphs \( G \) such that \( \varphi \) defines a linear order on \( G \), the vertex set \( V(\Lambda[G]) \) consists of all \( \bar{v} \in V(G)^{k+1} \) such that \( \varphi[G, \bar{v}, y_1, y_2] \) is a linear order of \( V(G) \). \( E(\Lambda[G]) \) is empty, and for all \( \bar{v} \in V(\Lambda[G]) \) we have \( \sigma^{\Lambda[G]}(\bar{v}) = \emptyset \), and \( \alpha^{\Lambda[G]}(\bar{v}) \) is the connected component of \( G \) that contains \( v \). It is easy to see that \( (V(\Lambda[G]), E(\Lambda[G])), \sigma^{\Lambda[G]}, \alpha^{\Lambda[G]} \) is a treelike decomposition of \( G \).

Let \( \lambda_\beta(\bar{x},y) \) be an IFP-formula that defines the bag of the node \( \bar{x} \) (see Lemma 5.4.1). We let

\[
\lambda_\ell(\bar{x},y_1,y_2) := \varphi(\bar{x}, y_1, y_2) \land \lambda_\beta(\bar{x},y_1) \land \lambda_\beta(\bar{x},y_2)
\]

Then for every graph \( G \) and every \( \bar{v} \in V(\Lambda[\mathbb{v}]) \) the binary relation \( \leq^{\Lambda[G]} = \lambda_\ell[G, \bar{v}, y_1, y_2] \in V(G)^{k+1} \) is the restriction of the linear order \( \varphi[G, \bar{v}, y_1, y_2] \) to \( \beta^{\Lambda[G]}(\bar{v}) \). Hence \( \Lambda[G] \) is an ordered treelike decomposition of \( G \).

We get the following corollary from the previous lemma and the Ordered Decomposition Lifting Lemma 7.1.7.

**Corollary 7.1.9.** Let \( B, C \) be classes of graphs such that \( C \) admits IFP-definable treelike decompositions over \( B \) and \( B \) admits IFP-definable orders. Then \( C \) admits IFP-definable ordered treelike decompositions.

### 7.1.2 Unions and Finite Differences

**Lemma 7.1.10 (Union Lemma for Definable Ordered Decompositions).** Let \( B, C \) be classes of graphs. If \( B \) and \( C \) admit IFP-definable ordered treelike decompositions then so does \( B \cup C \).

**Proof.** Let \( \Lambda^B \) and \( \Lambda^C \) be od-schemes defining ordered treelike decompositions on all graphs in \( B \) and \( C \), respectively. Without loss of generality we may assume that both \( \Lambda^B \) and \( \Lambda^C \) are \( \ell \)-dimensional, because we can always increase the dimension of an od-scheme artificially. Let \( \varphi_B \) be an IFP-formula that defines the class \( \mathcal{O}T^{\Lambda_B} \). Such a formula exists by Lemma 7.1.6.

Let \( \Lambda \) be the od-scheme defined as follows:

- \( \lambda_V(\bar{x}) := (\varphi_B \land \lambda^B_V(\bar{x})) \lor (\neg \varphi_B \land \lambda^C_V(\bar{x})) \);
- \( \lambda_E(\bar{x}, \bar{x}') := (\varphi_B \land \lambda^B_E(\bar{x}, \bar{x}')) \lor (\neg \varphi_B \land \lambda^C_E(\bar{x}, \bar{x}')) \);
- \( \lambda_\sigma(\bar{x}, y) := (\varphi_B \land \lambda^B_\sigma(\bar{x}, y)) \lor (\neg \varphi_B \land \lambda^C_\sigma(\bar{x}, y)) \);
- \( \lambda_\alpha(\bar{x}, y) := (\varphi_B \land \lambda^B_\alpha(\bar{x}, y)) \lor (\neg \varphi_B \land \lambda^C_\alpha(\bar{x}, y)) \);
- \( \lambda_\ell(\bar{x}, y_1, y_2) := (\varphi_B \land \lambda^B_\ell(\bar{x}, y_1, y_2)) \lor (\neg \varphi_B \land \lambda^C_\ell(\bar{x}, y_1, y_2)) \).

It is easy to see that \( \Lambda \) defines ordered treelike decompositions on all graphs in \( B \cup C \). □

It is worth noting that in Lemma 7.1.10 we need not impose any definability conditions on \( B \) and \( C \). The reason is that for an od-scheme \( \Lambda \) with \( B \subseteq \mathcal{O}T_\Lambda \) we can define \( \mathcal{O}T_\Lambda \) in IFP (by Lemma 7.1.6), and this is sufficient for the proof. This nicely illustrates why definable ordered treelike decompositions are often easier to handle than definable canonisations. It is not necessarily the case that if classes \( B, C \) admit IFP+C-definable canonisation then their...
union admits \( \text{IFP+C} \)-definable canonisation as well, simply because we cannot recognise (in an \( \text{IFP} \)-definable way) whether a transduction \( \Theta \) that canonises all graphs in \( \mathcal{B} \) canonises a given graph \( G \) (unless, of course, \( \mathcal{B} \) is \( \text{IFP} \)-definable and \( G \in \mathcal{B} \)).

As a corollary of Lemma 7.1.8 and Lemma 3.2.11 we get the following simple result.

**Corollary 7.1.11.** Every class of graphs that is finite up to isomorphism admits \( \text{IFP} \)-definable ordered treelike decompositions.

Combined with the Union Lemma, this yields the following.

**Corollary 7.1.12.** Let \( \mathcal{C} \) and \( \mathcal{C}^* \) be classes of graphs such that the symmetric difference \( \mathcal{C} \triangle \mathcal{C}^* \) is finite up to isomorphism. Then \( \mathcal{C} \) admits \( \text{IFP} \)-definable ordered treelike decompositions if and only if \( \mathcal{C}^* \) admits \( \text{IFP} \)-definable ordered treelike decompositions.

### 7.1.3 Tight Ordered Decompositions

**Definition 7.1.13.** An o-decomposition is **tight** if its underlying decomposition is tight.

**Lemma 7.1.14.** Let \( \Lambda \) be an od-scheme. Then there is an od-scheme \( \Lambda' \) such that for every graph \( G \in \text{OT}_\Lambda \) the o-decomposition \( \Lambda'[G] \) is a tight ordered treelike decomposition of \( G \).

**Proof.** We apply the construction of Lemmas 4.4.2 and 5.3.1 to \( \Lambda \). As the bags of the resulting tight decomposition are all subsets of bags of the original decomposition, they can be ordered by suitable restrictions of the orders of the bags of \( \Lambda \), and we obtain a tight ordered treelike decomposition. \( \square \)

### 7.2 Parametrised O-Decompositions Schemes

Similarly as parametrised d-schemes, we define **parametrised od-schemes**

\[
\Lambda(\mathcal{Z}) = \left( \lambda_V(\mathcal{Z}, \mathcal{x}), \lambda_E(\mathcal{Z}, \mathcal{x}), \lambda_\sigma(\mathcal{Z}, \mathcal{x}, \mathcal{y}), \lambda_\alpha(\mathcal{Z}, \mathcal{x}, \mathcal{y}), \lambda_{\leq}(\mathcal{Z}, \mathcal{x}, \mathcal{y}_1, \mathcal{y}_2) \right)
\]

and the o-decompositions \( \Lambda[G, \mathcal{P}] \) defined by \( \Lambda(\mathcal{Z}) \) on graph interpretations \((G, \mathcal{P})\) for \( \mathcal{Z} \). The class of all graph interpretations \((G, \mathcal{P})\) such that \( \Lambda[G, \mathcal{P}] \) is an ordered treelike decomposition is denoted by \( \text{OT}_\Lambda(\mathcal{Z}) \).

**Lemma 7.2.1.** Let \( \Lambda(\mathcal{Z}) \) be a parametrised od-scheme. Then there is an \( \text{IFP} \)-formula \( \lambda_{\text{OT}}(\mathcal{Z}) \) that defines the class \( \text{OT}_\Lambda(\mathcal{Z}) \).

**Proof.** Straightforward. \( \square \)

We extend disjoint unions from decompositions to o-decompositions in a straightforward way and observe that the disjoint union of a family of ordered treelike decompositions is an ordered treelike decomposition.

**Lemma 7.2.2.** Let \( \Lambda(\mathcal{Z}) \) be a parametrised od-scheme, where \( \mathcal{Z} \) is a tuple of individual variables. Then there is an od-scheme \( \Lambda' \) such that for all graphs \( G \) the o-decomposition \( \Lambda'[G] \) is isomorphic to the disjoint union of the o-decompositions \( \Lambda[G, \mathcal{P}] \) for all \( \mathcal{P} \in G^{\mathcal{Z}} \) with \((G, \mathcal{P}) \in \text{OT}_\Lambda(\mathcal{Z}) \).

**Proof.** Similar to the proof of Lemma 5.5.4. \( \square \)
A corollary of Lemma 7.2.2 is the following lemma, which allows us to eliminate parameters from definitions of ordered treelike decompositions.

**Lemma 7.2.3 (Parameter Elimination Lemma).** Let \( \Lambda(\overline{z}) \) be a parametrised od-scheme, where \( \overline{z} \) is a tuple of individual variables. Then there is an od-scheme \( \Lambda' \) such that for all graphs \( G \) it holds that

\[
G \in \mathcal{O}T_{\Lambda'}(A) \iff \text{there is a tuple } \overline{p} \in G^\overline{z} \text{ such that } (G, \overline{p}) \in \mathcal{O}T_{\Lambda(\overline{z})}.
\]

**Proof.** We let \( \Lambda' \) define the disjoint union of the o-decompositions \( \Lambda[G, \overline{p}] \) for all \( \overline{p} \in G^\overline{z} \) such that \((G, \overline{p}) \in \mathcal{O}T_{\Lambda(\overline{z})} \).

### 7.2.1 O-Decompositions of Graphs within Other Graphs

It is straightforward to generalise definitions of decompositions of graphs within other graphs (see Section 5.5.1) to o-decompositions. For the reader’s convenience, let me repeat the setting and definitions. Suppose that \( G, H \) are graphs with \( V(H) \subseteq V(G) \). Further suppose that \( \Delta = (D, \sigma, \alpha, \leq) \) is a decomposition of \( H \). Let \( \Lambda \) is an od-scheme such that \( \Lambda[G] = \Delta \). Then we say that \( \Lambda \) defines the o-decomposition \( \Delta \) of \( H \) within \( G \).

**Lemma 7.2.4 (Transduction Lemma for Definable O-Decompositions).** Let \( \Theta(\overline{Z}) \) be a simple 1-dimensional IFP-graph transduction and \( \Lambda(\overline{Z}') \) a parametrised od-scheme. Then there is a parametrised od-scheme \( \Lambda'(\overline{Z}, \overline{Z}') \) such that for all graph interpretations \( (G, P, P') \) for \( \overline{Z}, \overline{Z}' \) with \( (G, P) \in \mathcal{D}_{\Theta(\overline{Z})} \) and \( P' \in \Theta[G, P]^{\overline{Z}'} \) the od-scheme \( \Lambda'[\Theta[G, P], P'] \) on \( \Theta[G, P] \) within \( (G, P, P') \).

**Proof.** This is an immediate consequence of the Transduction Lemma (Fact 2.4.6).

As for definable (plain) decompositions, we can formulate parametrised versions of results on definable ordered treelike decompositions. The proofs of the unparametrised versions usually generalise without any problems. As an example, we state a parametrised version of the Ordered Decomposition Lifting Lemma 7.1.7. We take this opportunity to rephrase the lemma:

**Lemma 7.2.5 (Parametrised Ordered Decomposition Lifting Lemma).** Let \( \Lambda^1(\overline{Z}) \) be an \( \ell \)-dimensional parametrised d-scheme and \( \Lambda^2(\overline{x}) \) an od-scheme, where \( |\overline{x}| = \ell \). Then there is a parametrised od-scheme \( \Lambda(\overline{Z}) \) such that for all graph interpretations \( (G, \overline{P}) \in \mathcal{T}_{\Lambda^1(\overline{x})} \) the following holds.

Let \( \Delta^1 := (D^1, \sigma^1, \alpha^1) := \Lambda^1[G, \overline{P}] \). Suppose that for every \( \overline{v} \in V(D^1) \) the o-scheme \( \Lambda^2(\overline{x}) \) defines an ordered treelike decomposition of \( \tau^1(\overline{v}) \) within \( (G, \overline{v}) \). Then \( (G, \overline{P}) \in \mathcal{O}T_{\Lambda(\overline{Z})} \).

**Proof.** The proof is a straightforward extension of the proof of the Lemma 7.1.7 using the Decomposition Lifting Lemma 5.6.2 instead of its Corollary 5.6.3.

### 7.3 Extension Lemmas

Recall that “extension lemmas” are lemmas that allow us to “extend” definable treelike decompositions from graphs \( H, H', \ldots \) to other graphs \( G \). The simplest extension lemma that we need is the following Finite Extension Lemma. Remember that for all classes \( B \) of graphs and all \( k \in \mathbb{N}^+ \), by \( N_k(B) \) we denote the class of extensions of graphs in \( B \) by at most \( k \) vertices.

---

M. Grohe, *Definable Graph Structure Theory*
Lemma 7.3.1 (Finite Extension Lemma). Let $\mathcal{B}$ be a class of graphs that admits $\mathsf{IFP}$-definable ordered treelike decompositions. Then for every $k \in \mathbb{N}^+$ the class $\mathcal{N}_k(\mathcal{B})$ admits $\mathsf{IFP}$-definable ordered treelike decompositions.

It is an easy exercise to prove this lemma directly. We shall derive it as a corollary of the following extension lemma.

Lemma 7.3.2 (Ordered Extension Lemma). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be classes of graphs such that $\mathcal{A}$ admits $\mathsf{IFP}$-definable orders and $\mathcal{B}$ admits $\mathsf{IFP}$-definable ordered treelike decompositions. Let $\Theta(\mathcal{P})$ be a simple 1-dimensional $\mathsf{IFP}$-graph transduction such that for all $G \in \mathcal{C}$ there is a tuple $\vec{p} \in G^\mathcal{P}$ such that $(G, \vec{p}) \in \mathcal{D}_{\Theta(\mathcal{P})}$ and it holds that:

$$\Theta[G, \vec{p}] \in \mathcal{A} \quad \text{and} \quad G \setminus \Theta[G, \vec{p}] \in \mathcal{B}.$$  

Then $\mathcal{C}$ admits $\mathsf{IFP}$-definable ordered treelike decompositions.

Proof. Let $\mathit{ord}(\vec{p}, z_1, z_2)$ be an $\mathsf{IFP}$-formula that defines a linear order on all graphs in $\mathcal{A}$, and let $\Lambda$ be an o-scheme that defines an ordered treelike decomposition on all graphs in $\mathcal{B}$.

To explain the proof, we fix a graph $G \in \mathcal{C}$. Let $\vec{p} \in G^\mathcal{P}$ such that $(G, \vec{p}) \in \mathcal{D}_{\Theta(\mathcal{P})}$, and let $\vec{q} \in G^\mathcal{P}$. We let $H^\mathcal{P} := \Theta[G, \vec{p}]$ and $\leq_{\mathcal{P}q} := \mathit{ord}[H^\mathcal{P}, \vec{q}, y_1, y_2]$. Note that $\leq_{\mathcal{P}q}$ is not necessarily a linear order of $H^\mathcal{P}$, because not all choices of $\vec{p}$ yield a structure $H^\mathcal{P} \in \mathcal{A}$, and even if $H^\mathcal{P} \in \mathcal{A}$ the binary relation $\leq_{\mathcal{P}q}$ is a linear order only for some choices of the parameters $\vec{q}$.

But there is an $\mathsf{IFP}$-formula $\mathit{check-ord}(\vec{q}, \vec{p})$ such that $G \models \mathit{check-ord}[\vec{q}, \vec{p}]$ if and only if $\leq_{\mathcal{P}q}$ is a linear order of $V(H^\mathcal{P})$. Let $G^\mathcal{P} := G \setminus H^\mathcal{P}$ and $\Delta^\mathcal{P} = (D^\mathcal{P}, \sigma^\mathcal{P}, \alpha^\mathcal{P}, \leq^\mathcal{P}) := \Lambda[G^\mathcal{P}]$. Note that $\Delta^\mathcal{P}$ is not necessarily an ordered treelike decomposition of $G^\mathcal{P}$, because $G^\mathcal{P}$ is not contained in $\mathcal{B}$ for all $\vec{p}$. But there is an $\mathsf{IFP}$-formula $\mathit{check-dec}(\vec{p})$ such that $G \models \mathit{check-dec}[\vec{p}]$ if and only if $G^\mathcal{P} \in OT_\Lambda$. Let $P$ be the set of all $\vec{p} \in G^\mathcal{P}$ such that $(G, \vec{p}) \in \mathcal{D}_{\Theta(\mathcal{P})}$ and $\leq_{\mathcal{P}q}$ is a linear order of $H^\mathcal{P}$ and $G^\mathcal{P} \in OT_\Lambda$. By the assumptions of the lemma, $P$ is nonempty. In the following we denote elements of $P$ by $p, p'$. If $p = \overline{pq}$ we let $H^p := H^\mathcal{P}$ and $\leq_p := \leq_{\mathcal{P}q}$ and $G^p := G^\mathcal{P}$ and $\Delta^p := (D^p, \sigma^p, \alpha^p, \leq^p) := \Delta^\mathcal{P}$. We write $\leq^p$ instead of $\leq_{D^p}$ and $N^p$ instead of $N^\mathcal{P}_p$. For every $v \in V(G)$, we let $A_v$ be the connected component of $G$ that contains $v$. For every $p \in P$ and $v \in V(G)$ we let $M_v^p$ be the set of all nodes $t \in V(D^p)$ with $\sigma^p(t) = \emptyset$ and $\alpha^p(t) = V(A)$ for some connected component $A$ of $G^p$ with $A \subseteq A_v$. Note that $M_v^p \neq \emptyset$. Actually, $M_v^p$ contains at least one node for every connected component $A$ of $G^p$ with $A \subseteq A_v$. Furthermore, if $v \in V(G_p)$ then at least one such component exists. We define an o-decomposition $\Delta := (D, \sigma, \alpha, \leq)$ of $G$ as follows:

$$V(D) := \{vwpt \mid v, w \in V(G), p \in P, t \in V(D^p)\},$$

$$E(D) := \{(vwpt, vwpu) \mid v, w \in V(G) \text{ with } v \neq w, p \in P, t \in V(D^p), u \in M_v^p\} \cup \{(vwpt, vwpu) \mid v, w \in V(G) \text{ with } v \neq w, p \in P, tu \in E(D^p)\},$$

$$\sigma(vwpt) := \begin{cases} \emptyset & \text{if } v = w, \\ (\sigma^p(t) \cup H^p) \cap A_v & \text{otherwise,} \end{cases}$$

$$\alpha(vwpt) := \begin{cases} V(A_v) & \text{if } v = w, \\ \alpha^p(t) \cap V(A_v) & \text{otherwise,} \end{cases}$$

for all $vwpt \in V(D)$.

Observe that for all distinct $v, w \in V(G)$ and all $p \in P$ the digraph $D$ contains a copy of $D^p$ with vertex set $\{vwp\} \times V(D^p)$. We say that this copy is indexed by $vw$. We call
the nodes $vvtp$ for $v \neq w$ internal nodes. The copies of the $D^p$s are joined together at new external nodes of the form $vvpt$, which have an edge to the copies of all nodes $u \in M^p$ in all copies of $D^p$ indexed by $vw$ for some $w \neq v$. Note that all external nodes are $\preceq^D$-minimal. We will see that the definitions of $\sigma$ and $\alpha$ guarantee that for all external nodes $vvtp$ we have $\beta(vvtp) = V(H^p \cap A_v)$ and for all internal nodes $vvpt$ we have $\beta(vvpt) = (\beta^p(t) \cup V(H^p)) \cap A_v$.

We will define the linear orders $\preceq$ of the nodes in such a way that they coincide with $\preceq^p$ on $\beta^p(t)$ and with $\preceq_p$ on $H^p$.

If we just wanted to construct an ordered treelike decomposition of $G$, it would be sufficient to take one copy of $D^p$ and one external node for each connected component of $G$ and each $p$. The reason we take many more copies of the $D^p$ and many more external nodes is that we want to make the decomposition IFP-definable. The reason we introduce the vertices $w$ in the index tuples is to distinguish external nodes (with $v = w$) from internal nodes (with $v \neq w$).

Claim 1. $(D, \sigma, \alpha)$ is a treelike decomposition of $G$.

Proof. It is obvious from the construction that the digraph $D$ is acyclic: external nodes have in-degree 0 and thus cannot appear in any cycle, and if we delete all external nodes what remains is a disjoint union of copies of the acyclic graphs $D^p$. Thus $(D, \sigma, \alpha)$ satisfies (TL.1).

To prove (TL.2) let $x \in V(D)$. Suppose first that $x = vvpt$ is an external node. Then $\sigma(x) = \emptyset$ and $\alpha(x) = V(A_v)$. Obviously, we have $\sigma(x) \cap \alpha(x) = \emptyset$ and $N^G(\alpha(x)) = \emptyset$. Suppose next that $x = vvpt$ is an internal node. Then $\sigma(x) = (\sigma^p(x) \cup H_v) \cap V(A_v)$ and $\alpha(x) = \alpha^p(x) \cap V(A_v)$. As $\sigma^p(x) \cap \alpha^p(x) = \emptyset$ by (TL.2) for $\Delta^p$ and $V(H^p) \cap \alpha^p(x) \subseteq V(H^p) \cap V(G^p) = \emptyset$, we have $\sigma(x) \cap \alpha(x) = \emptyset$. To prove that $N^G(\alpha(x)) \subseteq \sigma(x)$, let $v \in \alpha(x)$ and $w \in N^G(v) \setminus \alpha(x)$. If $w \in V(G^p)$ then $v \in \sigma^p(t)$, because by (TL.2) for $\Delta^p$ we have $N^G_v(\alpha^p(t)) \subseteq \sigma^p(t)$. Otherwise, $w \in V(H^p)$. As we also have $w \in V(A_v)$, because $v \in V(A_v)$, this implies $v \in V(H^p) \cap V(A_v) \subseteq \sigma(x)$.

To prove (TL.3) let $x \in V(D)$ and $y \in N^+_V(x)$. Suppose first that $x = vvpt$ is an external node. Then $y = vvpu$ for some $w \neq v$ and $u \in M^p$. We have $\alpha(x) = \gamma(x) = V(A_v)$ and $\alpha(y) = \alpha^p(u) \cap V(A_v) = \alpha(x)$ and $\gamma(y) = (\gamma^p(u) \cap V(H^p)) \cap V(A_v) = \gamma(x)$. Suppose next that $x = vvpt$ is an internal node. Then $y = vvpu$ for some $u \in N^+_V(t)$, and we have

$$\alpha(x) = \alpha^p(t) \cap V(A_v) \quad \gamma(x) = (\gamma^p(t) \cup V(H^p)) \cap V(A_v),$$

$$\alpha(y) = \alpha^p(u) \cap V(A_v) \quad \gamma(y) = (\gamma^p(u) \cup V(H^p)) \cap V(A_v).$$

By (TL.3) for $\Delta^p$, it follows that $\alpha(y) \subseteq \alpha(x)$ and $\gamma(y) \subseteq \gamma(x)$.

To prove (TL.4) let $x \in V(D)$ and $y_1, y_2 \in N^p_V(x)$. Suppose first that $x = vvpt$ is an external node. Then for $i = 1, 2$ we have $y_i = vw, pu_i$ for some $w \neq v$ and $u_i \in M^p$. As $w_i \in M^p$, we have $\sigma(w_i) = \emptyset$, and $\alpha^p(u_i)$ is the vertex set of a connected component $A_i$ of $G^p$ with $A_i \subseteq A_v$. It follows that $\sigma(y_i) = V(H^p) \cap V(A_v)$ and $\alpha(y_i) = V(A_i)$. Hence if $A_1 = A_2$ we have $y_1 \parallel A_2$, and if $A_1 \neq A_2$ we have

$$\gamma(y_1) \cap \gamma(y_2) = \left(\left(V(H^p) \cap V(A_v)\right) \cup V(A_1)\right) \cap \left(\left(V(H^p) \cap V(A_v)\right) \cup V(A_2)\right)$$

$$= V(H^p) \cap V(A_v)$$

$$= \sigma(y_1) \cap \sigma(y_2),$$

because $V(A_1) \cap V(A_2) = \emptyset$ and $V(A_i) \cap V(H^p) \subseteq V(G^p) \cap V(H^p) = \emptyset$.

M. Grohe, *Definable Graph Structure Theory*
Suppose next that \( x = vwpt \) is an internal node. Then for \( i = 1, 2 \) we have \( y_i = vwpu_i \) for some \( u_i \in N_p^D(t) \) and thus \( \sigma(y_i) = (\sigma^p(u_i) \cup V(H^p)) \cap V(A_v) \) and \( \alpha(y_i) = \alpha^p(u_i) \cap V(A_v) \). By (TL.4) for \( \Delta^p \), either \( u_1 \parallel^\Delta u_2 \) or \( u_1 \perp^\Delta u_2 \). If \( u_1 \parallel^\Delta u_2 \) then \( y_1 \parallel^\Delta y_2 \). If \( u_1 \perp^\Delta u_2 \) then

\[
\gamma(y_1) \cap \gamma(y_2) = (\gamma^p(u_1) \cup V(H^p)) \cap V(A_v) \cap (\gamma^p(u_1) \cup V(H^p)) \cap V(A_v)
\]

\[
= \left( (\gamma^p(u_1) \cap \gamma^p(u_2)) \cup V(H^p) \right) \cap V(A_v)
\]

\[
= \left( (\sigma^p(u_1) \cap \sigma^p(u_2)) \cup V(H^p) \right) \cap V(A_v)
\]

\[
= (\sigma^p(u_1) \cup V(H^p)) \cap V(A_v) \cap (\sigma^p(u_2) \cup V(H^p)) \cap V(A_v)
\]

\[
= \sigma(y_1) \cap \sigma(y_2).
\]

To prove (TL.5), let \( A \) be a connected component of \( G \). Then for all \( v \in V(A) \), all \( p \in P \) and all \( t \in V(D^p) \) we have \( \sigma(vwpt) = \emptyset \) and \( \alpha(vwpt) = V(A_v) = V(A) \).

\[
\text{Claim 2}.
\]

(1) For all \( v \in V(G) \), all \( p \in P \), and all \( t \in V(D^p) \) we have \( \beta(vwpt) = V(H^p) \cap A_v \).

(2) For all \( v, w \in V(G) \) with \( v \neq w \), all \( p \in P \), and all \( t \in V(D^p) \) we have \( \beta(vwpt) = (\beta^p(t) \cup V(H^p)) \cap A_v \).

\[
\text{Proof.} \text{ To prove (1), let } v \in V(G), p \in P, \text{ and } t \in V(D^p), \text{ and let } x = vwpt. \text{ Then}
\]

\[
N^D_+(x) = \{vwpu \mid w \in V(G) \setminus \{v\}, u \in M^p_v\}.
\]

We have

\[
\beta(x) = \gamma(x) \setminus \bigcup_{y \in N^D_+(x)} \alpha(y) = V(A_v) \setminus \bigcup_{y \in N^D_+(x)} \alpha(y).
\]

As \( \alpha(y) \subseteq \alpha^p(u) \) for some \( u \in V(D^p) \) and thus \( \alpha(y) \cap V(H^p) = \emptyset \), we have \( V(H^p) \cap V(A_v) \subseteq \beta(x) \).

To prove the converse, let \( A^1, \ldots, A^m \) be the connected components of \( G^p \) contained in \( A_v \), and observe that \( \bigcup_{i=1}^{m} A^i = G^p \cap A_v \). By (TL.5) for every \( i \in [m] \) there is a node \( u_i \in V(D^p) \) such that \( \sigma^p(u_i) = \emptyset \) and \( \alpha^p(u_i) = V(A^i) \). By definition, we have \( u_i \in M^p_v \). Let \( w \in V(G) \setminus \{v\} \) be arbitrary and \( y_i := vwpu_i \). Then \( y_i \in N^D_+(x) \) and \( \alpha(y_i) = V(A_i) \). Thus

\[
\beta(x) = V(A_v) \setminus \bigcup_{y \in N^D_+(x)} \alpha(y) \subseteq V(A_v) \setminus \bigcup_{i=1}^{m} \alpha(y_i) = V(A_v) \setminus V(G^p) = V(A_v) \cap V(H^p).
\]

To prove (2), let \( v, w \in V(G) \) with \( v \neq w \), \( p \in P \), and \( t \in V(D^p) \). Let \( x := vwpt \). Note that

\[
N^D_+(x) = \{vwpu \mid u \in N^p_+(t)\}.
\]

Thus

\[
\beta(x) = \gamma(x) \setminus \bigcup_{y \in N^D_+(x)} \alpha(y)
\]

\[
= \left( (\gamma^p(t) \cup V(H^p)) \cap V(A_v) \right) \setminus \bigcup_{u \in N^p_+(t)} \left( \alpha^p(u) \cap V(A_v) \right).
\]
Lifting Lemma 7.1.7.

Proof. Follows from the CC Decomposition Lemma 5.1.8 and the Ordered Decomposition decompositions. Then $C$ class of all connected components of the graphs in $C$ admits $\text{IFP}$-definable ordered treelike decompositions. Let $\Theta(G,1,\ldots,k)$ be the simple $\text{IFP}$-graph transduction with $\theta_V(x_1,\ldots,x_k,y) := \bigvee_{i=1}^k y = x_i$ and $\theta_E(x_1,\ldots,x_k,y_1,y_2) := E(y_1,y_2)$. Then for all graphs $G$ and $v_1,\ldots,v_k \in V(G)$ it holds that

$$\Theta[G,v_1,\ldots,v_k] = G[\{v_1,\ldots,v_k\}] \in \mathcal{G}_k.$$ 

As the class $\mathcal{G}_k$ is finite up to isomorphism, by Lemma 3.2.11 it admits $\text{IFP}$-definable orders. We apply the Ordered Extension Lemma with $A:= \mathcal{G}_k$ and $C := \mathcal{N}_k(B) \setminus B$. Then we apply the Union Lemma 7.1.10 to $C$ and $B$. 

It would be even nicer if we could prove the strengthening of the Ordered Extension Lemma where we only require the class $A$ to admit $\text{IFP}$-definable ordered treelike decompositions instead of definable orders. Unfortunately, this stronger version of the lemma is false, as the following simple example shows.

Example 7.3.3. Let $A$ be the class of all graphs $G$ with $E(G) = \emptyset$ and $B := \mathcal{G}_1$. Both of these classes admit $\text{IFP}$-definable ordered treelike decompositions. Let $\Theta(x)$ be the simple $\text{IFP}$-graph transduction with $\theta_V(x,y) := y \neq x$ and $\theta_E(x,y_1,y_2) := \text{false}$. Then for every graph $G$ and every $v \in V(G)$ it holds that $\Theta[G,v] = (V(G) \setminus \{x\},\emptyset) \in A$ and $G \setminus \Theta[G,v] = G[\{v\}] \in B$. However, the class $C := \mathcal{G}$ of all graphs does not admit $\text{IFP}$-definable treelike decompositions.

Let me remark that requiring $\Theta[G,\overline{u}]$ to be an induced subgraph of $G$ does not help. I leave it to the reader to construct a counterexample based on the fact that the class of all connected bipartite graphs does not admit $\text{IFP}$-definable ordered treelike decompositions.

The next lemma allows us to lift decompositions from the connected components of a graph to the whole graph.

Lemma 7.3.4 (Component Lifting Lemma). Let $C$ be a class of graphs such that the class of all connected components of the graphs in $C$ admits $\text{IFP}$-definable ordered treelike decompositions. Then $C$ admits $\text{IFP}$-definable ordered treelike decompositions.

Proof. Follows from the CC Decomposition Lemma 5.1.8 and the Ordered Decomposition Lifting Lemma 7.1.7.

M. Grohe, Definable Graph Structure Theory
7.4 Canonisation via Definable Ordered Treelike Decompositions

**Theorem 7.4.1 (Canonisation Theorem).** Let $\mathcal{C}$ be a class of graphs that admits $\mathsf{IFP}$-definable ordered treelike decompositions. Then $\mathcal{C}$ admits $\mathsf{IFP+C}$-definable canonisation.

**Corollary 7.4.2.** Let $\mathcal{C}$ be a class of structures that admits $\mathsf{IFP}$-definable ordered treelike decompositions. Then $\mathsf{IFP+C}$ captures polynomial time on $\mathcal{C}$.

**Corollary 7.4.3 (Grohe and Marínó [53]).** For every $k \in \mathbb{N}$, $\mathsf{IFP+C}$ captures polynomial time on the class of all graphs of tree width at most $k$.

**Proof.** Recall from Example 7.1.5 that the class of all graphs of tree width at most $k$ admits $\mathsf{IFP}$-definable ordered treelike decompositions. □

The rest of this section is devoted to the proof of the Canonisation Theorem 7.4.1. The proof is based on the proof of Lemma 3.3.20 in Section 3.3.4 and I advise the reader to review that proof before continuing here.

#### 7.4.1 Sunflower Canonisation

In this subsection, we prove a generalisation of Lemma 3.3.20. In combinatorics, a *sunflower* is a family $S_1, \ldots, S_m$ of sets such that for all $i, j, i', j' \in [m]$ with $i \neq j, i' \neq j'$ it holds that $S_i \cap S_j = S_{i'} \cap S_{j'}$. The set $S_i \cap S_j = \bigcap_{i=1}^m S_i$ is called the *core* of the sunflower. For example, the vertex sets of the connected components of a graph form a sunflower, albeit a degenerate one with an empty core.

**Lemma 7.4.4 (Sunflower Canonisation Lemma).** Let $\Theta(\bar{x})$ be an $\mathsf{IFP+C}$-transduction from graphs to ordered graphs and $\chi(\bar{x}, y)$ an $\mathsf{IFP+C}$-formula. Then there is a normal $\mathsf{IFP+C}$-transduction $\Theta'$ from graphs to ordered graphs such that the following holds. Let $G$ be a graph and $P$ the set of all $p \in G$ with $(G, p) \in D_{\Theta(\bar{x})}$. For all $p \in P$, let $A_p$ be the induced subgraph of $G$ with vertex set $\chi[G, p, y]$, and let $B_p := \Theta[G, p]$. Suppose that the following conditions are satisfied.

(i) $G = \bigcup_{p \in P} A_p$.

(ii) There is a set $Q \subseteq V(G)$ such that for all $\bar{p}, \bar{q} \in P$, either $V(A_{\bar{p}}) = V(A_{\bar{q}})$ or $V(A_{\bar{p}}) \cap V(A_{\bar{q}}) = Q$.

(iii) For all $\bar{p} \in P$ the ordered graph $B_{\bar{p}}$ is an ordered copy of $A_{\bar{p}}$.

(iv) There is a linear order $\preceq_Q$ of $Q$ and for all $\bar{p} \in P$ an isomorphism $f_{\bar{p}}$ from $A_{\bar{p}}$ to the graph underlying $B_{\bar{p}}$ that maps $Q$ to an initial segment of the order $\preceq_{B_{\bar{p}}}$, such that

$$v \preceq_Q w \iff f_{\bar{p}}(v) \preceq_{B_{\bar{p}}} f_{\bar{p}}(w)$$

for all $v, w \in Q$.

Then $\Theta'$ canonises $G$. Furthermore, there is an isomorphism $f$ from $G$ to the graph underlying $\Theta'[G]$ that maps $Q$ to an initial segment of $\preceq_{\Theta'[G]}$ such that $v \preceq_Q w \iff f(v) \preceq_{\Theta'[G]} f(w)$ for all $v, w \in Q$.

Preliminary Version
Before we prove this lemma, we note that it implies Lemma \[3.3.20\]. Indeed, let \( \Theta(\overline{w}) \) be an IFP\( +C \)-transduction form graphs to ordered graphs, and let \( \chi(\overline{w}, y) \) be an IFP\( +C \)-formula. Let \( \Theta^1(\overline{w}) \) be the transduction obtained from \( \Theta(\overline{w}) \) by restricting the domain to all \((G, \overline{p})\) such that \( \chi(G, \overline{p}, y) \) is the vertex set of a connected component of \( G \) (as we have done it in the proof of Claim 1 in the proof of Lemma \[3.3.20\]). Then if \( G \) is a graph that satisfies conditions (i) and (ii) of Lemma \[3.3.20\] with respect to \( \Theta(\overline{w}) \) and \( \chi(\overline{w}, y) \), then \( G \) satisfies conditions (i)–(iv) of the Sunflower Canonisation Lemma \[7.4.4\] with respect to \( \Theta^1(\overline{w}) \) and \( \chi(\overline{w}, y) \) (if we let \( Q \) be the empty set). Hence the transduction \( \Theta' \) of Lemma \[7.4.4\] canonises \( G \).

**Proof of the Sunflower Canonisation Lemma \[7.4.4\]**. The proof is very similar to the proof of Lemma \[3.3.20\]. Throughout the proof, we fix a graph \( G \). As usually, the transduction \( \Theta' \) we shall define will not depend on \( G \). We define \( P \), the graphs \( A_{\overline{p}} \), and the ordered graph \( B_{\overline{p}} \) as in the statement of the lemma and assume that conditions (i)–(iv) are satisfied. We let \( Q := \bigcap_{\overline{p} \in P} V(A_{\overline{p}}) \). Then by (ii), for all \( \overline{p}, \overline{q} \in P \), either \( V(A_{\overline{p}}) \cap V(A_{\overline{q}}) = Q \) or \( V(A_{\overline{p}}) = V(A_{\overline{q}}) \). For all \( \overline{p} \in P \), we choose \( f_{\overline{p}} \) as in (iv). It is important to note that \( f_{\overline{p}} \) is not IFP\( +C \)-definable. Hence we are not allowed to use \( f_{\overline{p}} \) in the definition of the canonisation \( \Theta' \).

Without loss of generality we may assume that the transduction \( \Theta(\overline{w}) \) is normal and that in addition to (i)–(iv) the following “componentwise parameter independence” holds.

**(v)** For all \( \overline{p}, \overline{q} \in P \), if \( A_{\overline{p}} = A_{\overline{q}} \) then \( B_{\overline{p}} = B_{\overline{q}} \).

Indeed, if (v) does not hold we can simply restrict the domain of \( \Theta \) to those \( \overline{p} \) that define a lexicographically minimal \( B_{\overline{p}} \) among all \( B_{\overline{p}} \) for \( \overline{q} \in P \) with \( A_{\overline{p}} = A_{\overline{q}} \). This will guarantee that if \( A_{\overline{p}} = A_{\overline{q}} \) then \( B_{\overline{p}} \cong B_{\overline{q}} \), and then equality follows because the transduction is normal.

Let \( A_1, \ldots, A_m \) be an enumeration of \( \{A_{\overline{p}} \mid \overline{p} \in P\} \) (without repetitions). Then by (ii), for all \( i, j \in [m] \) with \( i \neq j \) we have \( V(A_i) \cap V(A_j) = Q \). For every \( i \in [m] \), let \( P_i \) be the set of all \( \overline{p} \in P \) such that \( A_{\overline{p}} = A_i \), and let \( B_i := B_{\overline{p}} \) for some \( \overline{p} \in P_i \). By (v), \( B_i \) does not depend on the choice of \( \overline{p} \). Let \( q := |Q| \), and note that by (iv), for all \( i, j \in [m] \) the initial segments of \( B_i \) and \( B_j \) of length \( q \) coincide, that is,

\[
B_i[0, q - 1] = B_j[0, q - 1].
\]

Without loss of generality we may assume that we have chosen the indices in our enumeration of the \( A_i \) in such a way that \( B_1 \leq_{s-lex} B_2 \leq_{s-lex} \ldots \leq_{s-lex} B_m \). Let \( 1 = i_1 < i_2 < \ldots < i_{\ell} < i_{\ell+1} = m + 1 \) such that

\[
B_1 = B_2 = \ldots = B_{i_2 - 1} <_{lex} B_{i_2} = \ldots = B_{i_3 - 1} <_{lex} \ldots <_{lex} B_{i_\ell} = \ldots = B_m.
\]

For every \( j \in [\ell] \), let \( J_j := \{i_j, i_j + 1, \ldots, i_{j+1} - 1\} \) and \( A^j := A_{i_j} \) and \( B^j := B_{i_j} \) and \( n_j := |B^j| \) and \( k_j := i_{j+1} - i_j \) and

\[
s_j := \sum_{j' = 1}^{j} (n_{j'} - q) \cdot k_j.
\]

Furthermore, we let \( s_0 := 0 \). To understand the definition of \( s_j \), note that the order of the graph \( \bigcup_{i \in J_j} A_i \) is \( q + (n_j - q) \cdot k_j \). Hence the order of the graph \( \bigcup_{j' = 1}^{j} \bigcup_{i \in J_{j'}} A_i = \bigcup_{j' = 1}^{j+1} A_i \) is \( q + s_j \).

Now we define \( C_j \) to be the ordered graph obtained by concatenating \( k_j \) copies of \( B^j \) and identifying their initial segments of length \( q \), so that \( C_j \) is an ordered copy of \( \bigcup_{i \in J_j} A_i \).
Then we shift the universe of the $C_j$ to make them all fit together. To be precise, for every $i \in [0, k_j - 1]$ we let $B_i^j$ be the image of $B_i^j$ under the mapping

$$a \mapsto \begin{cases} 
    a & \text{if } a \in [0, q - 1], \\
    s_{j-1} + (n_j - q) \cdot i + a & \text{if } a \in [q, n_j - 1].
\end{cases}$$

Then

$$V(B_i^j) = [0, q - 1] \cup [q + s_{j-1} + (n_j - q) \cdot i, q + s_{j-1} + (n_j - q) \cdot (i + 1) - 1].$$

We define $C_j$ by letting $V(C_j) := \bigcup_{i=0}^{k_j-1} V(B_i^j)$ and $E(C_j) := \bigcup_{i=0}^{k_j-1} E(B_i^j)$ and $\leq C_j := \leq |V(C_j)|$. Note that

$$V(C_j) = [0, q - 1] \cup [q + s_{j-1}, q + s_{j-1}].$$

Let $C$ be the ordered graph with $V(C) = \bigcup_{j=1}^{\ell} E(C_j)$ and $E(C) = \bigcup_{j=1}^{\ell} E(C_j)$ and $\leq C := \leq |V(C)|$. Then $C$ is an ordered copy of

$$\bigcup_{j=1}^{\ell} \bigcup_{i \in J_j} A_i = \bigcup_{i=1}^{m} A_i = G.$$

Indeed, $C$ is the desired canonical copy of $G$. It remains to construct an $\text{IFP}+\text{C}$-transduction $\Theta'$ (not depending on $G$) such that $\Theta'[G] = C$.

As in the proof of Lemma 3.3.20, we construct $\Theta'$ in a sequence of claims. As the claims here are almost identical to those in the proof of Lemma 3.3.20 and their proofs are very similar (just taking into account the modified definitions of $s_j$ and $C_j$), we omit the proofs.

Claim 1. For all $j \in [\ell]$ and $i, i' \in J_j$ we have $|P_i| = |P_i'|$.

For every $j \in [\ell]$, let $p_j := |P_i|$ for some (and hence for all) $i \in J_j$ and

$$P_j := \bigcup_{i \in J_j} P_i.$$

Then $|P_j| = p_j \cdot k_j$.

Claim 2. There is an $\text{IFP}+\text{C}$-formula $P(\pi, y)$ such that for all $\pi \in G^\pi$ and $j \in \text{Num}(G)$

$$G \models P[\pi, j - 1] \iff j \in [\ell] \text{ and } \pi \in P_j.$$

Claim 3. There is an $\text{IFP}+\text{C}$-formula $k(z, y)$ such that for all $k, j \in \text{Num}(G)$

$$G \models k[k, j - 1] \iff j \in [\ell] \text{ and } k = k_j.$$

Claim 4. There is an $\text{IFP}+\text{C}$-formula $s(z, y)$ such that for all $s, j \in \text{Num}(G)$

$$G \models s[s, j - 1] \iff j \in [\ell] \text{ and } s = s_j.$$
Claim 5. There is an \( \text{IFP+C}\)-transduction \( \Theta^4(y) \) from graphs to ordered graphs such that for all \( j \in \text{Num}(G) \) we have \( (G,j) \in \mathcal{D}_{\Theta^4(y)} \) if and only if \( j \in [0,\ell-1] \), and for all \( j \in [\ell] \) we have \( \Theta^4[G,j-1] = C_j \).

Now we can define the transduction \( \Theta' \) with \( \Theta'[G] = C \) exactly as in the proof of Lemma 3.3.20, again exploiting that \( V(C) = \bigcup_{j=1}^{\ell} V(C_j) \) and \( E(C) = \bigcup_{j=1}^{\ell} E(C_j) \) and \( \leq C = \leq |V(C)| \).

For the proof of the Canonisation Theorem 7.4.1, we actually need the following parametrised version of the Sunflower Canonisation Lemma 7.4.4.

Lemma 7.4.5 (Parameterised Sunflower Canonisation Lemma). Let \( \Theta(X,x) \) be an \( \text{IFP+C}\)-transduction from graphs to ordered graphs (where \( X \) is a tuple of variables that may contain both relation variables and individual variables) and \( \chi(X,x,y) \) an \( \text{IFP+C}\)-formula. Then there is an \( \text{IFP+C}\)-transduction \( \Theta'(X) \) from graphs to ordered graphs such that the following holds. Let \( G \) be a graph and \( P \in \Gamma \) with \( (G,P) \in \mathcal{D}_{\Theta(X,x)} \). For all \( p \in P \), let \( A_p \) be the induced subgraph of \( G \) with vertex set \( \chi[G,P,p,y] \), and let \( B_p := \Theta[G,P,p] \). Let

\[
H := \bigcup_{p \in P} A_p.
\]

Suppose that the following conditions are satisfied.

(i) There is a set \( Q \subseteq V(H) \) such that for all \( p,p' \in P \), either \( V(A_p) = V(A_{p'}) \) or \( V(A_p) \cap V(A_{p'}) = Q \).

(ii) For all \( p \in P \) the ordered graph \( B_p \) is an ordered copy of \( A_p \).

(iii) There is a linear order \( \leq Q \) of \( Q \) and for all \( p \in P \) an isomorphism \( f_p \) from \( A_p \) to the graph underlying \( B_p \) that maps \( Q \) to an initial segment of the order \( \leq B_p \), such that

\[
v \leq Q w \iff f_p(v) \leq B_p f_p(w)
\]

for all \( v,w \in Q \).

Then \( \Theta'[G,P] \) is an ordered copy of \( H \). Furthermore, there is an isomorphism \( f \) from \( H \) to the graph underlying \( \Theta'[G,P] \) that maps \( Q \) to an initial segment of \( \leq \Theta'[G,P] \) such that \( v \leq Q w \iff f(v) \leq \Theta'[G,P] f(w) \) for all \( v,w \in Q \).

Proof. The proof is completely analogous to the proof of Lemma 7.4.1 only the notation gets even more cluttered.

7.4.2 Proof of the Canonisation Theorem

Let \( \Lambda \) be an \( \ell \)-dimensional od-scheme. We shall define an \( \text{IFP+C}[\{E\}, \{E, \leq\}]\)-transduction \( \Theta \) that canonises the class \( \mathcal{OT}_\Lambda \). Let \( \text{dom}_\Lambda \) be an \( \text{IFP}\)-sentence that defines the class \( \mathcal{OT}_\Lambda \).
Our goal is to define the formulae $\varphi, \psi$.

For all nodes $t \in V(D)$ we let $n_t := |\gamma(t)|$ and $s_t := |\sigma(t)|$. Furthermore, we let $C_t := G[\gamma(t)]$.

Let $\pi$ be an $\ell$-tuple of vertex variables, and let $y_1, y_2$ be number variables. Furthermore, let $X, Y$ be relation variables whose types match the types of $\pi$ and $\pi y_1 y_2$, respectively. That is, $X$ ranges over subsets of $V(G)^{\ell}$ and $Y$ ranges over subsets of $V(G)^{\ell} \times \text{Num}(G)^2$. We shall define a simultaneous fixed-point formula

$$\text{ifp} \left( \begin{array}{c}
X\pi \\
Y\pi y_1 y_2
\end{array} \leftarrow \begin{array}{c}
\varphi(\pi, X, Y) \\
\psi(\pi, y_1, y_2, X, Y)
\end{array} \right) \pi.$$

Let $X^i, Y^i$, for $i \in \mathbb{N}$, be the stages of the fixed-point process in $G$. That is, $X^i := Y^i := \emptyset$ and

$$X^{i+1} := X^i \cup \{\pi \in V(G)^{\ell} \mid G \models \varphi(\pi, X^i, Y^i)\},$$

$$Y^{i+1} := Y^i \cup \{\pi j_1 j_2 \in V(G)^{\ell} \times \text{Num}(G)^2 \mid G \models \psi(\pi, j_1, j_2, X^i, Y^i)\}$$

for all $i \in \mathbb{N}$. We let $X^\infty := \bigcup_{i \in \mathbb{N}} X^i$ and $Y^\infty := \bigcup_{i \in \mathbb{N}} Y^i$ be the fixed points. For all $\ell$-tuples $\pi \in V(G)^{\ell}$ and all $i \in \mathbb{N} \cup \{\infty\}$, we let

$$Y^i(\pi) := \{(j_1, j_2) \in \text{Num}(G) \mid \pi j_1 j_2 \in Y^i\}.$$

In the following, we denote tuples in $V(G)^{\ell}$ by $s, t, u$ if they are intended to be nodes of $D$. Our goal is to define the formulae $\varphi, \psi$ such that the following conditions are satisfied.

(A) $X^i$ is the set of all nodes of $D$ of depth at most $i - 1$.

(B) For all nodes $t \in X^i$ it holds that $Y^i(t) \subseteq [0, n_t - 1]^2$.

Furthermore, for all nodes $t \in X^i$ there is a bijective mapping $f_t$ from $\gamma(t)$ to $[0, n_t - 1]$, which only depends on $t$ and not on $i$, such that the following conditions are satisfied.

(C) For all $v, w \in \gamma(t)$ it holds that $vw \in E(G) \iff (f_t(v), f_t(w)) \in Y^i(t)$.

(D) For all $v \in \gamma(t)$ it holds that $v \in \sigma(t) \iff f_t(v) \in [0, s_t - 1]$.

(E) For all $v, w \in \sigma(t)$ it holds that $v \leq_t w \iff f_t(v) \leq f_t(w)$.

Suppose we have defined $\varphi, \psi$ such that [A], [E] are satisfied. Then we can complete the proof of Theorem 7.4.1. Let $\Theta^1(\pi)$ be the following $\text{IFP+C}$-transduction from graphs to graphs.

- $\theta^1_{\text{dom}}(\pi) := \text{dom}_{\Lambda} \land \lambda_V(\pi)$.
  Then $(G, \overline{\pi}) \in D_{\Theta^1(\pi)}$ if and only if $G \in \mathcal{OT}_\Lambda$ and $\overline{\pi} \in V(D)$.

- $\theta^1_V(\pi, y) := \exists y' (\#x' \lambda_\gamma(\pi, x') = y' \land y < y').$
  Here $\lambda_\gamma(\pi, x') := \lambda_\sigma(\pi, x') \lor \lambda_\alpha(\pi, x')$ defines $\gamma(\pi)$. The formula $\theta^1_V(\pi, y)$ makes sure that $V(\Theta^1[G, \overline{\pi}]) = [0, |\gamma(\overline{\pi})| - 1] = [0, n_t - 1]$. 

Preliminary Version
\[ \theta_E(\overline{x}, y_1, y_2) := \text{ifp} \left( \begin{array}{c}
Y \overline{y}_1 y_2 \leftarrow \psi(\overline{x}, y_1, y_2, X, Y) \\
X \overline{x} \leftarrow \varphi(\overline{x}, Y, Y)
\end{array} \right) \overline{y}_1 y_2. \]

This formula makes sure that \( E(\Theta^1[G, \overline{v}]) = Y^\infty(\overline{v}) \).

- \( \theta_1(\overline{x}, y_1, y_2) := y_1 \leq y_2 \).

We want to apply Lemma 3.3.20 to \( \Theta(\overline{x}) \). It follows from (TL.5) that \( \Theta(\overline{x}) \) and \( G \) satisfy condition (i) of the lemma, and it follows from (C) that they satisfy (ii). Hence we can apply the lemma. The resulting transduction \( \Theta \) canonises \( G \).

It remains to define \( \varphi, \psi \) such that (A)–(E) are satisfied. We let

\[ \varphi(\overline{x}, X, Y) := \lambda_V(\overline{x}) \land \forall \overline{x}' \left( (\lambda_V(\overline{x}') \land \lambda_E(\overline{x}, \overline{x}')) \rightarrow X = \overline{x}' \right), \]

Then \( \overline{v} \) enters \( X^{i+1} \) if and only if all its children in \( D \) are in \( X^i \). By induction, this implies (A). The difficult part is to define \( \psi \). Let \( i \in \mathbb{N} \) and suppose that \( X^i, Y^i \) are defined and satisfy (A)–(E). For every \( u \in X^i \), let \( f_u : \gamma(u) \rightarrow [0, n_u - 1] \) be a bijective mapping that satisfies (C)–(E). Let

\[ D_u := ([0, n_u - 1], Y^j(u), \leq). \]

By the induction hypothesis, \( D_u \) is an ordered graph with the following properties.

- (F) \( D_u \) is an ordered copy of \( C_u \), and the mapping \( f_u \) is an isomorphism from the \( C_u \) to the graph underlying \( D_u \).
- (G) \( f_u(\sigma(u)) = [0, s_u - 1] \).
- (H) For all \( v, w \in \sigma(u) \) it holds that \( v \leq u w \iff f_u(v) \leq f_u(w) \).

Let \( t \in X^{i+1} = \varphi[G, \overline{x}, X^i, Y^j] \). It is our goal to define a mapping \( f_t \) and the relation \( Y^{i+1}(t) \) such that (B)–(E) are satisfied. Furthermore, we have to make sure that the definition of \( Y^{i+1}(t) \) from \( X^i, Y^j \) can be formalised in the logic \( \text{IFP} + \mathbb{C} \). Let me emphasise again that the mapping \( f_t \) will not be definable and that we are not allowed to use the mappings \( f_u \) for \( u \in X^i \) in the \( \text{IFP} + \mathbb{C} \)-definition of \( Y^{i+1} \).

**Step 1. Reordering the vertices in \( \sigma(u) \).**

Let \( u \in N^D_\beta(t) \). As \( \sigma(u) \subseteq \beta(t) \) by the \( \beta\gamma\sigma \)-Lemma 4.2.9, the vertices in \( \sigma(u) \) are linearly ordered by both \( \leq_t \) and \( \leq_u \). These two linear orders do not necessarily coincide. Let \( g_{t,u} : \gamma(u) \rightarrow \gamma(u) \) be the bijective mapping with \( v \leq_t w \iff g_{t,u}(v) \leq u g_{t,u}(w) \) for all \( v, w \in \sigma(t) \) and \( g_{t,u}(v) = v \) for all \( v \in \alpha(u) \). Furthermore, let \( f_{t,u}' := f_u \circ g_{t,u} \) and \( D_{t,u}' := f_{t,u}'(C_u) = f_{t,u}' \circ f_u^{-1}(D_u) \). Then

- (I) \( V(D_{t,u}') = [0, n_u - 1] \);
- (J) \( D_{t,u}' \) is an ordered copy of \( C_u \), and the mapping \( f_{t,u}' \) is an isomorphism from the \( C_u \) to the graph underlying \( D_{t,u}' \);
- (K) \( f_{t,u}'(\sigma(u)) = [0, s_u - 1] \);
- (L) for all \( v, w \in \sigma(u) \) it holds that \( v \leq_t w \iff f_{t,u}'(v) \leq f_{t,u}'(w) \).
7.4. Canonisation via Definable Ordered Treelike Decompositions

Step 3. Adding $\beta$

By a similar argument as in Step 1, we may further assume that Formally, we let Sunflower Canonisation Lemma 7.4.5 with $Q$ ordered graph isomorphic to $(\cdot \dot{\scriptstyle \gamma} \dot{\scriptstyle \chi})$

Note that $\Theta$ be an $\mathsf{IFP+C}$-transduction $\Theta^2(X,Y,\bar{x},\bar{x}')$ such that $\Theta^2[G,X^i,Y^i,t,u] = D't_u$.

Claim 1. There is an $\mathsf{IFP+C}$-transduction $\Theta^2(X,Y,\bar{x},\bar{x}')$ such that $\Theta^2[G,X^i,Y^i,t,u] = D't_u$.

Proof. Straightforward.

Step 2. Canonising the sunflower at $\sigma(u)$.

Let $u \in N^P(t)$, and let $P_{t,u}$ be the set of all $u' \in N^P(t)$ with $\sigma(u') = \sigma(u)$. Note that by $(\text{TL.4})$ the sets $\gamma(u')$ for $u' \in P_{t,u}$ form a sunflower with core $\sigma(u)$. In this step, we shall canonise the union $H_{t,u} := \bigcup_{u' \in P_{t,u}} C_{u'}$

by an application of the Parametrised Sunflower Canonisation Lemma 7.4.5. We let $\Theta^3(X,Y,\bar{x},\bar{x}',\bar{x}'')$ be an $\mathsf{IFP+C}$-transformation such that

- $(G,X^i,Y^i,t,u,u') \in D_{\Theta(\bar{x})}$ if $u' \in P_{t,u}$,
- $\Theta^3[G,X^i,Y^i,t,u,u'] = D't_{u'}$.

Formally, we let

$$
\begin{align*}
\theta^3_{\text{dom}}(X,Y,\bar{x},\bar{x}',\bar{x}'') &:= \text{dom}_A \land \varphi(\bar{x},X,Y) \land \lambda_E(\bar{x}',\bar{x}') \land \lambda_E(\bar{x},\bar{x}'') \\
& \land \forall y(\lambda_{\sigma}(\bar{x}',y) \leftrightarrow \lambda_\sigma(\bar{x}'',y)), \\
\theta^3_E(X,Y,\bar{x},\bar{x}',\bar{x}'',y) &:= \theta^3_E(X,Y,\bar{x},\bar{x}',\bar{x}'',y), \\
\theta^3_E(X,Y,\bar{x},\bar{x}',\bar{x}'',y_1,y_2) &:= \theta^3_E(X,Y,\bar{x},\bar{x}',\bar{x}'',y_1,y_2).
\end{align*}
$$

Note that $\Theta^3(X,Y,\bar{x},\bar{x}',\bar{x}'')$ (as $\Theta(\bar{x},\bar{x}')$ with $\bar{x} := (X,Y,\bar{x})$ and $\bar{x}'' := \bar{x}'$) and $\lambda_{\gamma}(\bar{x}'',y)$ (as $\chi(\bar{x},y,z)$) and $G$ and $\bar{P} := (X^i,Y^i,t,u)$ satisfy conditions (i)-(iv) of the Parametrised Sunflower Canonisation Lemma 7.4.5 with $Q = \sigma(u)$ and $\leq_Q := \leq_t \cap Q^2$. Hence there is a normal $\mathsf{IFP+C}$-transduction $\Theta^4(X,Y,\bar{x},\bar{x}')$ such that

(M) $I_{t,u} := \Theta^4[G,X^i,Y^i,t,u]$ is an ordered copy of $H_{t,u}$, and there is an isomorphism $h_{t,u}$ from $H_{t,u}$ to the graph underlying $I_{t,u}$ that maps $\sigma(u)$ to $[0, s_u - 1]$.

By a similar argument as in Step 1, we may further assume that

(N) For all $v, w \in \sigma(u)$ it holds that $v \leq_t w \iff h_{t,u}(v) \leq h_{t,u}(w)$.

Step 3. Adding $\beta(t)$.

In this step, we shall canonise $A_{t,u} := G[\beta(t)] \cup H_{t,u}$. Let $b_t := |\beta(t)|$, and let $B'_t$ be the normal ordered graph isomorphic to $(G[\beta(t)], \leq_t)$, and let $k_t$ be the isomorphism from $(G[\beta(t)], \leq_t)$ to $B'_t$. That is, $k_t$ is the unique order preserving mapping from $(\beta(t), \leq_t)$ to $(\{0, b_t - 1\}, \leq)$. Suppose that $k_t(\sigma(u)) = \{j_0, \ldots, j_{s_u - 1}\}$, where $j_0 < j_1 < \ldots < j_{s_u - 1}$. Let $I'_{t,u}$ be the image of $I_{t,u}$ under the mapping

$$
a \mapsto \begin{cases} 
  j_a & \text{if } a \in [0, s_u - 1], \\
  b_t - s_u + a & \text{if } a \in [s_u, n_u - 1].
\end{cases}
$$

Preliminary Version
Let \( L_{t,u} \) be the “ordered union” of \( B'_t \) and \( I'_{t,u} \), that is, the ordered graph with

\[
V(L_{t,u}) = V(B'_t) \cup V(I'_{t,u}) = [0, b_t - s_u + n_u - 1]
\]

and \( E(L_{t,u}) = E(B'_t) \cup E(I'_{t,u}) \) and \( \leq_{L_{t,u}} \subseteq |V(J_{t,u})| \). Then

(O) \( L_{t,u} \) is an ordered copy of \( A_{t,u} \), and there is an isomorphism \( \ell_{t,u} \) from \( A_{t,u} \) to the graph underlying \( L_{t,u} \) that maps \( \beta(t) \) to \([0, b_t - 1] \).

(P) For all \( v, w \in \beta(t) \) it holds that \( v \leq_t w \iff \ell_{t,u}(v) \leq \ell_{t,u}(w) \).

**Claim 2.** There is an \( \text{IFP} + \text{C} \)-transduction \( \Theta^5(X, Y, \bar{x}, \bar{x}') \) such that \( \Theta^5[G, X^i, Y^i, t, u] = L_{t,u} \).

**Proof.** Straightforward.

**Step 4. Canonising \( C_t \).**

Observe that the vertex sets of the graphs \( A_{t,u} \), for \( u \in N^D(t) \), form another sunflower with core \( \beta(t) \). Moreover, the union of these graphs is \( C_t \). Hence we can apply the Parametrised Sunflower Canonisation Lemma \[7.4.5\] again to canonise \( C_t \).

We first construct a formula \( \chi(\bar{x}, \bar{x}', y) \) that defines the vertex set of \( A_{t,u} \). We let

\[
\chi(\bar{x}, \bar{x}', z) := \lambda (\bar{x}, z) \land \neg \exists \bar{x}'' (\lambda_E(\bar{x}, \bar{x}'') \land \neg \forall y (\lambda_\sigma(\bar{x}'', y) \iff \lambda_\sigma(\bar{x}', y)) \land \lambda_\alpha(\bar{x}'', z))
\]

Observe that

\[
\chi[G, t, u, z] = \gamma(t) \setminus \bigcup_{u' \in N^D(t)} \alpha(u'') = V(A_{t,u}).
\]

Then \( \Theta^5(X, Y, \bar{x}, \bar{x}') \) (as \( \Theta(\bar{X}, \bar{y}) \) with \( \bar{X} := (X, Y, \bar{x}) \)) and \( \chi(\bar{x}, \bar{x}', z) \) and \( G \) and \( \bar{P} := (X^i, Y^i, t) \) satisfy conditions (i)–(iv) of the Parametrised Sunflower Canonisation Lemma \[7.4.5\]. Let \( \Theta^6(X, Y, \bar{x}) \) be the resulting transduction that canonises \( C_t = \bigcup_{u \in N^D(t)} A_{t,u} \).

**Step 5. The final touch.**

We cannot let \( \Theta^6[G, X^i, Y^i, t] \) be \( D_t \) because the vertices corresponding to those of \( \sigma(t) \) do not necessarily form an initial segment. This would lead to problems with (D).

However, similarly to Step 1 we can reorder the vertices in such a way that those of \( \sigma(t) \) appear first, ordered according to \( \leq_t \). We let \( \Theta^7(X, Y, \bar{x}) \) be a transduction that reflects the modified order and \( D_t := \Theta^7[G, X^i, Y^i, t] \). Then

(Q) \( D_t \) is an ordered copy of \( C_t \).

(R) There is an isomorphism \( f_t \) from \( C_t \) to the graph underlying \( D_t \) such that \( f_t(\sigma(t)) = [0, s_t - 1] \).

(S) For all \( v, w \in \sigma(t) \) we have \( v \leq_t w \iff v \leq w \).

To complete the proof, we let

\[
\psi(\bar{x}, y_1, y_2, X, Y) := \varphi(X, Y, \bar{x}) \land \theta^7_V(X, Y, \bar{x}, y_1) \land \theta^7_V(X, Y, \bar{x}, y_2) \land \theta^7_E(X, Y, \bar{x}, y_1, y_2).
\]

Then \( Y^{i+1}(t) = E(D_t) \). Condition (B) follows from \( V(D_t) = [0, n_t - 1] \), and conditions (C)–(E) follow from (Q)–(S).
Chapter 8

3-Connected Components

In this short chapter, we shall prove that the decomposition of graphs into their 3-connected components yields an $\mathsf{IFP}$-definable treelike decomposition. The main result is the 3CC-Decomposition Lemma 8.3.1.

8.1 Decomposition into 2-Connected Components

As a warm-up, we prove the 2CC Decomposition Lemma 8.1.1 stating that the decomposition of graphs into their 2-connected components (also known as blocks) can be described as an $\mathsf{IFP}$-definable treelike decomposition. We will never use this result except as a lemma in the proof of the 3CC Decomposition Lemma 8.3.1 and there we could avoid it without too much additional effort. The main reason we state and prove the 2CC Decomposition Lemma is that both the statement of the lemma and its proof serve as blueprints for two later results, the 3CC Decomposition Lemma 8.3.1 and the (much more complicated) Q4C Decomposition Lemma 10.2.4. The proof of the 2CC Decomposition Lemma that we give below is not the easiest proof for this lemma, but the proof that is easiest to generalise.

Before we state the the 2CC Decomposition Lemma, it may be worthwhile to recall the CC Decomposition Lemma 5.1.8 which is the actual starting point of our series of decomposition lemmas. Observe that for every graph $G$ the decomposition $\Delta_{cc} := \Lambda_{cc}[G]$ is treelike, has adhesion 0, and is tight. Moreover, all its torsos are induced subgraphs of $G$ and are connected.

Lemma 8.1.1 (2CC Decomposition Lemma). There is a $d$-scheme $\Lambda_{2cc}$ such that for all graphs $G$ the decomposition $\Delta_{2cc} := (D_{2cc}, \sigma_{2cc}, \alpha_{2cc}) := \Lambda_{2cc}[G]$ has the following properties.

(i) $\Delta_{2cc}$ is treelike.

(ii) The adhesion of $\Delta_{2cc}$ is at most 1.

(iii) $\Delta_{2cc}$ is tight.

(iv) For all $t \in V(D_{2cc})$ the torso $\tau_{2cc}(t)$ is an induced subgraph of $G$.

(v) For all $t \in V(\Delta)$ the torso $\tau_{2cc}(t)$ is in $Z_2^*$, that is, either 2-connected or a complete graph of order at most 2.
We call the torsos of the decomposition $\Delta_{2cc}$ the \textit{2-connected components} of $G$. We call a 2-connected component of order at least 3 a \textit{proper} 2-connected component. Moreover, if $H = \tau_{2cc}(t)$ is a 2-connected component of $G$, then we call $t$ an \textit{index} of $H$.

We introduce some terminology for the proof. For simplicity, we call a vertex $x \in V(G)$ a \textit{separator} of $G$ if the set $\{x\}$ is a separator of $G$, and we say that $x$ \textit{separates} $v \in V(G)$ from $w \in V(G)$ if $v$ and $w$ belong to different connected components of $G \setminus \{x\}$. Vertices that are separators are also known as \textit{cut vertices}. We say that a vertex $v \in V(G)$ is \textit{inseparable} from a vertex $w \in V(G)$ if $v$ and $w$ belong to the same connected component of $G$ and there is no vertex $x \in V(G)$ that separates $v$ from $w$.

\smallskip

\textbf{Proof of the 2CC Decomposition Lemma 8.1.1}. We shall define a scheme $\Lambda$ such that for all connected graphs $G$, the decomposition $\Delta := \Lambda[G]$ has the properties (i)–(v). It will be straightforward to define a 2-dimensional $d$-scheme $\Lambda$ of the CC Decomposition Lemma 5.1.8 and $\Lambda$. It is obvious that the resulting $d$-scheme has the desired properties.

To explain the definition of $\Lambda$, we fix a connected graph $G$. We shall define a decomposition $\Delta = (D, \sigma, \alpha)$ of $G$ satisfying (A). It will be straightforward to define a 2-dimensional $d$-scheme $\Lambda$, of course not depending on the specific graph $G$, such that $\Delta = \Lambda[G]$. If $G \in Z^2_2$, we let $\Delta$ be the trivial 2-dimensional treelike decomposition of $G$ (see Example 5.1.6). In the following, we assume that $G$ is not in $Z^2_2$. Then $G$ has at least one 1-separator.

For each $\tau = (v_1, v_2) \in V(G)^2$, we define a set $S_\tau$ and two induced subgraphs $A_\tau, H_\tau \subseteq G$.

We let $S_\tau := \begin{cases} \emptyset & \text{if } v_1 = v_2, \\ \{v_1\} & \text{otherwise}. \end{cases}$

We let $A_\tau$ be the connected component of $G \setminus S_\tau$ that contains $v_2$, and we let $H_\tau := G[V(A_\tau) \cup S_\tau]$.

The decomposition $\Delta$ will have two kinds of nodes: $r$-nodes (root nodes) and $c$-nodes (component-nodes). All nodes will be pairs $\tau = (v_1, v_2) \in V(G)^2$ such that

(A) $v_1$ is a separator of $G$.

Let $\tau = (v_1, v_2) \in V(G)^2$ be a pair satisfying (A).

(B) $\tau$ is an \textit{r-node} if $v_1 = v_2$.

(C) $\tau$ is a \textit{c-node} if $v_1 \neq v_2$.

We let $V_r$ and $V_c$ be the sets of $r$-nodes and $c$-nodes, respectively, and $V(D) := V_r \cup V_c$.

Note that for all $\tau \in V(D)$ we have

$$S_\tau = N(A_\tau) = \partial(H_\tau), \quad (8.1.1)$$

because either $S_\tau = \emptyset$ or $S_\tau$ is a separator of $G$.

To define the edge relation $E(D)$, let $\tau = (v_1, v_2), \bar{\tau} = (w_1, w_2) \in V(D)$. Then $\tau \bar{\tau} \in E(D)$ if one of the following two conditions is satisfied:

(D) $\tau \in V_r$ and $\bar{\tau} \in V_c$ and $v_1 = w_1$, or

(E) $\tau \in V_c$ and $\bar{\tau} \in V_c$ and $V(H_\tau) \subseteq V(H_{\bar{\tau}}) \setminus \{v_1\}$ and $w_1$ is inseparable from $v_1$ in $G$.
We define $\sigma, \alpha : V(D) \to 2^{V(G)}$ by

\[(F)\] $\sigma(\overline{v}) := S_{\overline{v}}$ and $\alpha(\overline{v}) := V(A_{\overline{v}})$, for all $\overline{v} \in V(D)$.

This completes the definition of the decomposition $\Delta$. It is straightforward to construct a $d$-scheme $\Lambda$ (not depending on the specific graph $G$) such that $\Delta = \Lambda[G]$.

Note that for all nodes $\overline{v} \in V(D)$ we have $\gamma(\overline{v}) = V(H_{\overline{v}})$.

Claim 1. $\Delta$ is a strict treelike decomposition.

Proof. It is immediate from the definitions that $\Delta$ satisfies (TL.2)

To prove (TL.3), let $\overline{v} = (v_1, v_2) \in V(D)$ and $\overline{w} \in N^+_D(\overline{v})$. Then $\overline{w} \in V_c$ and thus

$\alpha(\overline{w}) \subset \gamma(\overline{w}) = V(H_{\overline{w}}) \subseteq \begin{cases} V(G) = \alpha(\overline{w}) = \gamma(\overline{w}) & \text{if } \overline{v} \in V_r, \\ V(H_{\overline{w}}) \setminus \{v_1\} = \alpha(\overline{w}) \subseteq \gamma(\overline{w}) & \text{if } \overline{v} \in V_c. \end{cases}$

In both cases, this implies $\alpha(\overline{w}) \subset \alpha(\overline{v})$ and $\gamma(\overline{w}) \subseteq \gamma(\overline{v})$.

To prove (TL.4) let $\overline{v} = (v_1, v_2)$ and $\overline{w}_1 = (w_1^1, w_1^2), \overline{w}_2 = (w_2^1, w_2^2) \in N^+_D(\overline{v})$. For $i = 1, 2$, let $S_i := S_{\overline{w}_i}$ and $A_i := A_{\overline{w}_i}$ and $H_i := H_{\overline{w}_i}$. If $A_1 = A_2$ then $S_1 = N(A_1) = N(A_2) = S_2$ and thus $\overline{w}_1 \parallel \overline{w}_2$. In the following, we assume that $A_1 \neq A_2$. We shall prove that $V(H_1^1) \cap V(H_2^2) = S_1 \cap S_2$, which implies $\overline{w}_1 \perp \overline{w}_2$.

If $\overline{v} \in V_r$, then $v_1 = w_1^1 = w_1^2$, and thus both $A_1$ and $A_2$ are connected components of $G \\setminus \{v_1\}$. This implies that

$V(H_1^1) \cap V(H_2^2) = \{v_1\} = S_1 \cap S_2$.

Suppose that $\overline{v} \in V_c$. We shall prove that $V(H_1^1) \cap V(A_2^2) = \emptyset$. By symmetry, it follows that $V(A_1^1) \cap V(H_2^2) = \emptyset$ and thus $V(H_1^1) \cap V(H_2^2) = S_1 \cap S_2$.

Suppose for contradiction that $x \in V(H_1^1) \cap V(A_2^2)$. Let $P$ be a path from $x$ to $w_1^1$ in $H_1^1$. If $w_1^2 \notin V(P)$, we have $P \subseteq A_2^2$ and thus $w_1^2 \notin V(A_1^1)$. Then $w_1^2$ separates $w_1^1$ from $v_1$, which contradicts (E). Otherwise, we have $w_1^2 \in V(A_1^1)$, and $w_1^1$ separates $w_1^2$ from $v_1$. This also contradicts (E).

To prove (TL.5), just note that for every $\overline{v} \in V_r$ we have $\sigma(\overline{v}) = \emptyset$ and $\alpha(\overline{v}) = V(G)$. We have $V_r \neq \emptyset$, because $G$ is connected and not in $Z_2^*$ and thus has a 1-separator.

As $\sigma(\overline{v}) = S_{\overline{v}} \subseteq \{v_1\}$ for all $\overline{v} = (v_1, v_2) \in V(D)$, the adhesion of $\Delta$ is at most 1. It follows from (S.1.1) that $\Delta$ is tight. Thus we have proved (i)–(iii).

To prove (iv), just observe that for all $\overline{v} \in V(D)$ we have $\tau(\overline{v}) = G[\beta(\overline{v})]$, because the adhesion of $\Delta$ is at most one. Finally, (v) follows from the next claim.

Claim 2. For all $\overline{v} \in V(D)$ we have $G[\beta(\overline{v})] \in Z_2^*$.

Proof. Let $\overline{v} = (v_1, v_2) \in V(D)$.

Case 1: $\overline{v} \in V_r$.

Then $\gamma(\overline{v}) = V(G)$. For every $\overline{w} = (w_1, w_2) \in N^+_D(\overline{v})$, we have $\sigma(\overline{w}) = \{v_1\} = \{v_1\}$. Thus $v_1 \notin \alpha(\overline{w})$, which implies $v_1 \in \beta(\overline{w})$. Furthermore, for every $w \in V(G) \setminus \{v_1\}$, we have $(v_1, w) \in N^+_D(\overline{w})$ and $w \in \alpha(\overline{v}, w)$. Thus $w \notin \beta(\overline{v})$. It follows that $\beta(\overline{v}) = \{v_1\}$ and thus $G[\beta(\overline{v})] = K[\{v_1\}] \in Z_2^*$.

Case 2: $\overline{v} \in V_c$.

Let $A := A_{\overline{v}}$ and $H := H_{\overline{v}}$ and $B := G[\beta(\overline{v})]$. We have $v_1 \in \sigma(\overline{v}) \subseteq \beta(\overline{v}) = V(B)$. Let
Let $v \in V(A)$ with $v_1v \in E(G)$. As there is no $w_1$ separating $v$ from $v_1$, for all $\overline{w} \in N^D_P(\overline{v})$ we have $v \notin \alpha(\overline{w})$. Thus $v \in V(B)$. It follows that if $|B| = 2$ then $B = K[v_1, v] \in \mathcal{Z}_2^+$. In the following, we assume that $|B| \geq 3$.

Suppose for contradiction that $B$ is not 2-connected, and let $x \in V(B)$ be a separator of $B$. Without loss of generality we may assume that $x$ is inseparable from $v_1$ in $B$. Then $x$ is inseparable from $v_1$ in $G \supseteq B$. Let $y \in V(B)$ such that $x$ separates $y$ from $v_1$ in $B$. If $x$ separates $y$ from $v_1$ in $G$, then $(x, y) \in N^D_P(\overline{y})$ with $y \in \alpha(x, y)$, which contradicts $y \in V(B)$.

Thus $x$ does not separate $y$ from $v_1$ in $G$. Let $P \subseteq G \setminus \{x\}$ be a path from $y$ to $v_1$. Then $P \subseteq H$, because $\partial(H) = \{v_1\}$. Moreover, for all $\overline{w} = (w_1, w_2) \in N^D_P(\overline{v})$ we have $V(P) \cap V(A_{\overline{w}}) = \emptyset$, because $|N(A_{\overline{w}})| = 1$ and both endpoints of $P$ are in $V(G) \setminus V(A_{\overline{w}})$. Thus $P \subseteq B$, which is a contradiction. □

### 8.2 2-Separators of 2-Connected Graphs

In the proof of the 3CC Decomposition Lemma 8.3.1, stating that the decomposition of a graph into its 3-connected components is definable, we will try to follow the proof of the 2CC Decomposition Lemma 8.1.1 as closely as we can. But there will be one difficulty, arising from the fact that, as opposed to 1-separators, 2-separators of a graph may cross: we say that $\Delta$ is a 2-separator from $S$ if for any $x \in S$, $\Delta$ does not separate $x$ from $S$. As there is no 2-separator of size $1$, we will try to follow the proof of the 2CC Decomposition Lemma 8.1.1 (to verify [TL.4] for the decomposition $\Delta$).

**Example 8.2.1.** Consider the cycle $C_4 = ([4], \{12, 23, 34, 41\})$. The two 2-separators $\{1, 3\}$ and $\{2, 4\}$ of the cycle cross, and thus there is no obvious canonical way to decompose the cycle into “3-connected components.”

In this section, we analyse the structure of the 2-separators of a 2-connected graph.

**Lemma 8.2.2.** Let $S = \{v, w\}$ be a 2-separator of a 2-connected graph $G$. Then for every connected component $A$ of $G \setminus S$ the graph $\hat{A} := G[V(A) \cup S] + vw$ is a topological subgraph of $G$ that has a faithful image in $G$.

**Proof.** Let $A' \neq A$ be another connected component of $G \setminus S$. As $N(A') = S$, there is a path $P \subseteq G$ from $v$ to $w$ with all internal vertices in $A'$. Obviously, $G[V(A) \cup S] \cup P$ is a subdivision of $\hat{A}$ and thus $\hat{A}$ is a topological subgraph of $G$. It is easy to define a faithful image of $\hat{A}$ in $G$. □

**Definition 8.2.3.** Let $G$ be a connected graph and $W, X \subseteq V(G)$.

1. A *proper* $(W, X)$-separator is a set $S \subseteq V(G)$ such that $W \not\subset S$ and $X \not\subset S$ and $S$ is a $(W, X)$-separator, that is, there is no path from $W \setminus S$ to $X \setminus S$ in $G \setminus S$.

2. $X$ is *inseparable* from $W$ if there is no proper $(W, X)$-separator $S$ of size $|S| \leq |X|$.

Note that inseparability is not symmetric. Also note that a vertex $x$ is inseparable from a vertex $w$ (according to the definition given on page 166) if and only if the set $\{x\}$ is inseparable from the set $\{w\}$. The next lemma is almost trivial. The only reason for stating it in such a complicated way is to prepare for the following, less trivial lemma. The lemma is implicitly used in the proof of the 2CC Decomposition Lemma 8.1.1 (to verify [TL.4] for the decomposition $\Delta$).

M. Grohe, *Definable Graph Structure Theory*
Lemma 8.2.4. Let $G$ be a connected graph and $w, s_1, s_2 \in V(G)$ such that

(i) $w, s_1, s_2$ are pairwise distinct;

(ii) $s_1$ and $s_2$ are separators of $G$;

(iii) both $s_1$ and $s_2$ are inseparable from $w$.

For $i = 1, 2$, let $A_i$ be a connected component of $G \setminus \{s_i\}$ such that $w \not\in V(A_i)$. Then $V(A_1) \cap V(A_2) = \emptyset$ and $s_1 \not\in V(A_2)$ and $s_2 \not\in V(A_1)$.

Proof. See the proof of Claim 1, axiom [TL.4] in the proof of the 2CC Decomposition Lemma 8.1.1.

The direct analogue of Lemma 8.2.4 for 2-separators of 2-connected graphs does not hold. We can fix this by adding one additional condition.

Definition 8.2.5. Let $G$ be a connected graph and $W \subseteq V(G)$. A split extension of $W$ is a vertex $v \in V(G) \setminus W$ such that for every connected component $A$ of $G \setminus (W \cup \{v\})$ it holds that $N(A) \neq W \cup \{v\}$.

Example 8.2.6. Let $C$ be a cycle of length at least 5 and $w_1w_2 \in E(C)$ an arbitrary edge. Let $W := \{w_1, w_2\}$. Then all vertices $v \in V(C) \setminus W$ are split extensions of $W$ in $C$.

Now let $v_i$ be the neighbour of $w_i$ distinct from $w_3-i$, and let $S_i := \{v_i, w_3-i\}$. Then $S_i$ is a separator of $C$, and the connected components of $C \setminus S_i$ are $C[\{w_i\}]$ and $A_i := C \setminus \{v_i, w_i, w_3-i\}$. Furthermore, there is no proper $W$-$S_i$-separator in $C$.

Note that we have $A_1 \cap A_2 = C \setminus \{v_1, v_2, w_1, w_2\} \neq \emptyset$ and $V(A_1) \cap S_2 = \{v_2\} \neq \emptyset$ and $V(A_2) \cap S_1 = \{v_1\} \neq \emptyset$.

Lemma 8.2.7. Let $G$ be a 2-connected graph and $W \subseteq V(G)$ a clique that has no split extension in $G$. Let $S_1, S_2 \subseteq V(G)$ such that

(i) $W, S_1, S_2$ are pairwise distinct, and $W \not\subseteq S_i$ for $i = 1, 2$;

(ii) $S_1$ and $S_2$ are 2-separators of $G$;

(iii) both $S_1$ and $S_2$ are inseparable from $W$.

For $i = 1, 2$, let $A_i$ be a connected component of $G \setminus S_i$ with $W \cap V(A_i) = \emptyset$. Then $V(A_1) \cap V(A_2) = \emptyset$ and $V(A_1) \cap S_2 = \emptyset$ and $S_1 \cap V(A_2) = \emptyset$.

Proof. For $i = 1, 2$, let $X_i$ be the vertex set of the connected component of $G \setminus S_i$ with $W \cap X_i \neq \emptyset$, and let $Y_i := V(G) \setminus (S_i \cup X_i)$. Then $V(A_i) \subseteq Y_i$. Let $S_i := S_1 \cap S_2$ and $S_{iX} := S_i \cap X_{3-i}$, $S_{iY} := S_i \cap Y_{3-i}$ for $i = 1, 2$. Figure 8.1 illustrates the situation.

Claim 1. Either $S_{iY} \neq \emptyset$ or $S_{2Y} \neq \emptyset$.

Proof. Suppose for contradiction that $S_{iY} \neq \emptyset$ and $S_{2Y} \neq \emptyset$. Then for $i = 1, 2$, we have $S_{iX} \neq \emptyset$, because otherwise $S_{3-i}$ separates $S_i$ from $W$. Let $s_{iX} \in S_{iX}$ and $s_{iY} \in S_{iY}$. Then $S_{iX} = \{s_{iX}\}$ and $S_{iY} = \emptyset$ and $S_{iY} = \{s_{iY}\}$ and $S_i = \{s_{iX}, s_{iY}\}$, because $|S_i| \leq 2$. Now $\{s_{1X}, s_{2X}\}$ separates $S_1$ (and $S_2$) from $W$, and as $S_1$ is inseparable from $W$, it follows that $W \not\subseteq \{s_{1X}, s_{2X}\}$. As $W \not\subseteq S_1$ and $W \not\subseteq S_2$, it follows that $W = \{s_{1X}, s_{2X}\}$.

But then $s_{1Y}$ is a split extension of $W$. To see this, let $A$ be a connected component of $G \setminus (\{s_{1Y}\}) = G \setminus \{s_{1X}, s_{1Y}, s_{2X}\} = G \setminus (S_1 \cup S_{2X})$. Then either $A \subseteq X_1 \cap X_2$ or $A \subseteq X_1 \cap Y_2$.
or \( A \subseteq Y_1 \). If \( A \subseteq X_1 \cap X_2 \) then \( N^G(A) \subseteq S_1X \cup S_+ \cup S_2X = \{s_1X, s_2X\} \). If \( A \subseteq X_1 \cap Y_2 \), then \( N^G(A) \subseteq S_1Y \cup S_+ \cup S_2X = \{s_2X, s_1Y\} \). If \( A \subseteq Y_1 \), then \( N^G(A) \subseteq S_1Y = \{s_1Y\} \). In all three cases we have \( N^G(A) \subseteq W \cup \{s_1Y\} = W \). Thus \( s_1Y \) is indeed a split extension of \( W \). This contradicts the assumption of the lemma that \( W \) have no split extension.

Without loss of generality, we assume that

\[ S_{1Y} = \emptyset. \]

Then either \( V(A_2) \subseteq X_1 \cap Y_2 \) or \( V(A_2) \subseteq Y_1 \cap Y_2 \), because \( V(A_2) \subseteq Y_2 \) and \( A_2 \) is connected and there is no edge from \( X_1 \cap Y_2 \) to \( Y_1 \cap Y_2 \).

**Case 1:** \( V(A_2) \subseteq Y_1 \cap Y_2 \).

Then \( S_{2X} = \emptyset \), because \( S_2 = N^G(A_2) \) and there is no edge from \( Y_1 \) to \( X_1 \supseteq S_{2X} \). Thus \( S_1 \) separates \( S_2 \) from \( W \). This is impossible, because \( S_2 \) is inseparable from \( W \).

**Case 2:** \( V(A_2) \subseteq X_1 \cap Y_2 \).

Then \( S_{2Y} = \emptyset \), because \( S_2 = N^G(A_2) \) and there is no edge from \( X_1 \) to \( Y_1 \supseteq S_{2Y} \). Hence either \( V(A_1) \subseteq Y_1 \cap X_2 \) or \( V(A_1) \subseteq Y_1 \cap Y_2 \). As \( V(A_2) \subseteq X_1 \cap Y_2 \), the assertion of the lemma follows.

### 8.3 Decomposition into 3-Connected Components

**Lemma 8.3.1 (3CC Decomposition Lemma).** There is a \( d \)-scheme \( \Lambda_{3cc} \) such that for all graphs \( G \) the decomposition \( \Delta_{3cc} := (D_{3cc}, \sigma_{3cc}, \alpha_{3cc}) := \Lambda_{3cc}[G] \) has the following properties.

(i) \( \Delta_{3cc} \) is a treelike decomposition of \( G \).

(ii) The adhesion of \( \Delta_{3cc} \) is at most 2.

(iii) The decomposition \( \Delta_{3cc} \) is tight.

(iv) For all \( t \in V(D_{3cc}) \) the torso \( \tau_{3cc}(t) \) is a topological subgraph of \( G \) that has a faithful image.

(v) For all \( t \in V(D_{3cc}) \) the torso \( \tau_{3cc}(t) \) is in \( Z_3^* \), that is, either 3-connected or a complete graph of order at most 3.
We call the torsos of the decomposition \( \Delta_{3cc} \) the 3-connected components of \( G \). We call a 3-connected component of order at least 4 a proper 3-connected component. Moreover, if \( H = \tau_{3cc}(t) \) is a 3-connected component of \( G \), then we call \( t \) an index of \( H \).

Before we prove the lemma, we draw some consequences.

**Corollary 8.3.2.** Let \( C \) be a topological subgraph ideal. Then \( C \) admits IFP-definable treelike decompositions over \( \mathcal{Z}_3^* \cap C \).

**Corollary 8.3.3 (3CC Lifting Lemma).** Let \( C \) be a topological subgraph ideal such that \( \mathcal{Z}_3 \cap C \) admits IFP-definable ordered treelike decompositions. Then \( C \) admits IFP-definable ordered treelike decompositions.

**Proof.** By the Finite Extension Lemma 7.3.1, the class \( \mathcal{Z}_3^* \cap C \) admits IFP-definable ordered treelike decompositions. Now the corollary follows from Corollary 8.3.2 and the Ordered Decomposition Lifting Lemma 7.1.7.

The 3CC Decomposition Lemma 8.3.1 follows from the 2CC Decomposition Lemma 8.1.1 and the following lemma by means of the Decomposition Lifting Lemma 5.6.2. The precise argument requires some care; we will carry it out in detail below.

**Lemma 8.3.4.** There is a d-scheme \( \Lambda \) such that for all 2-connected graphs \( G \) the decomposition \( \Delta := \Lambda[G] \) satisfies conditions (i)–(v) of the 3CC Decomposition Lemma 8.3.1.

**Proof.** To explain the definition of \( \Lambda \), we fix a 2-connected graph \( G \). We shall define a decomposition \( \Delta = (D, \sigma, \alpha) \) of \( G \) satisfying (i)-(v). It will be straightforward to define a 3-dimensional d-scheme \( \Lambda \), of course not depending on the specific graph \( G \), such that \( \Delta = \Lambda[G] \). If \( G \in \mathcal{Z}_3^* \), we let \( \Delta \) be the trivial 3-dimensional treelike decomposition of \( G \) (see Example 5.1.6). In the following, we assume that \( G \) is not in \( \mathcal{Z}_3^* \).

For each triple \( v = (v_1, v_2, v_3) \in V(G)^3 \), we define a set \( S_v \) and two graphs \( A_v, H_v \). We let

\[
S_v := \begin{cases} 
\emptyset & \text{if } v_3 \in \{v_1, v_2\}, \\
\{v_1, v_2\} & \text{otherwise.}
\end{cases}
\]

We let \( A_v \) be the connected component of \( G \setminus S_v \) that contains \( v_3 \), and we let

\[
H_v := G[V(A_v) \cup S_v] \cup K[S_v].
\]

Note that if \( v_3 \in \{v_1, v_2\} \) then \( A_v = H_v = G \).

The decomposition \( \Delta \) will have three kinds of nodes: r-nodes (root nodes), s-nodes (split-extension nodes), and c-nodes (component-nodes). The torsos at the r-nodes and s-nodes will be complete graphs of order at most 3. Only the torsos at the c-nodes may be proper 3-connected graphs. All nodes will be triples \( v = (v_1, v_2, v_3) \in V(G)^3 \) such that

(A) \( \{v_1, v_2\} \) is a 2-separator of \( G \).

Let \( v = (v_1, v_2, v_3) \in V(G)^3 \) be a triple satisfying (A)

(B) \( v \) is an r-node if \( v_3 \in \{v_1, v_2\} \).

(C) \( v \) is an s-node if \( v_3 \notin \{v_1, v_2\} \) and \( v_3 \) is a split extension of \( S_v \) in \( H_v \).

(D) \( v \) is a c-node if \( v_3 \notin \{v_1, v_2\} \) and \( S_v \) has no split extension in \( H_v \).
We let $V_r, V_s, V_c$ be the sets of r-nodes, s-nodes, and c-nodes, respectively, and $V(D) := V_r \cup V_s \cup V_c$. Note that $V_r \neq \emptyset$, because $G \not\subseteq \mathcal{Z}_4^*$.

**Claim 1.** For all $\bar{v} \in V(D)$ we have $S_\bar{v} = N(A_\bar{v}) = \partial(H_\bar{v})$.

**Proof.** If $\bar{v} \in V_r$, then $A_\bar{v} = H_\bar{v} = G$ and thus $N(A_\bar{v}) = \partial(H_\bar{v}) = \emptyset = S_\bar{v}$.

If $\bar{v} \in V_s \cup V_c$, then $S_\bar{v}$ is a 2-separator of $G$ and $A_\bar{v}$ a connected component of $G \setminus S_\bar{v}$, and the assertion holds because $G$ is 2-connected.

**Claim 2.** For all $\bar{v} \in V(D)$, the graph $H_\bar{v}$ is a topological subgraph of $G$ that has a faithful image.

**Proof.** This follows immediately from Lemma 8.2.2.

To define the edge relation $E(D)$, let $\bar{v} = (v_1, v_2, v_3), \bar{w} = (w_1, w_2, w_3) \in V(D)$. Then $\bar{v}\bar{w} \in E(D)$ if one of the following conditions is satisfied:

- (E) $\bar{v} \in V_r$ and $\bar{w} \in V_s \cup V_c$ and $S_{\bar{v}} = \{v_1, v_2\} = \bar{v}$, or
- (F) $\bar{v} \in V_s$ and $\bar{w} \in V_s \cup V_c$ and $A_{\bar{v}}$ is a connected component of $H_\bar{v} \setminus \bar{v}$, or
- (G) $\bar{v} \in V_c$ and $\bar{w} \in V_s \cup V_c$ and $S_{\bar{v}} \subseteq V(H_\bar{v})$ and $S_\bar{v} \neq S_{\bar{w}}$ and $S_\bar{v}$ is inseparable from $S_{\bar{w}}$ in $H_\bar{v}$ and $A_{\bar{v}} \subseteq A_{\bar{w}}$.

We define $\sigma, \alpha : V(D) \to 2^{V(G)}$ as follows:

- (H) For all $\bar{v} \in V(D)$ we let $\sigma(\bar{v}) := S_{\bar{v}}$ and $\alpha(\bar{v}) := V(A_{\bar{v}})$.

This completes the definition of the decomposition $\Delta$. It is completely straightforward to construct a d-scheme $\Lambda$ (not depending on the specific graph $G$) such that $\Delta = \Lambda[G]$.

Observe that $\gamma(\bar{v}) = V(H_{\bar{v}})$ for all $\bar{v} \in V(D)$.

**Claim 3.** $\Delta$ is a strict treelike decomposition.

**Proof.** It is immediate from the definitions that $\Delta$ satisfies (TL.2). To prove (TL.3s), let $\bar{v} = (v_1, v_2, v_3), \bar{w} = (w_1, w_2, w_3) \in V(D)$ such that $\bar{v}\bar{w} \in E(D)$. If $\bar{v}$ is an r-node, then $\alpha(\bar{v}) = \gamma(\bar{v}) = V(G)$. Thus $\gamma(\bar{w}) \subseteq \gamma(\bar{v})$ and $\alpha(\bar{w}) \subseteq \alpha(\bar{v})$. Moreover, we have $S_{\bar{w}} \neq \emptyset$ and thus $\alpha(\bar{w}) \subseteq \gamma(\bar{w}) \subseteq \alpha(\bar{v})$. If $\bar{v}$ is an s-node, then $\alpha(\bar{w}) = V(A_{\bar{w}}) \subseteq V(H_{\bar{w}}) \setminus \bar{v} = V(A_{\bar{v}}) \setminus \{v_3\} \subseteq V(A_{\bar{w}}) = \alpha(\bar{v})$.

Furthermore, $N^G(A_{\bar{w}}) = N^{H_{\bar{w}}}(A_{\bar{w}})$ because $V(A_{\bar{w}}) \cap \partial^G(H_{\bar{w}}) \subseteq V(A_{\bar{w}}) \cap \bar{v} = \emptyset$ and thus $\sigma(\bar{w}) = N^G(A_{\bar{w}}) = N^{H_{\bar{w}}}(A_{\bar{w}}) \subseteq \bar{v} \subseteq \gamma(\bar{v})$.

Hence $\gamma(\bar{w}) = \alpha(\bar{w}) \cup \sigma(\bar{w}) \subseteq \gamma(\bar{v})$.

Finally, suppose that $\bar{v}$ is a c-node. Then $\alpha(\bar{w}) = V(A_{\bar{w}}) \subseteq V(A_{\bar{v}}) = \alpha(\bar{v})$ by (G). The inclusion is strict, because $S_{\bar{w}} \subseteq V(H_{\bar{w}})$ and $S_{\bar{w}} \neq S_{\bar{v}}$, which implies $S_{\bar{w}} \cap V(A_{\bar{v}}) \neq \emptyset$ and thus $V(A_{\bar{v}}) \setminus V(A_{\bar{w}}) \neq \emptyset$. Moreover, $\gamma(\bar{w}) = S_{\bar{w}} \cup V(A_{\bar{v}}) \subseteq V(H_{\bar{w}}) = \gamma(\bar{v})$.

To prove (TL.4), let $\bar{v} = (v_1, v_2, v_3) \in V(G)$ and $\bar{w} = (w_1, w_2, w_3) \subseteq V(G)$ and $\bar{w} = (w_1, w_2, w_3) \subseteq N_{\bar{w}}(\bar{v})$. For $i = 1, 2$, let $S^i := S_{\bar{w}}$ and $A^i := A_{\bar{w}}$ and $H^i := H_{\bar{w}}$. If $A^1 = A^3$, then $S^1 = N(A^1) = N(A^2) = S^2$ and thus $\bar{w}^1 \parallel \bar{w}^2$. So suppose that $A^1 \neq A^2$. We shall prove that $V(H^1) \cap V(H^2) = S^1 \cap S^2$, which implies $\bar{w}^1 \perp \bar{w}^2$.

If $\bar{v}$ is an r-node or an s-node, then both $A^1, A^2$ are connected components of $G \setminus \bar{v}$, and thus we have $V(A^1) \cap V(A^2) = \emptyset$ and $V(A^1) \cap S^{3-i} \subseteq V(A^i) \cap \bar{v} = \emptyset$. If $\bar{v}$ is a c-node, then
we have σ(τ) = S_τ = \{v_1, v_2\}. Assertion (iii) follows from Claim 1. Assertions (iv) and (v) follow from the next claim.

Claim 4. For all \(\varpi \in V(D)\) the torso \(\tau(\varpi)\) is a topological subgraph of \(G\) that has a faithful image in \(G\). Furthermore, \(\tau(\varpi) \in Z^*_3\).

Proof. Let \(\varpi = (v_1, v_2, v_3) \in V(D)\). Let \(S := S_\varpi\) and \(A := A_\varpi\) and \(H := H_\varpi\). By Claim 2, \(H\) is a topological subgraph of \(G\) that has a faithful image.

Case 1: \(\varpi\) is an r-node.

Then \(A = H = G\). Let \(A'\) be a connected component of \(H \setminus \{v_1, v_2\}\), and let \(H' := G[V(A') \cup \{v_1, v_2\}] \cup K[\{v_1, v_2\}]\). If there is a split extension \(w\) of \(\{v_1, v_2\}\) in \(H'\), let \(\varpi := (v_1, v_2, w)\). Then \(\varpi \in V_s\). Otherwise, that is, if there is no split extension of \(\{v_1, v_2\}\) in \(H'\), let \(w \in V(A')\) be arbitrary and \(\varpi := (v_1, v_2, w)\). Then \(\varpi \in V_c\). In both cases, \(\varpi \in N^D(\varpi)\) with \(\alpha(\varpi) = V(A')\). Hence for every connected component \(A'\) of \(H \setminus \{v_1, v_2\}\) there is a child \(\varpi \in N^D(\varpi)\) with \(\alpha(\varpi) = V(A')\). This implies \(\beta(\varpi) = (v_1, v_2)\). Moreover, \(\tau(\varpi) = K[\{v_1, v_2\}]\), because for every \(\varpi \in N^D(\varpi)\) we have \(\sigma(\varpi) = S_\varpi = \{v_1, v_2\}\).

Case 2: \(\varpi\) is an s-node.

Then \(v_3\) is a split extension of \(S = \{v_1, v_2\}\) in \(H\). As \(H\) has a faithful image in \(G\), there is a path \(P_{12}\) from \(v_1\) to \(v_2\) in \(G \setminus \overline{A}\). Furthermore, for \(i = 1, 2\), there is a path \(P_{i3}\) from \(v_i\) to \(v_3\) in \(H \setminus \{v_{3-i}\}\). To see this, let \(w \in V(A)\) such that \(v_iw \in E(G)\). Such a \(w\) exists because \(N^G(A) = S\). Let \(Q_i\) be a path from \(w\) to \(v_3\) in the connected graph \(A\). We let \(P_{i3} := v_iwQ_iv_3\). Note that the three paths \(P_{12}, P_{13}, P_{23}\) are internally disjoint, because if \(P_{i2} \cap A = \emptyset\), and if \(P_{i1}\) and \(P_{i3}\) have an internal vertex in common then there is a connected component \(A'\) of \(H \setminus \overline{v}\) with \(N^G(A') = \overline{v}\), which contradicts \(v_3\) being a split extension of \(v_{12}\). Hence \(K[\overline{v}]\) is a topological subgraph of \(G\) that has a faithful image in \(G\).

Let \(A'\) be a connected component of \(H \setminus \overline{v}\). Let \(S' := N^G(A')\). Then \(|S'| = 2\) because \(H\) is 2-connected and \(v_3\) is a split extension of \(v_{12}\). Let \(w_1, w_2\) such that \(S' = \{w_1, w_2\}\), and let \(H' := G[V(A') \cup S]\). If there is a split extension of \(S'\) in \(H'\), let \(w_3\) be such a split extension. Otherwise, let \(w_3 \in V(A')\) be arbitrary. Let \(\varpi = (w_1, w_2, w_3)\). Note that \(\{w_1, w_2\} = S' \subset \overline{v}\) is a separator of \(G\), because it separates \(V(A')\) from the vertex in \(\overline{v} \setminus S'\). Hence \(\varpi \in V_s \cup V_c\) and \(\varpi \in E(D)\). It follows that \(\beta(\varpi) = \overline{v}\). Moreover,

\[
\tau(\varpi) = G[\overline{v}] \cup K[\sigma(\varpi)] \cup \bigcup_{\varpi \in N^D(\varpi)} K[\sigma(\varpi)] = K[\overline{v}].
\]

To see this, note that \(v_1v_2, v_1v_3 \in E(\tau(\varpi))\) because \(\sigma(\varpi) = \{v_1, v_2\}\). To see that \(v_i^3, v_3 \in E(\tau(\varpi))\) for \(i = 1, 2\), consider the path \(P_{i3}\). If this path has length 1 then \(v_i^3, v_3 \in E(G)\). Otherwise, there is a connected component \(A'\) of \(H \setminus \overline{v}\) that contains all internal vertices of \(P'\). We have argued that there is a \(\varpi \in N^D(\varpi)\) such that \(\alpha(\varpi) = V(A')\) and \(\sigma(\varpi) = N^G(\alpha(\varpi)) = \{v_1, v_2\}\).
Case 3: $\overline{v}$ is a c-node.
Let $\overline{w}_1, \ldots, \overline{w}_m \in N_+^D(\overline{v})$ be a system of representatives of the $\|\cdot\|$-classes in $N_+^D(\overline{v})$. For $i = 1, \ldots, m$, let $A^i := A_{\overline{w}_i} = G[\alpha(\overline{w}_i)]$ and $S^i := S_{\overline{w}_i} = \sigma(\overline{w}_i)$. Then
\[
\beta(\overline{v}) = V(H) \setminus \bigcup_{i=1}^m V(A^i)
\]
and
\[
\tau(\overline{v}) = G[\beta(\overline{v})] \cup K[S] \cup \bigcup_{i=1}^m K[S^i].
\]
The graph $\tau(\overline{v})$ is a minor of $G$ with a faithful image, because $H$ has a faithful image in $G$ and thus there is a path $P$ between the two vertices in $S$ with all internal vertices in $V(G) \setminus V(H)$, and for all $i \in [m]$ there is a path $P^i$ between the two vertices in $S^i$ with all internal vertices in $V(A^i)$. These paths are mutually internally disjoint, because the $A^i \subset H$ are mutually disjoint by (TL.4) and all the paths are internally disjoint from $\beta(\overline{v})$. Hence we can contract them to obtain the edges of $\tau(\overline{v})$ not in $E(G)$.

Suppose for contradiction that $\tau(\overline{v})$ has a proper separator of order at most 2. We may choose such a separator $S'$ in such a way that $S'$ is inseparable from $S$ in $\tau(\overline{v})$. Let $w$ be in a different connected component of $\tau(\overline{v}) \setminus S'$ than (the clique) $S \setminus S'$. Then $S'$ is also a separator of $H$ that separates $w$ from $S$. Let $A'$ be the connected component of $H \setminus S'$ that contains $w$ and $H' := G[V(A') \cup S'] \cup K[S']$. Let $w_1, w_2$ such that $S' = \{w_1, w_2\}$. If $S'$ has a split extension in $H'$, let $w_3$ be such a split extension. Otherwise, let $w_3 \in V(A')$ be arbitrary. Then $\overline{w} = (w_1, w_2, w_3) \in V_s \cup V_e$ and $\overline{w} \in E(D)$. Hence $\overline{w} \parallel \overline{w}'$ for some $i \in [m]$, and we have $w \in \beta(\overline{v}) \cap V(A') = \beta(\overline{v}) \cap V(A^i) = 0$. This is a contradiction. \qed

Proof of the 3CC Decomposition Lemma 8.3.1. To explain the definition of $\Lambda_{3cc}$, we fix a graph $G$. Let $\Lambda^1 := \Lambda_{2cc}$ be the d-scheme of the 2CC Decomposition Lemma 8.1.1 and $\Delta^1 = (D^1, \sigma^1, \alpha^1) := \Lambda^1[G]$. For every $t_1 \in V(D^1)$, let $H_{t_1} := \tau^1(t_1)$. Then

(A) $\Delta^1$ is treelike.

(B) The adhesion of $\Delta^1$ is at most 1.

(C) $\Delta^1$ is tight.

(D) For all $t_1 \in V(D^1)$ the torso $H_{t_1}$ is an induced subgraph of $G$.

(E) For all $t_1 \in V(D^1)$ it holds that $H_{t_1} \in Z_2^e$.

We transform the d-scheme of Lemma 8.3.4 into a parametrised d-scheme $\Lambda^2(\overline{v}_1)$ such that for every node $t_1 = \overline{v}_1 \in V(D^1)$, the scheme $\Lambda^2(\overline{v}_1)$ defines a decomposition
\[
\Delta^2_{t_1} = (D^2_{t_1}, \sigma^2_{t_1}, \alpha^2_{t_1}) := \Lambda^2[G, \overline{v}_1]
\]
of $H_{t_1}$ within $(G, \overline{v}_1)$. This decomposition has the following properties.

(F) $\Delta^2_{t_1}$ is treelike.

(G) The adhesion of $\Delta^2_{t_1}$ is at most 2.
(H) The decomposition $\Delta^2_{t_1}$ is tight.

(I) For all $t_2 \in V(D^2_{t_1})$ the torso $J_{t_1t_2} := \tau^2_{t_1}(t_2)$ is a topological subgraph of $H_{t_1}$ that has a faithful image.

(J) For all $t_2 \in V(D^2_{t_1})$ it holds that $J_{t_1t_2} \in Z_3^*$.

Formally, $\Lambda^2(\overline{\alpha})$ can be defined by applying the Transduction Lemma for Definable Decompositions\textsuperscript{5.5.6} to the d-scheme of Lemma\textsuperscript{8.3.4} and the transduction $\Theta(\overline{\alpha})$ that maps a graph $G$ and a node $t_1 \in V(\Lambda^1[G])$ to the torso $H_{t_1}$ and then combining the resulting decomposition with one that takes care of the graphs isomorphic to $K_1$ and $K_2$, which are not 2-connected.

We apply the Decomposition Lifting Lemma\textsuperscript{5.6.2} to $\Lambda^1$ and $\Lambda^2(\overline{\alpha})$. Let $\Lambda$ be the resulting d-scheme and $\Delta := \Lambda[G]$. To simplify the notation, in the following we denote nodes of $\Delta$ by $t$, $\omega$ and variants like $\ell'$, nodes of $D$ by $t_1$, $u_1$ and variants, and nodes of decompositions $\Delta^2_{t_1}$ by $t_2$, $u_2$ and variants. We write $N^2_{t_1}(t_2)$ instead of $N^D_{t_1}(t_2)$. By the Decomposition Lifting Lemma, the decomposition $\Delta$ has the following properties.

(K) $\Delta$ is treelike.

(L) $V(D) \subseteq \{t_1t_2 \mid t_1 \in V(D^1), t_2 \in V(D^2_{t_1})\}$.

(M) For $t = t_1t_2 \in V(D)$, we have

$\tau(t) = J_t$.

Remember that $J_t = J_{t_1t_2} = \tau^2_{t_1}(t_2)$.

(N) For $t = t_1t_2 \in V(D)$,

- either $\sigma(t) = \sigma^2_{t_1}(t_2)$ and $\alpha(t) \cap \beta^1(t_1) = \alpha^2_{t_1}(t_2)$
- or there is a $u_2 \in N^2_{t_1}(t_2)$ such that $\sigma(t) = \sigma^2_{t_1}(u_2)$ and $\alpha(t) \cap \beta^1(t_1) = (\beta^1(t_1) \setminus \gamma^2_{t_1}(u_2))$
- or $\sigma(t) = \sigma^1(t_1)$ and $\alpha(t) = \alpha^1(t_1)$.

The decomposition $\Delta$ satisfies almost all assertions of the 3CC Decomposition Lemma.\textsuperscript{(i)} follows from $\text{[K]}$,\textsuperscript{(ii)} follows from $\text{[B]}$,\textsuperscript{(iii)} follows from $\text{[G]}$,\textsuperscript{(iv)} follows from $\text{[N]}$,\textsuperscript{(v)} follows from $\text{[D]}$,\textsuperscript{(v)} and $\text{[M]}$\textsuperscript{(v)}

Thus it only remains to prove\textsuperscript{(iii)} stating that the decomposition is tight. Unfortunately, the decomposition $\Delta$ is not necessarily tight; its components $\alpha(t)$ may be disconnected. To fix this, we will apply Lemma\textsuperscript{5.3.2}. The following two claims show that $\Delta$ satisfies the assumptions of that lemma.

Claim 1. For all $t \in V(D)$ it holds that $\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))$.

Proof. Let $t = t_1t_2 \in V(D)$, and let $H := H_{t_1}$. Then $V(H) = \beta^1(t_1)$. By\textsuperscript{(N)}, we are in one of the following three cases.

Case 1: $\sigma(t) = \sigma^2_{t_1}(t_2)$ and $\alpha(t) \cap \beta^1(t_1) = \alpha^2_{t_1}(t_2)$.

Then $\sigma(t) = N^H(\alpha(t) \cap V(H)) = \partial^H(\gamma(t) \cap V(H))$, because $\Delta^2_{t_1}$ is tight. It follows that $\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))$, because $H$ is an induced subgraph of $G$ by\textsuperscript{(D)}.

Preliminary Version
Case 2: $\sigma(t) = \sigma^2(t_1)(u_2)$ and $\alpha(t) \cap \beta^1(t_1) = \beta^1(t_1) \setminus \gamma_1^2(t_1)(u_2)$ for some $u_2 \in N^2_{+t_1}$.

Then $\sigma(t) = \partial H(\gamma_1^2(t_1)(u_2)) = N^H(V(H) \setminus \gamma_1^2(t_1)(u_2)) = N^H(\alpha(t) \cap V(H))$ and similarly $\sigma(t) = N^H(\alpha_1^2(t_1)(u_2)) = \partial H(V(H) \setminus \alpha_1^2(t_1)(u_2)) = \partial H(\gamma(t) \cap V(H))$. Again, it follows that $\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))$ by (D).

Case 3: $\sigma(t) = t_1$ and $\alpha(t) = \alpha^1(t_1)$.

Then $\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))$, because $\Delta^1$ is tight.

Claim 2. For all $t \in V(D)$ with $V(J_t) \neq \sigma(t)$ the graph $J_t \setminus \sigma(t)$ is connected and it holds that $N^H(V(J_t) \setminus \sigma(t)) = \sigma(t)$.

Proof. This is immediate, because $|\sigma(t)| \leq 2$ and $J_t \in Z^*_3$.

We can apply Lemma [5.3.2] to transform $\Delta$ into the desired decomposition $\Delta_{3cc}$.

Remark 8.3.5. It might be easier to prove the 3CC Decomposition Lemma 8.3.1 directly along the lines of the proof of Lemma 8.3.4 instead of proving the 2CC Decomposition Lemma 8.1.1 first and then applying the Decomposition Lifting Lemma 5.6.2, as we did in our proof. Such a direct proof can be found in [49]. However, almost the same construction based on the Decomposition Lifting Lemma will be applied again later, in the proof of the Q4C Decomposition Lemma 10.2.4 and there it would be much harder to give a direct proof.
Chapter 9

Graphs Embeddable in a Surface

In this chapter, we develop a definable structure theory for classes of graphs embeddable in a fixed surface. The main result of the chapter, and maybe the whole first part of the book, is the Definable Structure Theorem for Embeddable Graphs \[9.4.1\] stating that each class of graphs embeddable in a fixed surface admits \(\text{IFP}\)-definable ordered treelike decompositions. By the results of Chapter \[7\], this implies that \(\text{IFP} + C\) captures polynomial time on each such class.

Planar graphs play a special role, and we obtain slightly stronger results for them. We show that 3-connected planar graphs admit \(\text{IFP}\)-definable orders. This not only implies the Definable Structure Theorem for Planar Graphs immediately by the 3CC Lifting Lemma (Corollary \[8.3.3\]), but will also play a crucial role in several proofs in the second part of the book. In addition, we will prove that the class of planar graphs is \(\text{IFP}\)-definable.

9.1 Surfaces and Embeddings of Graphs

We review the definitions and some of the most basic facts about graph embeddings. For more background, I refer the reader to \[22\]. For planar graphs, \[29\] is also a good reference. Appendix B of \[29\] is a short introduction to surface topology that covers much of what is needed here (in a slightly different terminology).

9.1.1 Topological Prerequisites

We denote topological spaces by bold-face letters. In particular, \(\mathbb{R}^n\) denotes the Euclidean \(n\)-space, \(I\) denotes the closed unit interval \(\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}\) with the usual topology, \(S^n\) denotes the \(n\)-sphere, and \(B^n\) denotes the closed unit ball in \(\mathbb{R}^n\). In our notation, we do not distinguish between a topological space and its ground set, and we equip subsets of topological spaces with the usual induced topology and view them as subspaces. The boundary of a subspace (or subset) \(Y\) of a topological space \(X\) is denoted by \(bd_X(Y)\). The interior of \(Y\) is denoted by \(int_X(Y)\), and the closure \(Y \cup bd_X(Y)\) of \(Y\) is denoted by \(cl_X(Y)\). We omit the subscript \(X\) if \(X\) is clear from the context. We write \(X \simeq Y\) to denote that the topological spaces \(X\) and \(Y\) are homeomorphic. A \textit{closed disk} is a space homeomorphic to \(B^2\), and an \textit{open disk} is a space homeomorphic to \(\mathbb{R}^2\). A \textit{simple curve} in a topological space \(X\) is a homeomorphic image of the unit interval \(I\) in \(X\). If \(g \subseteq X\) is a simple curve and \(g : I \to g\) a homeomorphism, then the \textit{endpoints} of \(g\) are \(g(0), g(1)\). We also say that
Chapter 9. Graphs Embeddable in a Surface

$g$ is a curve from $g(0)$ to $g(1)$. We let $\bar{g} := g \setminus \{g(0), g(1)\}$ and call the elements of $\bar{g}$ the internal points of $g$. A simple closed curve in $X$ is a homeomorphic image of the 1-sphere $S^1$ in $X$. Let $g$ be a simple closed curve in the plane $R^2$. By the Jordan Curve Theorem, $R^2 \setminus g$ has precisely two arcwise connected components $A_1$, $A_2$, and $bd(A_1) = bd(A_2) = g$. A consequence of the Jordan Curve Theorem is the following useful fact (Lemma 4.1.2 of [29]).

**Fact 9.1.1.** Let $g_1, g_2, g_3$ be three simple curves in the plane that have the same endpoints $v_1, v_2$ and mutually disjoint interiors. Then $R^2 \setminus (g_1 \cup g_2 \cup g_3)$ has precisely three arcwise connected components $f_{12}, f_{13}, f_{23}$ with boundaries $g_1 \cup g_2, g_1 \cup g_3, g_2 \cup g_3$, respectively.

Furthermore, if $h$ is another simple curve with endpoints $x_1$ in the interior of $g_1$ and $x_2$ in the interior of $g_2$ such that $h \cap f_{12} = \emptyset$ then $h \cap f_{12} \neq \emptyset$.

Let $c$ be a simple closed curve and $x_1, x_2, \ldots, x_n \in c$. Then we say that $x_1, \ldots, x_n$ appear on $c$ in in cyclic order if there is a homeomorphism $f : S^1 \to c$ such that $x_i = f(\cos(2\pi i/n), \sin(2\pi i/n))$ for all $i \in [n]$. Note that for $n \leq 3$, all pairwise distinct vertices $x_1, \ldots, x_n \in c$ appear on $c$ in cyclic order.

**Corollary 9.1.2.** Let $D$ be a closed disk and $x_1, x_2, x_3, x_4 \in bd(D)$ points that appear on the simple closed curve $bd(D)$ in cyclic order. Let $g \subseteq D$ be a simple curve from $x_1$ to $x_3$ and $h \subseteq D$ a simple curve from $x_2$ to $x_4$. Then $g \cap h \neq \emptyset$.

**Surfaces**

A surface is an arcwise connected compact 2-manifold, possibly with boundary. The classification theorem for surfaces without boundary (see, for example, Theorem 3.1.3 of [22]) says that, up to homeomorphism, there are only two infinite families $(S_g)_{g \in N}$ and $(N_h)_{h \in N^+}$ of surfaces constructed in a fairly simple way. Every orientable surface $S$ is homeomorphic to a surface $S_g$, for some $g \geq 0$, obtained from the 2-sphere by adding $g$ handles, and every nonorientable surface $S$ is homeomorphic to a surface $N_h$, for some $h \geq 1$, obtained from the 2-sphere by adding $h$ crosscaps (see Figure [9.1]). The number $g$ or $h$, respectively, is called the (orientable or nonorientable) genus of $S$. It is convenient to define the Euler genus $eg(S)$ of a surface $S$ without boundary to be $2g$ if $S$ is homeomorphic to $S_g$ and $h$ if $S$ is homeomorphic to $N_h$.

**Example 9.1.3.** $S_1$ is a torus (and every torus is homeomorphic to $S_1$). $S_2$ is a double torus, $S_3$ is a triple torus (also known as “pretzel”), et cetera. $N_1$ is a projective plane, and $N_2$ a Klein bottle.

Every surface $S$ with boundary can be obtained from a surface $\overline{S}$ without boundary by deleting the interior of finitely many mutually disjoint closed disks $D_1, \ldots, D_q \subseteq \overline{S}$, that is, $S = \overline{S} \setminus \bigcup_{i=1}^q int_S(D_i)$. Then the boundary of $S$ is the set $bd_S(S)$, which is a disjoint union of the simple closed curves $bd_S(D_1), \ldots, bd_S(D_1)$. We call these simple closed curves the cuffs of $S$. Even without explicit reference to a surface $\overline{S} \supseteq S$ we denote the boundary of $S$ by $bd(S)$ and the interior by $int(S)$. By $cf(S)$ we denote the number of cuffs of $S$. If a surface $S$ is obtained from a surface $S'$ by deleting the interior of a closed disk $D$ and $c := bd(D)$ is the corresponding cuff of $S$, we conversely say that $S'$ is obtained from $S$ by gluing a disk to the cuff $c$.

If $S$ is a surface and $D, D' \subseteq S$ are closed disks that have an empty intersection with $bd(S)$, then the two surfaces $S \setminus int(D)$ and $S \setminus int(D')$ are homeomorphic. If we combine

M. Grohe, Definable Graph Structure Theory
9.1. Surfaces and Embeddings of Graphs

this observation with the classification of surfaces without boundary, we see that every surface is homeomorphic to a surface $S_{g,q}$ obtained from $S_g$ by deleting the interior of $q$ mutually disjoint closed disks or to a surface $N_{h,q}$ obtained from $N_h$ by deleting the interior of $q$ mutually disjoint closed disks, for some $g, h, q \in \mathbb{N}$. We define the Euler genus $\text{eg}(S)$ of a surface $S$ to be $2g$ if $S$ is homeomorphic to $S_{g,q}$ for some $q \in \mathbb{N}$ and $h$ if $S$ is homeomorphic to $N_{h,q}$ for some $q \in \mathbb{N}$.

Note that the 2-sphere can be denoted by either $S_0$ or $S^2$. We usually use $S_0$. Similarly, both $B^2$ and $S_{0,1}$ denote closed disks, and we usually prefer the notation $S_{0,1}$. A cylinder is a space homeomorphic to $S_{0,2}$, and a Möbius strip a space homeomorphic to $N_{1,1}$.

(Non-)Separating and (Non-)Contractible Curves in Surfaces without Boundary

Let $g$ be a simple closed curve in a surface $S$. Then $g$ is nonseparating if $S \setminus g$ is arcwise connected, and separating otherwise. The curve $g$ is contractible if it is the boundary of a closed disk in $S$, and noncontractible otherwise. If $S$ is not homeomorphic to the sphere $S_0$, then the disk $D$ such that $g = \partial(D)$ is uniquely determined. In this case, we call the open disk $\text{ins}(g) := \text{int}(D)$ the inside of $g$.

Example 9.1.4. Figure 9.2 shows four simple closed curves on a double torus. Two of them are separating, and one is contractible.

Fact 9.1.5. Every surface of positive Euler genus contains a nonseparating simple closed curve.

(The following fact summarises the properties of (non)separating simple closed curves in surfaces without boundary most important for us (the fact combines Lemmas B.4 and B.5 of [29]).

Fact 9.1.6. Let $S$ be a surface without boundary and $g$ a simple closed curve in $S$.

\footnote{The standard definition of contractibility says that $g$ is contractible if there is a continuous function $g : S^1 \to S$ that is a homeomorphism from $S^1$ onto $g$ and that is homotopic to a constant function. The two definitions are equivalent by a result of [33]. We will briefly discuss homotopy in Section 15.1.1.}
Chapter 9. Graphs Embeddable in a Surface

Figure 9.2. Simple closed curves on a double torus

(1) If \( g \) is nonseparating then it is noncontractible, and there is a surface \( S' \) such that \( S \setminus g \) is homeomorphic to a space obtained from \( S' \) by deleting one or two disjoint closed disks. Furthermore, \( \text{eg}(S') < \text{eg}(S) \).

(2) If \( g \) is separating, then \( S \setminus g \) has precisely two arcwise connected components \( A_1, A_2 \), and \( \text{bd}(A_1) = \text{bd}(A_2) = g \). Furthermore, for \( i = 1, 2 \), there is a surface \( S_i \) such that \( A_i \) is homeomorphic to a space obtained from \( S_i \) by deleting a closed disk, and \( \text{eg}(S_1) + \text{eg}(S_2) = \text{eg}(S) \).

The surfaces \( S' \) in (1) and \( S_1, S_2 \) in (2) are unique up to homeomorphism. We call them the surface(s) obtained from \( S \) by cutting along \( g \) and capping the holes (see Appendix B of [29] for a description of the construction of \( S' \) or \( S_1, S_2 \) from \( S \) and \( g \)).

Note that a contractible curve \( g \) in a surface \( S \) is separating, and one of the surfaces \( S_1, S_2 \) obtained from \( S \) by cutting along \( g \) and capping the holes is a sphere, because a sphere is the only surface \( S' \) such that deleting a closed disk from it yields an open disk. If \( S_1 \) is a sphere, then \( \text{eg}(S_1) = 0 \) and thus \( \text{eg}(S_2) = \text{eg}(S) \). Facts [9.1.5] and [9.1.6] have the following consequences for noncontractible curves.

**Corollary 9.1.7.** Let \( S \) be a surface without boundary of positive Euler genus.

(1) \( S \) has a noncontractible simple closed curve.

(2) Let \( g \) be a noncontractible simple closed curve in \( S \). Then either \( g \) is separating, and the two surfaces \( S_1, S_2 \) obtained from \( S \) by cutting along \( g \) and capping the holes have smaller Euler genus than \( S \), or \( g \) is nonseparating, and the surface \( S'' \) obtained from \( S \) by cutting along \( g \) and capping the holes has smaller Euler genus than \( S \).

Noncontractible curves in surfaces with boundary will be discussed in Section [15.1.2].

**Overlapping Disks in Surface**

**Fact 9.1.8.** Let \( S \) be a surface, and let \( g_1, g_2, g_3 \subseteq S \) be simple curves that have the same endpoints and mutually disjoint interiors. If the simple closed curves \( g_1 \cup g_2 \) and \( g_2 \cup g_3 \) are contractible, then the simple closed curve \( g_1 \cup g_3 \) is contractible as well.

For a proof, see Proposition 4.3.1 of [92].

M. Grohe, *Definable Graph Structure Theory*
Fact 9.1.8 can be read as saying that if we have two closed disk in a surface, and
the intersection of these disks is a segment of their boundaries, then their union is also a closed
disk. We need the following generalisation of this fact.

Fact 9.1.9. Let $S$ be a surface, and let $D, D' \subseteq S$ be closed disks such that $\text{bd}(D') \cap D$ is
a simple curve. Then $D \cup D'$ is a closed disk.

While intuitively clear, this not a standard fact, so let me sketch a proof. Let $P' = \text{bd}(D') \cap D$. By assumption, $P'$ is a simple curve. Let $x, y$ be its endpoints. Then $x, y \in \text{bd}(D) \cap \text{bd}(D')$. Let $Q'$ be the segment of $\text{bd}(D')$ from $x$ to $y$ internally disjoint from $P'$, and let $P, Q$ be the two segments of $\text{bd}(D)$ from $x$ to $y$.

I claim that either $P \subseteq D'$ or $Q \subseteq D'$. Suppose for contradiction that there are points $z \in P \setminus D'$ and $z' \in Q \setminus D'$. Let $D_0 \subseteq D$ be a closed disk with $\text{bd}(D_0) \cap \text{bd}(D) = \{z, z'\}$. Let us say that a segment $P'_0$ of $P'$ is an inner segment if its two endpoints are in $\text{bd}(D)$ and its internal points are in $\text{int}(D)$. There may be infinitely many inner segments. However, a straightforward compactness argument shows that only finitely many inner segments $P'_1, \ldots, P'_k$ of $P'$ have a nonempty intersection with $D_0$.

Let $Z \subseteq D_0$ be a simple curve from $z$ to $z'$ with all internal points in the interior of $D_0$. We can find another curve $Z' \subseteq Z \cup P'_1 \cup \ldots \cup P'_k$ from $z$ to $z'$ such that for every $i \in [k]$, the intersection $Z \cap P'_i$ is either empty or a single point or a simple curve. (To achieve this, for example, for $P'_1$, consider the first and last points $v, v'$ of $Z$ in $P'_1$. Modify the curve by going from $z$ to $v$ along $Z$, then from $v$ to $v'$ along $P'_1$, and then from $v'$ to $z'$ along $Z$.) As $P' \cap Z \subseteq P' \cap D_0 \subseteq P'_1 \cup \ldots \cup P'_k$, we also have $P' \cap Z' \subseteq P'_1 \cup \ldots \cup P'_k$. Now we can slightly perturb the curve $Z'$ and obtain another curve $Z''$ from $z$ to $z'$ with all internal points in $\text{int}(D)$ such that the intersection $Z'' \cap P'$ consists of finitely many points $z_1, \ldots, z_n$, appearing on $Z''$ in this order. We may further assume that in each $z_i$ the two curves “cross” and not just “touch”. (That is, in arbitrarily small neighbourhoods of each $z_i$ on $P'$ we find points of both arcwise connected components of $D \setminus Z''$.) It follows from Corollary 9.1.2 that the two endpoints $x, y$ of $P'$ are in different arcwise connected components of $D \setminus Z''$. This implies that $n$ must be odd. For $j \in [2, n]$, let $Z_j$ be the open segment of $Z''$ between $z_{j-1}$ and $z_j$. Moreover, let $Z_1$ be the half-open segment from $z$ to $z_1$ (containing $z$, but not $z_1$), and let $Z_{n+1}$ be the half-open segment from $z_n$ to $z'$ (containing $z'$, but not $z_n$). As $Z \cap \text{bd}(D') = \{z_1, \ldots, z_n\}$, for all $j \in [n + 1]$ either $Z_j \subseteq \text{int}(D')$ or $Z_j \cap D' = \emptyset$. Thus $Z_0 \cap D' = \emptyset$ and $Z_{n+1} \cap D' = \emptyset$, because $z, z' \not\in D'$. Furthermore, for all $j \in [n]$ we have $Z_j \subseteq \text{int}(D')$ if and only if $Z_{j+1} \cap D' = \emptyset$, because the curves $P'$ and $Z''$ cross in $z_j$ and $P' \subseteq \text{bd}(D')$. However, this contradicts $n$ being odd, which proves the claim that either $P \subseteq D'$ or $Q \subseteq D'$.

Without loss of generality we may assume that $P \subseteq D'$. Now consider the three internally disjoint simple curves $Q, P, Q'$ from $x$ to $y$. The simple closed curve $Q \cup P$ is contractible, because it is the boundary of the disk $D$. The simple closed curve $P \cup Q'$ is contractible, because it is contained in the disk $D'$. Thus by Fact 9.1.8 the simple closed curve $Q \cup Q'$ is contractible as well. However, $Q \cup Q'$ is the boundary of $D \cup D'$, which proves that $D \cup D'$ is a closed disk.

---

2A similar argument appears in Thomassen’s proof that every surface can be triangulated [121] (also see Theorem 3.1.1 of [92]).
9.1.2 Graph Embeddings

Let \( X \) be a topological space. An embedded graph in \( X \) is a pair \( G = (V(G), E(G)) \) where \( V(G) \) is a finite subset of \( X \) and \( E(G) \) a set of simple curves in \( X \) such that for all \( e \in E(G) \), both endpoints and no internal point of \( e \) are in \( V(G) \), and any two distinct \( e, e' \in E(G) \) have at most one endpoint and no internal points in common. By \( G \) we denote both the set \( V(G) \cup \bigcup_{e \in E(G)} e \) and the subspace of \( X \) it induces; recall that we do not distinguish between a topological space and its ground set in our notation. As for abstract graphs, we denote the components of a unique edge \( e \) of \( X \) in our notation. As for abstract graphs, we denote the underlying graph of a graph with vertex set \( V \) and edge set \( E \). Two embedded graphs \( G \) and \( H \) are isomorphic (we write: \( G \cong H \)) if their underlying graphs are isomorphic, and they are homeomorphic (we write: \( G \simeq H \)) if there is a homeomorphism \( h \) from \( X \) to \( Y \) with \( h(V) = H \) and \( h(V(G)) = V(H) \).

An embedding of a graph \( G \) in \( X \) is an isomorphism from \( G \) to a graph \( G' \) embedded in \( X \), and \( G \) is embeddable in \( X \) if it has an embedding in \( X \). The class of all graphs embeddable in \( X \) is denoted by \( E_X \). If \( \Pi \) is an embedding of \( G \) in \( X \) that maps \( G \) to the embedded graph \( G' \), then we denote \( G' \) by \( \Pi(G) \). Furthermore, for each edge \( e \in E(G) \) we let \( \Pi(e) \) be the edge of \( \Pi(G) \) corresponding to \( e \). Two embeddings \( \Pi, \Pi' \) of a graph \( G \) in spaces \( X, X' \) are homeomorphic (we write: \( \Pi \simeq \Pi' \)) if there is a homeomorphism \( h \) from \( X \) to \( X' \) such that \( \Pi' = h \circ \Pi \) and \( \Pi'(e) = h(\Pi(e)) \) for all \( e \in E(G) \).

A graph \( G \) is 2-cell embedded in a space \( X \) if all faces \( f \in F(G) \) are open disks. An embedding \( \Pi \) of \( G \) in \( X \) is 2-cell if \( \Pi(G) \) is 2-cell embedded in \( X \).

**Embeddings in Surfaces Without Boundary**

In the following, let \( S \) be a surface without boundary, and let \( G \) be graph embedded in \( S \). Then every edge of \( G \) is incident to at least one and at most two faces. Furthermore, an edge \( e \in E(G) \) is incident to a face \( f \in F(G) \) if and only if \( e \subseteq \bd(f) \). Hence for every face \( f \in F(G) \) there is a subgraph \( H \subseteq G \) such that \( H = \bd(f) \). We denote \( H \) by \( Bd(f) \) and call it the boundary subgraph of \( f \). We call a subgraph \( H \subseteq G \) a facial subgraph if it is the boundary subgraph of some face \( f \in F(G) \).

Faces and facial subgraphs of graphs embedded in a surface can be more complicated than one might think. The following example illustrates a few typical cases.

**Example 9.1.10.** Figure 9.3(a) shows an embedding of \( K_4 \) in the torus that is not 2-cell, because it has a face homeomorphic to a cylinder.
For every vertex \( v \in V(G) \) there is a cyclic permutation \( \pi_v \) of the set \( E(v) \) of all edges incident with \( v \), such that in any “sufficiently small” closed disk around \( v \) the edges intersect the boundary of the disk in the cyclic order given by \( \pi_v \). More precisely, if \( D \subseteq S \) is a closed disk with \( v \in D \) and \( e \cap \text{bd}(D) \neq \emptyset \) for every \( e \in E(v) \), and if for every \( e \in E(v) \) we let \( e_0 \) be the segment of \( e \) from \( v \) to \( \text{bd}(D) \) and \( x_e \in \text{bd}(D) \) be the endpoint \( e_0 \), then the vertices \( x_e \) appear on \( \text{bd}(D) \) in the cyclic order given by \( \pi_v \). This order does not depend on the disk \( D \) (see Lemma 3.2.2 of [92] for a proof). Note that if \( \pi_v \) is such a cyclic permutation, then \( \pi_v^{-1} \) is another one corresponding to the opposite orientation of the boundary of the disk \( D \).

Clearly, for all edges \( e = vw \) there is a face \( f \in F(G) \) such that both \( e \) and \( \pi_v(e) \) are incident with \( f \); just take the face \( f \in F(G) \) that contains the segment of \( \text{bd}(D) \setminus \{x_{e'} \mid e' \in E(v)\} \) from \( x_e \) to \( x_{\pi_v(e)} \). This is not unique if \( E(v) = \{e, \pi_v(e)\} \). By a similar argument, for every face \( f \) incident with \( v \) and every edge \( e \in E(v) \) incident with \( f \), either \( \pi_v(e) \) is incident with \( f \) or \( \pi_v^{-1}(e) \) is incident with \( f \).

The cyclic orders around the vertices are often used to give a purely combinatorial definition of embeddings of graphs in surfaces (see Chapter 4 of [92]). We can also use them to define facial walks. Let \( f \) be a face of a graph \( G \) embedded in a surface, \( B \) a connected component of \( \text{Bd}(f) \), and \( e_1 = v_1v_2 \in E(B) \). We define a closed walk \( v_1e_1v_2e_2v_3\ldots v_{m-1}e_{m-1}v_me_mv_{m+1} = v_1 \) by letting \( e_{i+1} \in \{\pi_{v_{i+1}}(e_i), \pi_{v_{i+1}}^{-1}(e_i)\} \) such that \( e_{i+1} \) is incident with \( f \). We stop the walk as soon as the next edge would be \( e_1 = v_1v_2 \) again in the same direction, so that we have \( (v_i, v_{i+1}) \neq (v_1, v_2) \) for all \( i \in [2, m] \). Then \( V(B) = \{v_1, \ldots, v_m\} \) and \( E(B) = \{e_1, \ldots, e_m\} \). Every edge \( e \) may appear at most twice in this walk (once in each direction), and if it appears twice then \( f \) is the only face incident with \( e \). Of course, if \( \text{bd}(f) \) is connected, then modulo orientation and starting edge there is only one facial walk associated with \( f \).

---

There is a slight ambiguity here because it may happen that both \( \pi_{v_{i+1}}(e_i) \) and \( \pi_{v_{i+1}}^{-1}(e_i) \) are incident with \( f \). This can be resolved, but is not so important for us.
Euler’s Formula and Faces with Small Boundaries

Fact 9.1.11 (Euler’s formula). Let $G$ be a graph embedded in a surface $S$ without boundary, and let $C(G)$ be the set of connected components of $G$. Then

$$|V(G)| - |E(G)| + |F(G)| - |C(G)| + 1 \geq 2 - \text{eg}(S)$$

(9.1.1)

If $G$ is 2-cell embedded in $S$, then equality holds.

The integer $\chi(S) := 2 - \text{eg}(S)$ is known as the Euler characteristic of $S$. In the literature, Euler’s formula is usually only stated for 2-cell embedded graphs. As 2-cell embedded graphs are connected, in this special case we have

$$|V(G)| - |E(G)| + |F(G)| = 2 - \text{eg}(S).$$

(9.1.2)

Let us sketch how to derive the inequality (9.1.1) for arbitrary embedded graphs from equality (9.1.2).

Consider a graph $G$ embedded in a surface $S$. We can find a connected supergraph $G' \supseteq G$ that is 2-cell embedded in $S$. For example, we can let $G'$ be a sufficiently fine triangulation of $S$. Of course the fact that such a triangulation exists needs to be proved, but I ask the reader to trust me that it does (also see Chapter 3 of [92]). By Euler’s formula for 2-cell embedded graphs, equality (9.1.2) holds for $G'$.

We can obtain $G$ from $G'$ by a sequence deletions of edges and isolated vertices, and we will show that each edge or vertex deletion can only increase the left-hand side of (9.1.1). More precisely, there is a sequence $G' = G_0 \supset G_1 \supset \ldots \supset G_m = G$ such that for each $i \in [m]$, the graph $G_i$ is obtained from the graph $G_{i-1}$ by deleting either an isolated vertex or an edge. We shall prove that in both cases we have

$$|V(G_i)| - |E(G_i)| + |F(G_i)| - |C(G_i)| \geq |V(G_{i-1})| - |E(G_{i-1})| + |F(G_{i-1})| - |C(G_{i-1})|.$$  

(9.1.3)

If $G_i$ is obtained from $G_{i-1}$ by deleting an isolated vertex, then we reduce both the number of vertices and the number of connected components by one, whereas we leave the numbers of edges and faces unchanged. Hence (9.1.3) holds with equality. Suppose that $G_i$ is obtained from $G_{i-1}$ by deleting an edge $e \in E(G_{i-1})$. Then $|V(G_i)| = |V(G_{i-1})|$ and $|E(G_i)| = |E(G_{i-1})| - 1$. Furthermore, $|F(G_i)| \geq |F(G_{i-1})| - 1$, because $e$ is incident to at most 2 faces of $G_{i-1}$, and $|C(G_i)| \leq |C(G_{i-1})| + 1$, because $e$ connects to at most two connected components of $G_i$. It may happen that $|F(G_i)| = |F(G_{i-1})| - 1$ and that $|C(G_i)| = |C(G_{i-1})| + 1$, but not both at the same time. Indeed, suppose that $|C(G_i)| = |C(G_{i-1})| + 1$. Let $A', A''$ be the two connected components of $G_i$ that contain an endvertex of $e$, and let $v \in V(A), v' \in V(A')$ be the endvertices of $e$. Let $v_1 e_1 v_2 \ldots v_{m+1} = v_1$ be the facial walk of $f$ with $v_1 := v, e_1 := e, v_2 := v'$. As $v_2 = v' \in V(A')$ and $v_{m+1} = v \in V(A)$, there must be at least one $i \in [2, m]$ such that $v_i \in V(A')$ and $v_{i+1} \in V(A)$. Since $e$ is the only edge between $A'$ and $A$, we must have $e_i = e$. Thus $e$ appears twice in the facial walk, once in each direction, and this implies that $f$ is the only face incident with $e$. Hence by deleting $e$ we do not increase the number of faces, that is, $|F(G_i)| = |F(G_{i-1})|$. This completes the proof (9.1.3).

Inequality (9.1.1) follows, because

$$|V(G)| - |E(G)| + |F(G)| - |C(G)| + 1 \geq |V(G')| - |E(G')| + |F(G')| - |C(G')| + 1 = 2 - \text{eg}(S).$$

where the inequality follows by induction from (9.1.3) and the inequality follows from (9.1.2). This completes our proof sketch.
9.1. Surfaces and Embeddings of Graphs

It is well-known that Euler’s formula can be used to prove that sufficiently large surface graphs of minimum degree at least 3 have facial subgraphs with at most 6 vertices. We need a slight generalisation of this fact to graphs of minimum degree 2. Recall that a branch vertex of a graph $G$ is a vertex of degree at least 3 and that an isolated path path in $G$ is a path $P \subseteq G$ that has no internal branch vertices.

**Lemma 9.1.12.** Let $G$ be a graph of minimum degree at least 2 embedded in a surface $S$ without boundary. Then either $G$ has at most $14(\text{eg}(S) - 2)$ branch vertices or there is a facial subgraph $H$ of $G$ that contains at most 6 branch vertices.

**Proof.** Suppose that every facial subgraph $H$ of $G$ contains more than 6 branch vertices. We shall prove that $G$ has at most $14(\text{eg}(S) - 2)$ branch vertices.

Let $V_2 := \{v \in V(G) \mid \deg(v) = 2\}$ and $V_{\geq 3} := \{v \in V(G) \mid \deg(v) \geq 3\}$. Then

$$2|V_2| + 3|V_{\geq 3}| \leq \sum_{v \in V(G)} \deg(v) = 2 \cdot |E(G)|$$

and thus

$$|V_{\geq 3}| \leq \frac{2}{3}(|E(G)| - |V_2|). \tag{9.1.4}$$

Let $P(G)$ be the set of all isolated paths in $G$ of length at least 1 and with both endpoints in $V_{\geq 3}$. Observe that for every $p \in P(G)$,

(i) $|E(p)| - |V(p) \cap V_2| = 1$;

(ii) for all $f \in F(G)$, if $E(p) \cap E(\text{Bd}(f)) \neq \emptyset$ then $p \subseteq \text{Bd}(f)$;

(iii) there are at most two faces $f \in F(G)$ with $p \subseteq \text{Bd}(f)$.

Moreover, observe that (i) implies

$$|E(G)| - |V_2| = |P(G)| \tag{9.1.5}$$

For every facial subgraph $H$ of $G$, let $P(H) = \{p \in P(G) \mid p \subseteq H\}$. If $H$ is a facial subgraph, then the minimum degree of $H$ is at least 2. To see this, let $f$ be the face such that $\text{Bd}(f) = H$. Suppose that $v \in V(H)$ is a vertex with $\deg^H(v) \leq 1$. Then $\deg^G(v) \leq 1$ as well, because no edge of $G$ has a nonempty intersection with the face $f$ that “surrounds” $v$. This is a contradiction, because we assumed $G$ to be of minimum degree 2.

It follows that every vertex in $V(H) \cap V_{\geq 3}$ is an endvertex of at least two paths in $P(H)$. This implies

$$|V(H) \cap V_{\geq 3}| \leq |P(H)|, \tag{9.1.6}$$

Recall that all facial subgraphs of $G$ have more than 6 branch vertices. Then

$$7|F(G)| \leq \sum_{f \in F(G)} |V(\text{Bd}(f)) \cap V_{\geq 3}|$$

$$\leq \sum_{f \in F(G)} |P(\text{Bd}(f))| \tag{by (9.1.6)}$$

$$\leq 2|P(G)| \tag{by (iii)}. $$

Thus

$$|F(G)| \leq \frac{2}{7}|P(G)| \tag{9.1.7}$$
By Euler’s formula \([9.1.1]\),

\[
2 - \text{eg}(S) + C(G) - 1 \leq |V(G)| - |E(G)| + |F(G)|
\]

\[
= |V_{\geq 3}| - |P(G)| + |F(G)| \quad \text{(by \([9.1.5]\))}
\]

\[
\leq \frac{2}{3}|P(G)| - |P(G)| + \frac{2}{7}|P(G)| \quad \text{(by \([9.1.4]\), \([9.1.5]\), and \([9.1.7]\))}
\]

\[
= -\frac{1}{21}|P(G)|.
\]

Then \(|P(G)| \leq 21(\text{eg}(S) - C(G) - 1) \leq 21(\text{eg}(S) - 2)\), which implies

\[
|V_{\geq 3}| \leq \frac{2}{3}|P(G)| \quad \text{(by \([9.1.4]\) and \([9.1.5]\))}
\]

\[
\leq 14(\text{eg}(S) - 2).
\]

\[\square\]

**Embeddings in Surfaces with Boundary**

Unfortunately, even the most basic observation we made for graphs embedded in surfaces without boundary fails for surfaces with boundary. Let \(G\) be a graph embedded in a surface \(S\) and \(e \in E(G)\). Then it may well be that there is a face \(f \in F(G)\) incident with \(e\) such that \(e \not\subseteq \text{bd}(f)\). If this happens, then there is no boundary subgraph \(H \subseteq G\) such that \(H = \text{bd}(f)\). To avoid this, we restrict our attention to a special form of embeddings in surfaces with boundary.

Let \(S\) be a surface, possibly with nonempty boundary, and let \(G\) be a graph embedded in \(S\). A subset \(R \subseteq S\) is \(G\)-normal if \(R \cap G \subseteq V(G)\). The graph \(G\) is \(G\)-normal embedded in \(S\) if \(\text{bd}(S)\) is \(G\)-normal. That is, \(G\) is normally embedded in \(S\) if and only if the interior of all edges of \(G\) has an empty intersection with the boundary of \(S\). An embedding \(\Pi\) of an abstract graph \(H\) in \(S\) is \(H\)-normal if \(\Pi(H)\) is normally embedded in \(S\). In the following, we assume that \(G\) is normally embedded in \(S\). Then for every edge \(e \in E(G)\) and every face \(f \in F(G)\), the edge \(e\) is incident to \(f\) if and only if \(E \subseteq \text{bd}(f)\). Hence there is a well-defined boundary subgraph \(\text{bd}(f) \subseteq G\) such that \(\text{bd}(f) = \text{bd}(f) \setminus (\text{bd}(S) \setminus V(G))\). Furthermore, every edge is incident to at least one and at most two faces.

**Minimal Embeddings and Noncontractible Subgraphs**

Let \(G\) be a graph normally embedded in a surface \(S\). We say that \(G\) is minimally embedded in \(S\) if there is no embedding of \(G\) in a surface \(S'\) of smaller Euler genus or of the same Euler genus, but with fewer cuffs. Informally, we say that such an \(S'\) is simpler than \(S\). We give a slightly more refined definition of a surface being simpler than another surface in Section \([15.1.2]\). A minimal embedding of \(G\) is an embedding \(\Pi\) of \(G\) in some surface \(S\) such that \(\Pi(G)\) is minimally embedded in \(S\). Obviously, if \(G\) is minimally embedded in \(S\) then \(S\) is a surface without boundary, because otherwise we could glue disks on the cuffs of \(S\) and obtain an embedding of \(G\) in a surface of the same genus with fewer cuffs. Hence every graph \(G\) has an embedding in some surface \(S\) without boundary. We define the Euler genus of \(G\), denoted by \(\text{eg}(G)\), to be the least \(g \in \mathbb{N}\) such that \(G\) has an embedding in a surface of Euler genus \(g\). The class of all graphs of Euler genus at most \(g\) is denoted by \(\mathcal{E}_g\).

**Fact 9.1.13.** Every minimal embedding of a connected graph is 2-cell.
Let $G$ be embedded in a surface $S$ without boundary. A cycle $C \subseteq G$ is noncontractible if the simple closed curve $C$ is noncontractible in $S$. It follows from Corollary 9.1.7 that if $C \subseteq G$ is a noncontractible cycle then every connected component of $G \setminus C$ is embeddable in a surface that is simpler than $S$.

**Fact 9.1.14.** Let $G$ be a graph embedded in a surface $S$ without boundary. Then either there is a closed disk $D \subseteq S$ such that $G \subseteq D$ or $G$ contains a noncontractible cycle.

If $G$ is 2-connected, the disk $D$ can be chosen in such a way that there is a cycle $C \subseteq G$ such that $bd(D) = C$.

Even though this fact is well-known, I could not find an explicit reference. Therefore, let me sketch a proof. Let $S$ be a surface. We first prove the stronger statement for 2-connected graphs by a simple induction using Fact 9.1.9 and the fact that every 2-connected graph can be constructed from a cycle by repeatedly adding paths with both endvertices, but no internal vertices in the graph constructed so far. The statement of Fact 9.1.14 for arbitrary graphs follows from the statement for 2-connected graphs, because by adding edges we can easily extend every embedded graph to a 2-connected graph still embedded in the same surface.

**Facial Cycles**

Let $G$ be a graph normally embedded in a surface $S$. A cycle $C \subseteq G$ is chordless if $C$ is an induced subgraph of $G$. A cycle $C \subseteq G$ is nonseparating if $G \setminus C$ is connected.

**Lemma 9.1.15.** Let $G$ be a graph embedded in a surface $S$ and $C \subseteq G$ a contractible, chordless, and nonseparating cycle of $G$. Then $C$ is a facial cycle.

**Proof.** As $C$ is contractible, $S \setminus C$ has two arcwise connected components $A_1$ and $A_2$, both with boundary $C$. As $C$ is nonseparating, either $V(G) \cap A_1 = \emptyset$ or $V(G) \cap A_2 = \emptyset$. Say, $V(G) \cap A_2 = \emptyset$. Then $V(G) \subseteq cl(A_1)$. Let $e \in E(G)$. If both endvertices of $e$ are in $V(C)$, then $e \in E(C)$ because $C$ is chordless, and hence $e \subseteq C \subseteq cl(A_1)$. And if at least one endvertex of $e$ is in $A_1$, then clearly we also have $e \subseteq cl(A_1)$. It follows that $G \subseteq cl(A_1)$, and hence $A_2$ is a face of $G$ with boundary subgraph $C$. 

In the following, let $S$ be a surface without boundary. Recall that a subset $X \subseteq S$ is $G$-normal if $X \cap G \subseteq V(G)$, that is, $X$ has an empty intersection with the interior of all edges of $G$. Suppose now that $\text{eg}(S) > 0$. Recall that by Corollary 9.1.7 there is a noncontractible simple closed curve in $S$. The representativity $\rho(G)$ of $G$ is the maximum $r \in \mathbb{N}$ such that every $G$-normal noncontractible simple closed curve $g$ intersects $G$ in at least $r$ vertices. We extend the definition to graphs embedded in the 2-sphere $S_0$ by letting $\rho(G) := \infty$ for every graph embedded in $S_0$. Intuitively, an embedded graph of high representativity locally looks like a graph embedded in the plane. It is not hard to see that a connected embedded graph of representativity at least 1 is 2-cell embedded. Furthermore, if the representativity of an embedded graph $G$ is at least 3 then for every vertex $v \in V(G)$ there is open disk $D_v \subseteq S$ that contains the subgraph $B(v) := \bigcup_f Bd(f)$, where the union ranges over all faces $f \in F(G)$ incident with $v$.

We say that $G$ is polyhedrally embedded in $S$ if $G$ is 3-connected and embedded in $S$ with representativity $\rho(G) \geq 3$. An embedding $\Pi$ of a graph $G$ in a surface $S$ is polyhedral if $\Pi(G)$ is polyhedrally embedded in $S$. 

Preliminary Version
Figure 9.4. Edge contraction in a surface

Example 9.1.16. Figure 9.3 (b) shows a 2-cell embedding of the 3-connected graph $K_5$ in the torus that is not polyhedral, because its representativity is 1.

Fact 9.1.17 (Robertson and Vitray [114]). Let $G$ be polyhedrally embedded in a surface $S$ without boundary.

1. All facial subgraphs of $G$ are chordless and nonseparating cycles.
2. For any two facial cycles $C, C'$ of $G$ it holds that $|E(C) \cap E(C')| \leq 1$.
3. All edges of $G$ are contained in exactly two facial cycles.

(For a proof, see Propositions 5.5.13. and Propositions 5.5.12 of [92].)

Surface Minors

The following fact states that we can contract an edge of a graph embedded in a surface in such a way that only edges incident with one of the two endvertices of the contracted edge are affected, and all other edges and vertices remain unchanged. Figure 9.4 illustrates the fact. To state this formally, we need some terminology. Let $\Pi, \Pi'$ be normal embeddings of graphs $G, G'$, respectively, in a surface $S$. Let $H \subseteq G \cap G'$. We say that $\Pi$ and $\Pi'$ coincide on $H$ if the restrictions $\Pi|_{V(H) \cup E(H)}$ and $\Pi'|_{V(H) \cup E(H)}$ are equal. We say that $\Pi$ and $\Pi'$ strongly coincide on $H$ if they coincide on $H$ and for every subgraph $J \subseteq H$, there is a face $f \in F(\Pi)$ such that $\Pi(J) \subseteq Bd(f)$ if and only if there is a face $f' \in F(\Pi')$ such that $\Pi'(J) \subseteq Bd(f')$.

Fact 9.1.18. Let $G$ be a graph embedded in a surface $S$ and $e \in E(G)$ an edge with endvertices $v, w$. Then the minor $G/e$ of $G$ is isomorphic to a graph $G^*$ embedded in $S$ with the following properties:

1. $G \setminus w \subseteq G^*$ and $V(G) \setminus \{w\} = V(G^*)$.
2. The embeddings of $G$ and $G^*$ strongly coincide on $G \setminus w$.

For a proof, see the remarks on p.103 of [92]. Combined with the fact that subgraphs of embedded graphs are embedded in the same space, the corollary shows that we may view minors of graphs embedded in a surface $S$ as graphs embedded in $S$ such that vertices and edges not affected by contractions or deletions remain unchanged. In the following, we always view minors of embedded graphs in this way.

M. Grohe, Definable Graph Structure Theory
Corollary 9.1.19. Let \( \Pi \) be an embedding of a graph \( G \) in a surface \( S \), and let \( H \subseteq G \). Then there is an embedding \( \Pi^* \) of \( G/H \) in \( S \) that strongly coincides with \( \Pi \) on \( G \setminus H \).

Algorithmic Aspects

It is NP-hard to decide whether a given graph can be embedded in a given surface (where the surface is given say, by specifying its genus and whether it is orientable or not) [120]. However, we have the following:

Fact 9.1.20 (Filotti, Miller, and Reif [36]). For every surface \( S \), the class \( \mathcal{E}_S \) is decidable in polynomial time.

As \( \mathcal{E}_S \) is a minor ideal, the fact also follows from Robertson and Seymour’s [110] general result that all minor ideals are decidable in cubic time (Corollary 2.2.4). Mohar [91] proved that the classes \( \mathcal{E}_S \) are even decidable in linear time.

9.1.3 Plane and Planar Graphs

A plane graph is a graph embedded in the sphere \( S_0 \). Of course it would be more natural to call graphs embedded in \( \mathbb{R}^2 \) “plane”. However, a graph is embeddable in \( S_0 \) if and only if it is embeddable in \( \mathbb{R}^2 \), and it will be much more convenient for us to work with embeddings in \( S_0 \) because \( \mathbb{R}^2 \) is not compact and hence not a surface according to the definition given in the previous subsection. A planar graph is a graph that is embeddable in \( S_0 \). The class of all planar graphs is denoted by \( \mathcal{P} \). Hence \( \mathcal{P} = \mathcal{E}_{S_0} = \mathcal{E}_0 \).

Fact 9.1.21. Let \( G \) be a 2-connected plane graph. Then for all faces \( f \in F(G) \) the closure \( \text{cl}(f) \) is a closed disk and thus \( \text{Bd}(f) \) is a cycle.

Observe that every embedding of a 3-connected graph in \( S_0 \) is polyhedral. Hence the following fact due to Whitney follows directly from Lemma 9.1.15 and Fact 9.1.17(1).

Fact 9.1.22. Let \( G \) be a 3-connected plane graph and \( C \subseteq G \) a cycle. Then \( C \) is a facial cycle if and only if \( C \) is chordless and nonseparating.

From this fact and a version of the Jordan-Schönflies Theorem one obtains the following theorem (see [29] for a proof).

Fact 9.1.23 (Whitney’s Theorem [128]). Any two embeddings of a 3-connected graph in the sphere are homeomorphic.

In the following, we call the chordless and nonseparating cycles of a 3-connected planar graph \( G \) facial cycles of \( G \) even without specifying an embedding.

Fact 9.1.24 (Kuratowski’s Theorem). A graph \( G \) is planar if and only if neither \( K_5 \leq G \) nor \( K_{3,3} \leq G \).

Actually, this version of Kuratowski’s Theorem is due to Wagner [126]. Kuratowski’s original theorem [82] is stated for topological subgraphs instead of minors.

We say that two disjoint subgraphs \( H, I \) of a graph \( G \) are adjacent if there is an edge \( e \in E(G) \) with one endvertex in \( H \) and one endvertex in \( I \). The following lemma is an easy consequence of Kuratowski’s Theorem.
Lemma 9.1.25. Let $G$ be a graph embedded in a closed disk $D$.

(1) There are no mutually disjoint and mutually adjacent connected subgraphs $H_1, \ldots, H_4 \subseteq G$ such that for all $i \in [4]$ it holds that $V(H_i) \cap \text{bd}(D) \neq \emptyset$.

(2) There are no mutually disjoint connected subgraphs $H_1, \ldots, H_3, I_2, I_2 \subseteq G$ such that for all $i \in [3], j \in [2]$ it holds that $H_i$ is adjacent to $I_j$, and for all $i \in [3]$ it holds that $V(H_i) \cap \text{bd}(D) \neq \emptyset$.

Proof. To prove (1), suppose for contradiction that there are mutually disjoint and mutually adjacent subgraphs $H_1, \ldots, H_4 \subseteq G$ such that for all $i \in [4]$ it holds that $V(H_i) \cap \text{bd}(D) \neq \emptyset$. For $i \in [4]$, let $v_i \in V(H_i) \cap \text{bd}(D)$.

Let $G^*$ be the graph obtained from $G$ by adding a new vertex $v$ and edges from $v$ to $v_i$. Then $G^*$ is planar and contains $K_5$ as a minor, which is impossible.

To prove (2), we argue similarly with $K_{3,3}$ instead of $K_5$. □

Fact 9.1.26. Let $G, H$ be two planar graphs such that either $|G \cap H| \leq 1$ or $G \cap H = K_2$, that is, $|G \cap H| = 2$ and the two vertices in $V(G) \cap V(H)$ are adjacent in both $G$ and $H$.

Then $G \cup H$ is planar.

(See [57], Theorem 1.6.7 for a proof.)

9.2 Angles

Let $G$ be a graph embedded in a surface $S$. An angle of a face $f \in F(G)$ is a triple $(v_1, v_2, v_3) \in V(G)^3$ such that $v_1 \neq v_3$ and $v_1v_2, v_2v_3 \in E(Bd(f))$. The set of all angles of a face $f$ is denoted by $\angle(f)$. We let $\angle(G) := \bigcup_{f \in F(G)} \angle(f)$. An angle $\overline{w} := (w_1, w_2, w_3) \in \angle(G)$ is aligned with an angle $\overline{v} := (v_1, v_2, v_3) \in \angle(G)$ (we write $\overline{v} \cap^G \overline{w}$ and usually omit the superscript $G$ if $G$ is clear from the context) if $w_1 = v_2$ and $w_2 = v_3$ and there is a face $f \in F(G)$ such that $(v_1, v_2, v_3), (w_1, w_2, w_3) \in \angle(f)$.

Note that the angle relation is symmetric with respect to its first and third argument, that is, $(v_1, v_2, v_3) \in \angle(G) \iff (v_3, v_2, v_1) \in \angle(G)$. Moreover, $(v_1, v_2, v_3) \cap (w_1, w_2, w_3) \iff (w_3, w_2, w_1) \cap (v_3, v_2, v_1)$.

If $\Pi$ is an embedding of a graph $G$ in surface $S$, then we let $\angle(\Pi) := \{ \overline{v} \in V(G)^3 \mid \Pi(\overline{v}) \in \angle(\Pi(G)) \}$. For a 3-connected planar graph $G$, it will be convenient to denote by $\angle(G)$ the set $\angle(\Pi)$ of angles of some embedding $\Pi$ of $G$ in the sphere. This is well-defined because by Whitney’s Theorem (Fact 9.1.23 actually Fact 9.1.22 suffices here) all embeddings $\Pi$ of $G$ in the sphere have the same angles. Moreover, for every facial (that is, chordless and nonseparating) cycle $C$ of a 3-connected planar graph $G$ we define $\angle(C)$ to be the set of all triples $(v_1, v_2, v_3) \in V(C)^3$ such that $v_1 \neq v_3$ and $v_1v_2, v_2v_3 \in E(C)$. Then $\angle(G) = \bigcup_C \angle(C)$, where the union ranges over all facial cycles of $G$.

Angles are tailored towards 3-connected embedded graphs in which all facial subgraphs are cycles, because then they have the nice properties stated in the following two lemmas. Recall that graphs polyhedrally embedded in a surface are 3-connected (by definition), and their facial subgraphs are cycles (by Fact 9.1.17). Thus the lemmas apply to these.

Lemma 9.2.1. Let $G$ be a 3-connected graph embedded in a surface such that all facial subgraphs of $G$ are cycles.

(1) For every every angle $\overline{v} \in \angle(G)$ there is a unique face $f \in F(G)$ such that $\overline{v} \in \angle(f)$. 

M. Grohe, Definable Graph Structure Theory
9.2. Angles

(2) For every angle $\varpi \in V(G)$ there is a unique angle $\overline{\varpi} \in \angle(G)$ such that $\varpi \sim \overline{\varpi}$.

It is interesting to note that even (1) does not necessarily hold in 3-connected embedded graphs in which not all facial subgraphs are cycles, because it may happen that there is a vertex $v$ incident with four edges $e_1 = vw_1, \ldots, e_4 = vw_4$ and two faces $f_1, f_2$ as in Figure 9.5.

Figure 9.5. A node and its incident faces in an embedding that has unexpected angles (such a configuration can, for example, occur in a graph embedded in a torus).

Then not only ($v$ vertex

Let Lemma 9.2.2.

Lemma 9.2.1. To prove (1), suppose for contradiction that $\varpi = (v_1, v, v_2) \in \angle(f_1) \cap \angle(f_2)$ for distinct faces $f_1, f_2 \in F(G)$. Let $e_1 := vv_1$ and $e_2 := vv_2$ be the two edges of this angle. Both of these edges are incident with the faces $f_1$ and $f_2$ and thus with no other face. Let $\pi_v$ be the cyclic permutation of the edges incident with $v$, and suppose that $e := \pi_v(e_1) \neq e_2$. Then $e$ is incident with either $f_1$ or $f_2$, contradicting the assumption that $Bd(f_1)$ and $Bd(f_2)$ are cycles. Hence $\pi_v(e_1) = e_2$ and similarly $\pi_v(e_2) = e_1$. It follows that $e_1$ and $e_2$ are the only edges incident with $v$. Hence $\{v_1, v_2\}$ separates $v$ from $V(G) \setminus \{v_1, v_2\}$, which contradicts $G$ being 3-connected.

To prove (2), let $\varpi = (v_1, v_2, v_3) \in \angle(f)$, and let $C := Bd(f)$. Then there is a unique vertex $v_4 \in N^C(v_3) \setminus \{v_2\}$, and thus $(v_2, v_3, v_4)$ is the unique angle aligned with $\varpi$.

Lemma 9.2.2. Let $G$ be a graph of minimum degree 3 embedded in a surface such that all facial subgraphs of $G$ are cycles. Then for every $v \in V(G)$ the graph

$$C_v := (N(v), \{ww' \mid (w, v, w') \in \angle(G)\})$$

is a cycle.

Proof. Let $v \in V(G)$, and let $\pi_v$ by the cyclic permutation of the edges incident with $v$ induced by the embedding. Suppose that the edges incident with $v$ are $e_1, \ldots, e_m, e_{m+1} = e_1$, where $\pi_v(e_i) = e_{i+1}$ for all $i \in [m]$. Then for all $i \in [m]$, there is a face $f_i$ incident with $e_i$ and $e_{i+1}$, and every face $f$ incident with $v$ is among $f_1, \ldots, f_m$. Furthermore, the faces $f_i$ are mutually distinct, because if $f_i = f_j$ for some $i \neq j$, then $e_i, e_{i+1}, e_j, e_{j+1} \in E(Bd(f_i))$. This contradicts $Bd(f_i)$ being a cycle.

For every $i \in [m]$, let $e_i := vv_i$. Then $(w_i, v, w_{i+1}) \in \angle(G)$, and all angles of the form $(w, v, w')$ are among $(w_1, v, w_2), \ldots, (w_m, v, w_{m+1})$. It follows that $E(C_v) = \{w_iw_{i+1} \mid i \in [m]\}$, and as $m = |N(v)| \geq 3$ and $w_{m+1} = w_1$, it follows that $C_v$ is a cycle.

To define angles in a meaningful way for arbitrary graphs embedded in a surface, we could define them with respect to facial walks instead of faces. But there is no need for us to do this here.
The following three “Angle Lemmas” are the main results of this section. The first two of these lemmas are fairly abstract and do not refer to embedded graphs or angles explicitly, but their typical applications, in particular Corollary 9.2.4, do.

**Lemma 9.2.3 (First Angle Lemma).** For all IFP-formulae \( \angle(\pi_1, y_1, y_2, y_3) \) and \( \text{align}(\pi_2, y_1, y_2, y_3, z_1) \) there is an IFP-formula \( \text{ord}(\pi_1, \pi_2, y_1, y_2, z_1, z_2) \) such that the following holds. Let \( G \) be a connected graph of minimum degree at least 2 and \( \pi_1 \in G^\pi_1, \pi_2 \in G^\pi_2, v_1, v_2, v_3 \in V(G) \). Let

\[
\angle := \{(w, v, w') \in V(G)^3 \mid w, w' \in N(v) \text{ and } w \neq w' \text{ and } G \models \text{angle}[\pi_1, w, v, w']\}.
\]

For all \( v \in V(G) \), let

\[ C_v := (N(v), \{ww' \mid (w, v, w') \in \angle\}). \]

Suppose that:

(i) for all \( v \in V(G) \) and \( w, w' \in N(v) \), if \( (w, v, w') \in \angle \) then \( (w', v, w) \in \angle \);

(ii) for all \( v \in V(G) \), if \( \deg(v) = 2 \) then \( C_v \cong K_2 \) and if \( \deg(v) \geq 3 \) then \( C_v \) is a cycle;

(iii) for all triples \( (w_1, w_2, w_3) \in \angle \) there is exactly one \( w_4 \in N(w_3) \) such that \( (w_2, w_3, w_4) \in \angle \) and \( G \models \text{align}[\pi_2, w_1, w_2, w_3, w_4] \);

(iv) \( (v_1, v_2, v_3) \in \angle \).

Then \( \text{ord}[G, \pi_1, \pi_2, v_1, v_2, v_3, z_1, z_2] \) is a linear order of \( V(G) \).

**Proof.** Let \( G \) be a connected graph of minimum degree at least 2 and \( \pi_1 \in G^\pi_1, \pi_2 \in G^\pi_2 \) and \( v_1, v_2, v_3 \in V(G) \) such that conditions (i)–(iv) are satisfied. For every \( \pi = (w_1, w_2, w_3) \in \angle \), let \( f(\pi) \) be the unique vertex \( w_4 \in N(w_3) \) such that \( (w_2, w_3, w_4) \in \angle \) and \( (w_1, w_2, w_3, w_4) \in \text{align}[G, \pi_2, y_1, y_2, y_3, y_4] \).

We shall define an increasing sequence \( V_1 \subset V_2 \subset \ldots \subset V_n = V(G) \) of subsets of \( V(G) \) and an increasing sequence \( \leq_1 \leq_2 \leq_3 \leq \ldots \leq_n \subseteq V(G)^2 \) of binary relations such that for each \( i \in [n] \) the following two conditions are satisfied:

(A) \( \leq_i \) is a linear order of \( V_i \);

(B) for every \( v \in V_i \), there is a triple \( \pi \in V_i^3 \cap \angle \) such that \( v \in \pi \).

We let \( V_1 := \{v_1, v_2, v_3\} \) and define \( \leq_1 \) by \( v_1 \leq_1 v_2 \leq_1 v_3 \). For the inductive step, suppose that \( V_i \) and \( \leq_i \) are defined and that \( V_i \neq V(G) \). Let \( v \in V_i \) be minimal with respect to the linear order \( \leq_i \) such that \( N(v) \not\subseteq V_i \). Such a \( v \) exists because \( G \) is connected.

**Case 1:** There are \( w, w' \in N(v) \cap V_i \) such that \( (w, v, w') \in \angle \).

Let \( (w, w') \in V_i^2 \) be lexicographically minimal with respect to \( \leq_i \) such that \( (w, v, w') \in \angle \). Since \( w, w' \in N(v) \cap V_i \) and \( N(v) \not\subseteq V_i \), the degree of \( v \) is at least 3. Thus by (ii), \( C_v \) is a cycle with vertex set \( N(v) \). Let \( w'' \) be the first vertex in \( N(v) \setminus V_i \) on the cyclic walk around \( C_v \) starting with \( w \) followed by \( w' \). We let \( V_{i+1} := V_i \cup \{w''\} \), and we let \( \leq_{i+1} \) be the extension of \( \leq_i \) with \( v' \leq_{i+1} w'' \) for all \( v' \in V_i \). Then (A) is obviously satisfied. Let \( w'' \) be the predecessor of \( w'' \) on the cyclic walk around \( C_v \). Then \( (w'', v, w'') \in V_{i+1}^3 \cap \angle \). Hence (B) is also satisfied.
Case 2: There are no \( w, w' \in N(v) \cap V_i \) such that \((w, v, w') \in \angle\).

Then by (B) and (i), there are \( w, v' \in V_i \) such that \((w, v', v) \in \angle\). We choose such \((w, v')\) lexicographically minimal with respect to \(\leq_i\). Let \(w' := f(w, v', v)\). Then \((v', v, w') \in \angle\), and thus \(w' \notin V_i\) by the assumption of Case 2. We let \(V_{i+1} := V_i \cup \{w'\}\), and we let \(\leq_{i+1}\) be the extension of \(\leq_i\) with \(w'' \leq_{i+1} w'\) for all \(w'' \in V_i\). Then (A) and (B) are obviously satisfied.

Let \(n \in \mathbb{N}\) such that \(V_n = V(G)\). (Actually, \(n = |V(G)| - 2\).) Then \(\leq_n\) is a linear order of \(V(G)\). It is easy to see that there is an IFP-formula \(\text{ord}(\overline{x}_1, \overline{x}_2, y_1, y_2, y_3, z_1, z_2)\), which uses the formulae \(\text{angle}(\overline{x}_1, y_1, y_2, y_3)\) and \(\text{align}(\overline{x}_2, y_1, y_2, y_3, y_4)\) as building blocks, and which does not depend on the specific graph \(G\), such that

\[ \leq_n = \text{ord}[G, \overline{x}_1, \overline{x}_2, v_1, v_2, v_3, z_1, z_2]. \]

**Corollary 9.2.4.** For all IFP-formulae \(\text{angle}(y_1, y_2, y_3)\) and \(\text{align}(y_1, y_2, y_3, y_4)\) there is an IFP-formula \(\text{ord}(x_1, x_2, x_3, z_1, z_2)\) such that the following holds. Suppose that \(G\) is a graph polyhedrally embedded in some surface such that for all \(w_1, w_2, w_3, w_4 \in V(G)\)

\(\begin{align*}
(i) & \quad G \models \text{angle}[w_1, w_2, w_3] \iff (w_1, w_2, w_3) \in \angle(G); \\
(ii) & \quad G \models \text{align}[w_1, \ldots, w_4] \iff (w_1, w_2, w_3) \prec (w_2, w_3, w_4).
\end{align*}\)

Then for every angle \((v_1, v_2, v_3) \in \angle(G)\) the binary relation \(\text{ord}[G, v_1, v_2, v_3, z_1, z_2]\) is a linear order of \(V(G)\).

**Proof.** Let \(G\) be a graph polyhedrally embedded in some surface, satisfying (i) and (ii) for all \(w_1, \ldots, w_4 \in V(G)\), and let \((v_1, v_2, v_3) \in \angle(G)\). We need to prove that conditions (i)--(iv) of the First Angle Lemma \[9.2.3\] are satisfied. Lemma \[9.2.3\](i) follows immediately from the definition of angles. Lemma \[9.2.3\](ii) follows from Lemma \[9.2.2\] and Lemma \[9.2.3\](iii) follows from Lemma \[9.2.1\](2). Lemma \[9.2.3\](iv) follows from \((v_1, v_2, v_3) \in \angle(G)\).

The following “local” version of the First Angle Lemma will be used in Chapter \[13\].

**Lemma 9.2.5 (Second Angle Lemma).** For all IFP-formulae \(\text{angle}(\overline{x}_1, y_1, y_2, y_3)\) and \(\text{align}(\overline{x}_2, y_1, y_2, y_3, y_4)\) there is an IFP-formula \(\text{ord}(\overline{x}, z_1, z_2)\) such that the following holds. Let \(G\) be a connected graph of minimum degree at least 2 and \(\overline{v}_1 \in G_{\overline{x}_1}, \overline{v}_2 \in G_{\overline{x}_2}\). Let

\[ \angle := \{ (w, v, w') \in V(G)^3 \mid w, w' \in N(v) \text{ and } w \neq w' \text{ and } G \models \text{angle}[\overline{v}_1, w, v, w'] \}. \]

For all \(v \in V(G)\), let

\[ C_v := \{ N(v), \{ww' \mid (w, v, w') \in \angle \} \}. \]

Let \(U\) be the set of all \(v \in V(G)\) such that:

\(\begin{align*}
(i) & \quad \text{for all } w, w' \in N(v), \text{ if } (w, v, w') \in \angle \text{ then } (w', v, w) \in \angle; \\
(ii) & \quad \text{if } \deg(v) = 2 \text{ then } C_v \cong K_2 \text{ and if } \deg(v) \geq 3 \text{ then } C_v \text{ is a cycle}; \\
(iii) & \quad \text{for all } w, w' \in N(v), \text{ if } (w, v, w') \in \angle \text{ then there is exactly one } w'' \in N(w') \text{ such that } (v, w', w'') \in \angle \text{ and } G \models \text{align}[\overline{v}_2, w, v, w', w''].
\end{align*}\)

Then for every connected component \(A\) of \(G[U]\) there is a tuple \(\overline{v} \in G_{\overline{x}}\) such that \(\text{ord}[G, \overline{v}, z_1, z_2]\) is a linear order of \(V(A) \cup N(A)\).
The proof is an easy modification of the previous one. Let \( G \) be a connected graph of minimum degree at least 2, \( \overline{v}_1 \in G^{\overline{P}1}, \overline{v}_2 \in G^{\overline{P}2} \), and let \( U \) be the set of all \( v \in V(G) \) satisfying (i)–(iii). Let \( A \) be a connected component of \( G[U] \), and let \( W := V(A) \). Let \( \angle_A \) be the set of all triples \((v_1, v_2, v_3) \in \angle \) such that \( v_2, v_3 \in V(A) \). Note that if \((v_1, v_2, v_3) \in \angle_A \) then \( v_1, v_2, v_3 \in W \). Further note that \( \angle_A \neq \emptyset \) (this follows from (ii)). For every \( \overline{v} = (v_1, v_2, v_3) \in \angle_A \), let \( f(\overline{v}) \) be the unique vertex \( v_4 \in N(v_3) \) such that \((v_2, v_3, v_4) \in \angle \) and \( G \models \text{align}(\overline{v}_1, v_1, v_2, v_3, v_4) \).

We shall define an increasing sequence \( W_1 \subseteq W_2 \subseteq \ldots \subseteq W_n = W \) of subsets of \( W \) and an increasing sequence \( \leq_1 \subseteq \leq_2 \subseteq \ldots \subseteq \leq_n \subseteq W^2 \) of binary relations such that for each \( i \in [n] \) the following two conditions are satisfied:

(A) \( \leq_i \) is a linear order of \( W_i \);

(B) for every \( w \in W_i \), there is a triple \( \overline{w} = (w_1, w_2, w_3) \in W_i^3 \cap \angle_A \) such that \( w \in \overline{w} \).

We choose an arbitrary triple \((v_1, v_2, v_3) \in \angle_A \) and let \( W_1 := \{v_1, v_2, v_3\} \). We define \( \leq_1 \) by \( v_1 \leq_1 v_2 \leq_1 v_3 \). For the inductive step, suppose that \( W_i \) and \( \leq_i \) are defined and that \( W_i \neq W \). Let \( V_i := W_i \cap V(A) \). As \( W_i \neq W \) and \( V(A) \) is connected, there is a \( v \in V_i \) such that \( N(v) \not\subseteq W_i \). Let \( v \in V_i \) be minimal with respect to the linear order \( \leq_i \) such that \( N(v) \not\subseteq W_i \).

Case 1: There are \( w, w' \in N(v) \cap W_i \) such that \((w, v, w') \in \angle_A \).

Let \( (w, w') \in W_i^2 \) be lexicographically minimal with respect to \( \leq_i \) such that \((w, v, w') \in \angle_A \). Since \( w, w' \in N(v) \cap W_i \) and \( N(v) \not\subseteq W_i \), the degree of \( v \) is at least 3. Thus by (ii), \( C_v \) is a cycle with vertex set \( N(v) \). Let \( w'' \) be the first vertex in \( N(v) \setminus W_i \) on the cyclic walk around \( C_v \) starting with \( w \) followed by \( w' \). We let \( W_{i+1} := W_i \cup \{w''\} \), and we let \( \leq_{i+1} \) be the extension of \( \leq_i \) with \( v' \leq_{i+1} w'' \) for all \( v' \in W_i \). Then (A) is obviously satisfied. Furthermore, let \( w''' \) be the predecessor of \( w'' \) on the cyclic walk around \( C_v \). Then \((w''', v, w'') \in W_{i+1}^3 \cap \angle_A \). Hence (B) is also satisfied.

Case 2: There are no \( w, w' \in N(v) \cap W_i \) such that \((w, v, w') \in \angle_A \).

Then by (B) and (i), there are \( w \in W_i \) and \( v' \in V_i \) such that \((w, v', v) \in \angle_A \). We choose such \((w, v', v) \) lexicographically minimal with respect to \( \leq_i \). Let \( w' := f(w, v', v) \).

Then \((w', v, w') \in \angle_A \), and thus \( w' \not\in W_i \) by the assumption of Case 2. We let \( W_{i+1} := W_i \cup \{w'\} \), and we let \( \leq_{i+1} \) be the extension of \( \leq_i \) with \( w'' \leq_{i+1} w' \) for all \( w'' \in W_i \). Then (A) and (B) are obviously satisfied.

Let \( n \in \mathbb{N} \) such that \( W_n = W \). Then \( \leq_n \) is a linear order of \( W \). It is easy to see that there is an IFP-formula \( \text{ord}(\pi_1, \pi_2, y_1, y_2, y_3, z_1, z_2) \), which uses the formulae \( \text{angle}(\pi_1, y_1, y_2) \) and \( \text{align}(\pi_2, y_1, y_2, y_3, y_4) \) as building blocks, and which does not depend on the specific graph \( G \), the component \( A \), or the triple \((v_1, v_2, v_3) \), such that

\[
\leq_n = \text{ord}(G, \overline{v}_1, \overline{v}_2, v_1, v_3, z_1, z_2).
\]

Our third angle lemma, which we will use in the proof of Theorem 9.4.1, is less abstract than the first two. It is a variant of Corollary 9.2.4 that applies in a situation where we cannot define all angles of an embedded graph by an IFP-formula, but only some of them.

M. Grohe, Definable Graph Structure Theory
Lemma 9.2.6 (Third Angle Lemma). For all IFP-formulae $\text{angle}(y_1, y_2, y_3)$, $\text{align}(y_1, y_2, y_3, y_4)$, and $\text{rest-ord}(\overline{v}, y_1, y_2)$ there is an IFP-formula $\text{ord}(\overline{v}, z_1, z_2)$ such that the following holds. Let $G$ be a graph polyhedrally embedded in some surface, $\overline{v} \in G^{\overline{v}}$ and $F_0 \subset F(G)$ a family of faces of $G$ such that:

(i) for all $v_1, v_2, v_3 \in V(G)$,

$$G \models \text{angle}[v_1, v_2, v_3] \iff (v_1, v_2, v_3) \in \angle(F_0) := \bigcup_{f \in F_0} \angle(f);$$

(ii) for all $v_1, v_2, v_3, v_4 \in V(G)$,

$$G \models \text{align}[v_1, \ldots, v_4] \iff (v_1, v_2, v_3, v_4) \in \angle(F_0) \text{ and } (v_1, v_2, v_3) \vee (v_2, v_3, v_4);$$

(iii) $\text{rest-ord}(G, \overline{v}, y_1, y_2)$ is a linear order of the set $W := \{ w \in V(G) \mid \text{there is an edge } e \in E(G) \text{ incident with } w \text{ such that there is at most one face } f \in F_0 \text{ incident with } e \}$.

Then $\text{ord}(G, \overline{v}, z_1, z_2)$ is a linear order of $V(G)$.

Note that this lemma only applies if $F_0$ is a strict subset of $F(G)$. If $F_0 = F(G)$, we can apply the Corollary to the First Angle Lemma to define a linear order.

Proof. Let $G$ be a graph polyhedrally embedded in some surface, $\overline{v} \in G^{\overline{v}}$, and $F_0 \subset F(G)$ such that (i)–(iii) are satisfied. Let $W$ be defined as in (iii) and $\leq_W := \text{rest-ord}(G, \overline{v}, y_1, y_2)$. By Fact 9.1.17 all facial subgraphs of $G$ are cycles. Hence $G$ satisfies the assumptions of Lemmas 9.2.1 and 9.2.2. For every angle $\overline{v} := (v_1, v_2, v_3) \in \angle(G)$ we let $f(\overline{v})$ be the unique vertex such that $\overline{v} \vdash (v_2, v_3, f(\overline{v}))$. Note that if $\overline{v} \in \angle(F_0)$ then $(v_2, v_3, f(\overline{v})) \in \angle(F_0)$. For every $v \in V(G)$, let $C_v$ be the cycle with vertex set $N(v)$ defined in Lemma 9.2.2.

It follows from by Fact 9.1.17(3) that $W$ is the union of the vertex sets of all facial cycles of $G$ that are not contained in $F_0$. Hence $W \neq \emptyset$, because $F_0 \subset F(G)$. For all $v \in V(G)$, we let

$$Q_v := (N(v), \{ wu' \mid (w, v, w') \in \angle(F_0) \}).$$

Then $Q_v \subseteq C_v$. As $C_v$ is a cycle, if $Q_v \neq C_v$ then $Q_v$ is a disjoint union of paths, possibly of length 0, with both endvertices in $W$. Furthermore, we have

$$Q_v = C_v \iff v \in V(G) \setminus W.$$

We define an increasing sequence $V_1 \subset V_2 \subset V_3 \subset \ldots \subset V_n = V(G)$ of nonempty subsets of $V(G)$ and an increasing sequence $\leq_1 \subset \leq_2 \subset \ldots \subset \leq_n \subseteq V(G)^2$ of binary relations such that for each $i \in [n]$

(A) $\leq_i$ is a linear order of $V_i$;

(B) for every $v \in V_i \setminus W$, there is an angle $\overline{w} \in V_i^3 \cap \angle(F_0)$ such that $v \in \overline{w}$. 

We let $V_1 := W$ and $\leq_1 := \leq_W$. For the inductive step, suppose that $V_i$ and $\leq_i$ are defined and that $V_i \neq V(G)$. Let $v \in V_i$ be minimal with respect to the linear order $\leq_i$ such that $N(v) \not\subseteq V_i$. 

Preliminary Version
Case 1: $v \notin W$ and there are $w, w' \in N(v) \cap V_i$ such that $(w, v, w') \in \angle(F_0)$.
Then $Q_v = C_v$ is a cycle, and we can argue exactly as in Case 1 of the proof of the Angle Lemma 9.2.3.

Case 2: $v \notin W$ and there are no $w, w' \in N(v) \cap V_i$ such that $(w, v, w') \in \angle(F_0)$.
In this case, we can argue as in Case 2 of the proof of the Angle Lemma 9.2.3.

Case 3: $v \in W$.
Then $Q_v \neq C_v$ is a disjoint union of paths with both endvertices in $W$. We can use the linear order of $W$ to define a linear order on $N(v) = V(Q_v)$. We first order the paths lexicographically by their endvertices, and then we order each path of positive length linearly from the smaller to the larger endvertex. Let $w$ be the first vertex in $N(v) \setminus V_i$ with respect to this linear order. We let $V_{i+1} := V_i \cup \{w\}$, and we let $\leq_{i+1}$ be the extension of $\leq_i$ with $w' \leq_{i+1} w$ for all $w' \in V_i$.

It is easy to see that the order $\leq_n$ is IFP-definable. 

9.3 Planar Graphs

In this section, we shall prove the following theorem:

**Theorem 9.3.1.** The class $Z_3 \cap P$ of 3-connected planar graphs graphs admits IFP-definable orders.

**Corollary 9.3.2.** The logic IFP captures PTIME on the class $Z_3 \cap P$.

**Proof.** This follows from Theorem 9.3.1 and the Immerman-Vardi Theorem (via Lemma 3.2.6).

The following corollary follows from Theorem 9.3.1 by Lemma 7.1.8 and the 3CC-Lifting Lemma (Corollary 8.3.3), for the latter recalling that the class planar graphs is closed under taking minors.

**Corollary 9.3.3 (Definable Structure Theorem for Planar Graphs).** The class $P$ of all planar graphs admits IFP-definable ordered treelike decompositions.

**Corollary 9.3.4.** The logic IFP+C captures PTIME on $P$.

With slightly more effort, we can also use Theorem 9.3.1 to prove that planarity is IFP-definable.

**Theorem 9.3.5.** The class $P$ is IFP-definable.

**Proof.** Let $\Lambda_{3cc}$ be the d-scheme obtained from the 3CC Decomposition Lemma 8.3.1. Then for every $G \in P$ the decomposition $\Lambda_{3cc}[G]$ is a treelike decomposition of $G$ over $P \cap Z_3^*$ of adhesion at most 2.

It follows from Theorem 9.3.1 that there is an IFP-formula $\text{ord}(\pi, y_1, y_2)$ that defines an order on every graph $G \in P \cap Z_3^*$. Let $Q \supseteq P \cap Z_3^*$ be the class of all graphs $G$ such that $\text{ord}$ defines an order on $G$. By Lemma 3.2.6 IFP captures PTIME on $Q$. Hence the polynomial time decidable class $P \cap Z_3^* \subseteq Q$ is IFP-definable.

M. Grohe, Definable Graph Structure Theory
Let $\mathcal{P}' \supseteq \mathcal{P}$ be the class of all graphs $G$ such that $\Lambda_{3cc}[G]$ is a treelike decomposition of $G$ of adhesion at most 2 over $\mathcal{P} \cap Z_{3}^{*}$. If follows from the Definability Lifting Lemma 5.4.3 that $\mathcal{P}'$ is IFP-definable.

**Claim 1.** $\mathcal{P} = \mathcal{P}'$.

**Proof.** We already know that $\mathcal{P} \subseteq \mathcal{P}'$. To prove the converse inclusion, let $G \in \mathcal{P}'$. Let $\Delta = (D, \sigma, \alpha) := \Lambda_{3cc}[G]$. By induction on $D$, starting from the leaves, we shall prove that for all $t \in V(D)$ the graph

$$H_t := G[\gamma(t)] \cup K[\sigma(t)]$$

is planar. As by (TL.5) for every connected component $A$ of $G$ there is a $t \in V(D)$ such $G[\gamma(t)] = A$, this will show that all connected components of $G$ are planar. Hence $G$ is planar as well.

For the base step, note that for all leaves $t$ we have $H_t = \tau(t) \in \mathcal{P} \cap Z_{3}^{*} \subseteq \mathcal{P}$. For the inductive step, let $t \in V(D)$, and let $u_1, \ldots, u_m \in N_{D}^{\Delta}(t)$ by a system of representatives of the $\Vert \Delta$ classes in $N_{D}^{\Delta}(t)$. That is, for each $u \in N_{D}^{\Delta}(t)$ there is exactly one $i \in [m]$ such that $u \Vert \Delta u_i$. Let $H^0 := \tau(t)$ and $H^i := H^{i-1} \cup H_{u_i}$ for all $i \in [m]$. Note that for all $i \in [m]$,

$$H^{i-1} \cap H_{u_i} = K[\sigma(u_i)],$$

because $\sigma(u_i)$ is a clique in both $\tau(t)$ and $H_{u_i}$ and $\alpha(u_i) \cap \alpha(u_j) = \emptyset$ for $j \neq i$ and $\alpha(u_i) \cap V(\tau(t)) = \alpha(u_i) \cap \beta(t) = \emptyset$. Recall that $|\sigma(u_i)| \leq 2$. As $H_0$ is planar and $H_{u_i}$ is planar for all $i \in [m]$ by the induction hypothesis, a straightforward induction based on Fact 9.1.26 shows that $H^i$ is planar for all $i \in [m]$. In particular, $H^m \supseteq H_t$ is planar.

We shall give two different proofs of Theorem 9.3.1. We will see in the next section that the theorem follows from the proof of Theorem 9.4.1 as a special case. In this section, we give a direct and arguably simpler proof. The idea of this proof is also important in the proof of the Definable Structure Theorem 17.2.1 or more specifically Lemma 13.3.3. The key to the proof is the following lemma.

**Lemma 9.3.6.** There are IFP-formulae planar-angle$(x_1, x_2, x_3)$ and planar-aligned$(x_1, x_2, x_3, x_4)$ such that for every 3-connected plane graph $G$ and all $v_1, v_2, v_3, v_4 \in V(G)$ we have:

$$G \models \text{planar-angle}[v_1, v_2, v_3] \iff (v_1, v_2, v_3) \in \angle(G),$$

$$G \models \text{planar-aligned}[v_1, v_2, v_3, v_4] \iff (v_1, v_2, v_3) \cap (v_2, v_3, v_4).$$

The Lemma will be proved in Subsection 9.3.1 below.

**Proof of Theorem 9.3.1.** The theorem follows from Lemma 9.3.6 by means of Corollary 9.2.4 to the First Angle Lemma.

### 9.3.1 Proof of Lemma 9.3.6

In this subsection, we make the following assumption.

**Assumption 9.3.7.** $G$ is a 3-connected plane graph and $v_1, v_2, v_3 \in V(G)$ such that $v_1, v_3 \in N(v_2)$ and $v_1 \neq v_3$. 

Preliminary Version
Figure 9.6. The graph from Examples 9.3.8 and 9.3.11

We let $C(v_1, v_2, v_3)$ be the set of all cycles $C \subseteq G$ with $v_1 v_2, v_2 v_3 \in E(C)$. Observe that $C(v_1, v_2, v_3)$ contains a facial cycle of $G$ if and only if $(v_1, v_2, v_3)$ is an angle of $G$. We will analyse the set $C(v_1, v_2, v_3)$ to find an IFP-definable criterion for deciding whether it contains a facial cycle. We will actually be able to define this cycle in IFP, and this will give us the desired formulae $\text{planar-angle}(x_1, x_2, x_3)$ and $\text{planar-aligned}(x_1, x_2, x_3, x_4)$.

For each cycle $C \in C(v_1, v_2, v_3)$, we define the inside of $C$ to be the set

$$I(C) := \{v_2\} \cup \{v \in V(G) \mid \text{each path from } v \text{ to } v_2 \text{ has a nonempty intersection with } C \setminus \{v_2\}\}.$$  

Note that $V(C) \subseteq I(C)$. Inclusion of the sets $I(C)$ defines a quasiorder on the set $C(v_1, v_2, v_3)$. We shall see that if $C(v_1, v_2, v_3)$ contains a facial cycle then this facial cycle is the unique minimal element of this quasiorder.

**Example 9.3.8.** Let $G$ be the 3-connected planar graph shown in Figure 9.6. Consider the following cycles from $C(5, 0, 10)$ (we specify the cycles as sequences of vertices):

- $C_1 := 5, 0, 10, 9, 8, 7, 6, 5$
- $C_2 := 5, 0, 10, 30, 29, 28, 27, 7, 6, 5$
- $C_3 := 5, 0, 10, 9, 8, 28, 27, 26, 25, 5$
- $C_4 := 5, 0, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 1, 2, 3, 4, 5$

We have

$$I(C_1) = \{0, 10, 9, 8, 7, 6, 5\},$$

M. Grohe, *Definable Graph Structure Theory*
Thus $I(C_1) \subseteq I(C_2) \subseteq I(C_4)$ and $I(C_1) \subseteq I(C_3) \subseteq I(C_4)$. The sets $I(C_2)$ and $I(C_3)$ are incomparable. 

**Lemma 9.3.9.** Let $C, C' \in \mathcal{C}(v_1, v_2, v_3)$ such that $V(C) \subseteq I(C')$. Then $I(C) \subseteq I(C')$.

**Proof.** Let $P$ be a path from a vertex $v \in I(C)$ to $v_2$. Then $P$ has a nonempty intersection with $V(C) \setminus \{v_2\}$. Let $w \in (V(C) \setminus \{v_2\}) \cap V(P)$. Then $w \in I(C')$, and hence the path $wPv_2 \subseteq P$ has a nonempty intersection with $V(C') \setminus \{v_2\}$. 

**Lemma 9.3.10.** Let $C \in \mathcal{C}(v_1, v_2, v_3)$ be a facial cycle. Then for all $C' \in \mathcal{C}(v_1, v_2, v_3)$ it holds that

$$I(C) \subseteq I(C').$$

**Proof.** Let $C' \in \mathcal{C}(v_1, v_2, v_3) \setminus \{C\}$. Let $P := C \setminus \{v_2\}$ and $P' := C' \setminus \{v_2\}$. By Lemma 9.3.9 it suffices to prove that $V(C) \subseteq I(C')$. Suppose that $u \in V(C) \setminus I(C')$. Then $u \notin V(C')$ and thus $u \notin V(P')$. Let $u_1$ be the last vertex of $P'$ contained in $v_1Pu$ (possibly $u_1 = v_1$), and let $u_2$ be the first vertex of $u_1P'v_3$ contained in $uPv_3$ (possibly $u_2 = v_3$); see Figure 9.7. Let $P_1 := u_1Pu_2$, $P_2$ the segment of $C$ from $u_1$ to $u_2$ that does not contain $u$, and $P_3 := u_1P'u_2$. Note that $P_1, P_2, P_3$ are three simple curves in the plane with endpoints $u_1$, $u_2$ and mutually disjoint interiors and that $P_1 \cup P_2 = C$ is the boundary of a face. To prove that $u \in I(C')$, we must prove that every path from $u$ to $v_2$ has a nonempty intersection with $V(P')$. Let $Q$ be a path from $u$ to $v_2$. Then $Q$ is a simple curve from an interior point of $P_1$ to an interior point of $P_2$. It has an empty intersection with the face bounded by $C = P_1 \cup P_2$. Hence by Fact 9.1.1 it has a nonempty intersection with $P_3 \subseteq C'$. This shows that $u \in I(C')$ and hence that $I(C) \subseteq I(C')$. 

\[\text{Figure 9.7. Proof of Lemma 9.3.10}\]
We define a sequence \((W_n)_{n \in \mathbb{N} \cup \{\infty\}}\) of subsets of \(V(G)\) by \(W_0 := \emptyset\),

\[
W_{n+1} := W_n \cup \{v \in N(W_n \cup \{v_2\}) \mid \exists C \in \mathcal{C}(v_1, v_2, v_3) : V(C) \cap (W_n \cup \{v\}) = \emptyset\}
\]

for \(n \in \mathbb{N}\) and

\[
W_\infty := \bigcup_{n \in \mathbb{N}^+} W_n.
\]

Observe that for every \(n \in \mathbb{N}^+ \cup \{\infty\}\) and every \(w \in W_n\) there is a path from \(v_2\) to \(w\) in \(G[W_n \cup \{v_2\}]\).

**Example 9.3.11.** Let \(G\) be the 3-connected planar graph shown in Figure 9.6. We first compute the sets \(W_n\) for the angle \((v_1, v_2, v_3) := (5, 0, 10)\):

- \(W_1 = \{15, 20\}\),
- \(W_2 = \{15, 20, 14, 16, 35, 19, 1, 40\}\),
- \(W_3 = \{15, 20, 14, 16, 35, 19, 1, 40, 13, 34, 17, 36, 18, 39, 2, 21\}\),
- \(\vdots\)
- \(W_6 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 1, 2, 3, 4, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 21, 22, 23, 24\}\)
- \(W_7 = W_6 \cup \{25, 30\}\)
- \(\vdots\)
- \(W_9 = V(G) \setminus \{0, 5, 6, 7, 8, 9, 10\}\)
- \(W_n = W_9\) for all \(n \geq 9\).

Thus \(W_\infty \setminus V(G)\) is the vertex set of the facial cycle determined by the angle \((5, 0, 10)\).

Let us also compute the sets \(W_n\) for the triple \((v_1, v_2, v_3) := (5, 0, 15)\), which is not an angle. We have

- \(W_1 = \{10, 20\}\)
- \(W_2 = \{10, 20, 9, 11, 30, 1, 19, 40\}\)
- \(W_n = W_2\) for all \(n \geq 2\).

**Lemma 9.3.12.** Let \(C \in \mathcal{C}(v_1, v_2, v_3)\) be a facial cycle. Then

\[
V(C) = V(G) \setminus W_\infty.
\]

**Proof.** To see that \(V(C) \subseteq V(G) \setminus W_\infty\), we prove that \(V(C) \cap W_n = \emptyset\) by induction on \(n\). This is trivial for \(n = 0\). For the induction step, suppose that \(V(C) \cap W_n = \emptyset\) and let \(v \in W_{n+1} \setminus W_n\). Then there is a cycle \(C' \in \mathcal{C}(v_1, v_2, v_3)\) such that \(V(C') \cap (W_n \cup \{v\}) = \emptyset\). Furthermore, there is a path from \(v_2\) to \(v\) in \(G[W_n \cup \{v_2, v\}]\). Hence \(v \notin I(C')\). By Lemma 9.3.10 we have \(V(C) \subseteq I(C) \subseteq I(C')\) and thus \(v \notin V(C)\). This shows that \(W_{i+1} \cap V(C) = W_i \cap V(C) = \emptyset\).

To see that \(V(C) \supseteq V(G) \setminus W_\infty\), let \(w \in V(G) \setminus W_\infty\). Suppose for contradiction that \(w \notin V(C)\). As \(G\) is 3-connected, \(\deg(v_2) \geq 3\). Hence there is a \(w' \in N(v_2) \setminus \{v_1, v_3\}\). As \(C\) is nonseparating (by Fact 9.1.22), there is a path \(P\) from \(w'\) to \(w\) in \(G \setminus C\). A straightforward induction on the length of \(P\) shows that \(V(P) \subseteq W_\infty\). Thus \(w \in W_\infty\), which is a contradiction. \(\square\)

M. Grohe, *Definable Graph Structure Theory*
Proof of Lemma [9.3.6] It is not hard to formalise the inductive definition of the set \( W_\infty \) in the logic \( \text{IFP} \). For once, let us carry out the details; the reader may safely skip them. Let
\[
\text{possible}(X, x_1, x_2, x_3, x) := \neg (X(x) \lor x = x_2) \land \text{ifp} \left( Y(y) \leftarrow \neg X(y) \land \neg y = x \land \neg y = x_2 \land (y = x_1 \lor \exists y' (Y(y') \land E(y', y))) \right)(x_3).
\]
Then for any set \( W \subseteq V(G) \) and any vertex \( v \in V(G) \) we have
\[
G \models \text{possible}[W, v_1, v_2, v_3, v] \iff v \notin W \cup \{v_2\} \text{ and there is a path from } v_1 \text{ to } v_3 \text{ in } G \setminus (W \cup \{v_2\}) \iff \text{there is a } C \in \mathcal{C}(v_1, v_2, v_3) \text{ with } V(C) \cap (W \cup \{v\}) = \emptyset.
\]
Let
\[
\text{next}(X, x_1, x_2, x_3, x) := \text{possible}(X, x_1, x_2, x_3, x) \land \exists x' \left( (X(x') \lor x' = x_2) \land E(x', x) \right).
\]
Then \( \text{next}[G, W, v_1, v_2, v_3, x] = W_{i+1} \) for all \( i \in \mathbb{N}^+ \). Now we let
\[
\text{w-infty}(x_1, x_2, x_3, x) := \text{ifp} \left( X(x) \leftarrow \text{next}(X, x_1, x_2, x_3, x) \right)(x),
\]
\[
\text{candidate}(x_1, x_2, x_3, x) := \neg \text{w-infty}(x_1, x_2, x_3, x).
\]
Then \( \text{w-infty}[G, v_1, v_2, v_3, x] = W_\infty \). By Lemma [9.3.12] if \( \mathcal{C}(v_1, v_2, v_3) \) contains a facial cycle \( C \) then
\[
V(C) = \text{candidate}[G, v_1, v_2, v_3, x]
\]
Next, we want to state that a set of vertices is the vertex set of a chordless and nonseparating cycle. We define a few auxiliary formulae:
\[
\text{deg}_2(X) := \forall x (X(x) \to \exists y_1 \exists y_2 (y_1 \neq y_2 \land X(y_1) \land X(y_2) \land E(x, y_1) \land E(x, y_2) \land \forall y' ((X(y') \land E(x, y')) \to (y' = y_1 \lor y' = y_2))))
\]
states that all vertices in \( X \) have exactly two neighbours in \( X \).
\[
\text{conn}(X) := \forall x \forall x' \left( (X(x) \land X(x')) \to \text{ifp} \left( Z(z) \leftarrow z = x \lor \exists z' (Z(z') \land E(z', z) \land X(z)) \right)(x') \right)
\]
states that \( X \) is connected.
\[
\text{Hence } \text{cl-cycle}(X) := \text{deg}_2(X) \land \text{conn}(X) \text{ states that } X \text{ is the vertex set of a chordless cycle.}
\]
\[
\text{non-sep}(X) := \text{conn}(\neg X), \text{ the formula obtained from } \text{conn}(X) \text{ by replacing } X \text{ by } \neg X \text{ everywhere, states that the complement of } X \text{ is connected and hence that } X \text{ is nonseparating.}
\]
\[
\text{Hence } \text{cl-ns-cycle}(X) := \text{cl-cycle}(X) \land \text{non-sep}(X) \text{ states that } X \text{ is the vertex set of a chordless and nonseparating cycle.}
\]
By Fact [9.1.22] for every set \( W \subseteq V(G) \) we have
\[
G \models \text{cl-ns-cycle}[W] \iff G[W] \text{ is a facial cycle.}
\]
Now let \( \text{planar-angle}(x_1, x_2, x_3) \) be the formula obtained from \( \text{cl-ns-cycle}(X) \) by replacing every atom of the form \( X(x'') \), for any variable \( x'' \), by the formula \( \text{candidate}(x_1, x_2, x_3, x'') \). (Note that none of the variables \( x_1, x_2, x_3 \) appears in the formula \( \text{cl-ns-cycle}(X) \). Hence there are no name clashes.) Then \( G \models \text{planar-angle}[v_1, v_2, v_3] \) if and only if \( C(v_1, v_2, v_3) \) contains a facial cycle. Recall that \( C(v_1, v_2, v_3) \) contains a facial cycle if and only if \( (v_1, v_2, v_3) \) is an angle.

Two angles \( (v_1, v_2, v_3) \) and \( (v_2, v_3, v_4) \) are aligned if and only if they determine the same facial cycle. We let \( \text{planar-aligned}(x_1, x_2, x_3, x_4) := \text{planar-angle}(x_1, x_2, x_3) \land \text{planar-angle}(x_2, x_3, x_4) \land \forall x (\text{candidate}(x_1, x_2, x_3, x) \leftrightarrow \text{candidate}(x_2, x_3, x_4, x)) \).

\[ \square \]

### 9.4 Graphs on Arbitrary Surfaces

In this section, we shall prove the following theorem:

**Theorem 9.4.1 (Definable Structure Theorem for Embeddable Graphs).** For every surface \( S \), the class \( \mathcal{E}_S \) of all graphs embeddable in \( S \) admits \( \text{IFP} \)-definable ordered treelike decompositions.

**Corollary 9.4.2.** For every surface \( S \), the logic \( \text{IFP}+\text{C} \) captures \( \text{PTIME} \) on the class \( \mathcal{E}_S \).

Recall that we proved the class of planar graphs to be \( \text{IFP} \)-definable. For all surfaces \( S \) except the sphere, it is an open question whether the class of all graphs embeddable in \( S \) is \( \text{IFP} \)-definable. As a consequence of the Definable Structure Theorem 9.4.1, at least we can prove \( \text{IFP}+\text{C} \)-definability.

**Corollary 9.4.3.** For every surface \( S \), the class \( \mathcal{E}_S \) is \( \text{IFP}+\text{C} \)-definable.

**Proof.** Let \( \Lambda \) be a d-scheme such that \( \mathcal{E}_S \subseteq \mathcal{O}_T^\Lambda \). By Lemma 7.4.2, \( \text{IFP}+\text{C} \) captures \( \text{PTIME} \) on \( \mathcal{O}_T^\Lambda \), and as \( \mathcal{E}_S \) is decidable in polynomial time by Fact 9.1.20, this implies that it is definable in \( \text{IFP}+\text{C} \). \[ \square \]

As every graph embeddable in a surface \( S \) is also embeddable in the surface \( \overline{S} \) without boundary obtained from \( S \) by gluing disks to all cuffs of \( S \), without of loss of generality we may restrict our attention to surfaces without boundary. Hence for the rest of this chapter, all surfaces are assumed to be without boundary.

Our proof of Theorem 9.4.1 is by induction on the Euler genus. We could take Theorem 9.3.1 for planar graphs as the induction basis, but actually we do not have to, because for genus 0 our construction will never use the induction hypothesis. We will see that this gives an alternative proof of Theorem 9.3.1 (see Remark 9.4.18).

Recall that for every \( g \in \mathbb{N} \), by \( \mathcal{E}_g \) we denote the class of all graphs of Euler genus at most \( g \), and by \( \mathcal{U}(\mathcal{E}_g) \) the class of all disjoint unions of graphs in \( \mathcal{E}_g \). We let \( \mathcal{E}_{-1} := \emptyset \). Then \( \mathcal{U}(\mathcal{E}_{-1}) = \emptyset \). The following lemma takes care of the inductive step of our proof:

**Lemma 9.4.4.** Let \( g \in \mathbb{N} \), and suppose that there is an od-scheme \( \Lambda^{g-1} \) that defines ordered treelike decompositions on \( \mathcal{U}(\mathcal{E}_{g-1}) \). Then for every surface \( S \) of Euler genus \( g \) there is an od-scheme \( \Lambda \) that defines ordered treelike decompositions on \( \mathcal{E}_S \).
Proof of Theorem 9.4.1. By induction on \( g \geq -1 \), we prove that \( \mathcal{U}(\mathcal{E}_g) \) admits definable ordered treelike decompositions. The induction basis \( g = -1 \) is trivial. For the inductive step, let \( g \geq 0 \). Up to homeomorphism, there are at most two surfaces, say, \( S \) and \( N \), of Euler genus \( g \). It follows from the induction hypothesis and Lemma 9.4.4 that there are od-schemes \( \Lambda_S \) and \( \Lambda_N \) such that \( \mathcal{Z}_3 \cap \mathcal{E}_S \subseteq \mathcal{OT}_{\Lambda_S} \) and \( \mathcal{Z}_3 \cap \mathcal{E}_N \subseteq \mathcal{OT}_{\Lambda_N} \). By the Union Lemma for Definable Ordered Decompositions 7.1.10, the class \( \mathcal{Z}_3 \cap \mathcal{E}_g = (\mathcal{Z}_3 \cap \mathcal{E}_S) \cup (\mathcal{Z}_3 \cap \mathcal{E}_N) \) admits definable ordered treelike decompositions. Then it follows from the 3CC Lifting Lemma (Corollary 8.3.3) that \( \mathcal{U}(\mathcal{E}_g) \) admits definable ordered treelike decompositions.

The rest of this section is devoted to a proof of Lemma 9.4.4. Until the end of the section, we make the following assumption:

**Assumption 9.4.5.** \( S \) is a surface without boundary and of Euler genus \( g \in \mathbb{N} \). Furthermore, \( \Lambda^{g-1} \) is an od-scheme such that \( D^{g-1} := \mathcal{OT}_{\Lambda^{g-1}} \supseteq \mathcal{U}(\mathcal{E}_{g-1}) \).

### 9.4.1 Defining the Faces

In addition to Assumption 9.4.5, throughout Subsection 9.4.1 we assume:

**Assumption 9.4.6.** \( G \) is a 3-connected graph in \( \mathcal{E}_g \setminus D_{g-1} \) that is polyhedrally embedded in \( S \).

It is our goal to define an od-scheme \( \Lambda^g \) that defines an ordered treelike decomposition of \( G \). The od-scheme \( \Lambda^g \) may depend on \( g \) and \( \Lambda^{g-1} \), but of course not on the specific graph \( G \). The proof strategy is as follows. We will iteratively define sets \( F_i \) of facial cycles of \( G \). Either we succeed to define all (or almost all) facial cycles of \( G \), then we can define an ordering on \( G \) using the Third Angle Lemma 9.2.6. Or on the way we find a cycle \( C \) such that \( G \setminus C \in D_{g-1} \). We call such a cycle a *reducing cycle*. Then we delete the cycle \( C \) and apply the od-scheme \( \Lambda^{g-1} \) to the resulting graph \( G \setminus C \). We apply the Ordered Extension Lemma 7.3.2 and combine a definable order on \( C \) with the ordered treelike decomposition of \( G \setminus C \) to an ordered treelike decomposition of \( G \).

We define \( F_i \subseteq F(G) \) by induction on \( i \in \mathbb{N} \). In addition, we define sets

\[
\begin{align*}
\text{IS}_i, \text{BS}_i, \text{XS}_i & \subseteq S, \\
\text{IV}_i, \text{BV}_i, \text{ XV}_i & \subseteq V(G), \\
\text{IE}_i, \text{BE}_i, \text{XE}_i & \subseteq E(G).
\end{align*}
\]

and a graph \( H_i \). Once we have defined \( F_i \), to define \( \text{IS}_i, \ldots, \text{XE}_i, H_i \), we look at the closed subset

\[
\bigcup_{f \in F_i} \text{cl}(f)
\]

of \( S \). We let \( \text{IS}_i, \text{BS}_i, \text{XS}_i \) be the interior, boundary, and “exterior” (that is, complement in \( S \)) of this set. We let \( \text{IV}_i, \text{BV}_i, \text{ XV}_i \) and \( \text{IE}_i, \text{BE}_i, \text{XE}_i \) be the vertices and edges of \( G \) in the respective sets \( \text{IS}_i, \text{BS}_i, \text{XS}_i \), and we let \( H_i \) be the subgraph in \( \text{BS}_i \cup \text{XS}_i \). The idea is that at stage \( i \) of our induction we know the faces of \( G \) in \( \text{IS}_i \), and thus it remains to determine the faces of the graph \( H_i \). The formal definitions are as follows.

(A) We let \( F_0 := \emptyset \).
In the following, suppose that $F_i$ is defined for some $i \geq 0$.

(B) We let $IS_i := \text{int}(\bigcup_{f \in F_i} \text{cl}(f))$, $BS_i := \text{bd}(IS_i)$, and $XS_i := S \setminus (IS_i \cup BS_i)$.

(C) We let $IV_i := V(G) \cap IS_i$, $BV_i := V(G) \cap BS_i$, and $XV_i := V(G) \cap XS_i$.

(D) We let $BE_i := \{e \in E(G) \mid e \subseteq BS_i\}$, $IE_i := \{e \in E(G) \mid e \subseteq IS_i \cup BS_i\}$, $BE_i$, and $XE_i := E(G) \setminus (IE_i \cup BE_i)$.

Note that $IE_i$ is the set of all edges of $G$ only incident to faces in $F_i$, $BE_i$ is the set of all edges of $G$ both incident to a face in $F_i$ and a face in $F(G) \setminus F_i$, and $XE_i$ is the set of all edges not incident to any faces in $F_i$.

(E) We let $H_i := (V(G) \setminus IV_i, E(G) \setminus IE_i)$.

Observe that $H_0 = G$. As a subgraph of $G$, the graph $H_i$ is embedded in $S$. The faces of $H_i$ are the faces of $G$ not in $F_i$ and the arcwise connected components of $IS_i$.

A cycle $C$ of $H_i$ is suitable (in $H_i$) if $C$ is chordless and nonseparating in $G$ and $V(C)$ contains at most 36 branch vertices of $H_i$. Recall that a cycle $C$ of $G$ is reducing if $G \setminus C \in \mathcal{D}_{g-1}$; otherwise it is nonreducing. Note that every noncontractible cycle is reducing, because $U(\mathcal{E}_{g-1}) \subseteq \mathcal{D}_{g-1}$. We can now complete the inductive definition of the sets $F_i$.

(F) We let

$$F_{i+1} := F_i \cup \{f \in F(G) \mid \text{Bd}(f) \subseteq H_i \text{ is suitable and nonreducing}\}$$

Note that a face whose boundary is a suitable facial cycle does not necessarily end up in $F_{i+1}$, because the boundary may still be a reducing cycle.

As $G$ is finite and $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$, there is an $m \in \mathbb{N}$ such that $F_i = F_m$ for all $i \geq m$. We let $F_\infty := F_m$. Furthermore, we let $IS_\infty := IS_m$ and define $BS_\infty, XS_\infty, IV_\infty, BV_\infty, XV_\infty, IE_\infty, BE_\infty, XE_\infty$, and $H_\infty$ similarly.

**Lemma 9.4.7.** For all $i \in \mathbb{N}$ the graph $H_i$ has minimum degree at least 2.

**Proof.** Let $v \in V(H_i) = BV_i \cup XV_i$. Then there is at least one face $f \in F(G) \setminus F_i$ incident with $v$. The two edges on the boundary cycle of $f$ that are incident with $v$ are both contained in $BE_i \cup XE_i = E(H_i)$. Hence $\deg_{H_i}(v) \geq 2$. \hfill \square

**Lemma 9.4.8.** Let $i \in \mathbb{N}$. If $H_i$ has more than $\max\{2, 84g\}$ branch vertices, then $H_i$ contains a suitable facial cycle $C$ such that $C = \text{bd}(f)$ for a face $f \in F(G) \setminus F_i$.

**Proof.** Suppose that $H_i$ has more than $\max\{2, 84g\}$ branch vertices. As $84g > 14g - 2$, by Lemma 9.1.12, the graph $H_i$ has a facial cycle with at most 6 branch vertices. However, the face of $H_i$ bounded by this cycle may be an arcwise connected component of $IS_i$. Let $\mathfrak{A}$ be the set of all arcwise connected components of $IS_i$. Let us call $A \in \mathfrak{A}$ small if its boundary contains at most 6 branch vertices of $H_i$, and let $\mathfrak{S} \subseteq \mathfrak{A}$ be the set of all small arcwise connected components of $IS_i$. We define a new graph $J$ as follows.

- $V(J)$ consists of all vertices in $XV_i$, all vertices in $BV_i$ that are contained in the boundary of an $A \in \mathfrak{A} \setminus \mathfrak{S}$, all vertices in $BV_i$ that are branch vertices of $H_i$ and contained in the boundary of an $A \in \mathfrak{S}$, and a new vertex $v_A$ for every $A \in \mathfrak{S}$.

M. Grohe, *Definable Graph Structure Theory*
• The edge set \( E(J) \) consists of all edges in \( XE_i \), all edges in \( BE_i \) that are contained in the boundary of an \( A \in \mathcal{A} \setminus \mathcal{S} \), and for every \( A \in \mathcal{S} \) new edges between \( v_A \) and all branch vertices of \( H_i \) contained in the boundary of \( A \).

We can embed \( J \) in \( S \) by placing the new vertices \( v_A \) and the interior of the edges incident with \( v_A \) in \( A \). This embedding is not unique, possibly not even up to homeomorphism. Nevertheless, we fix one such embedding and henceforth view \( J \) as a graph embedded in \( S \).

Claim 1. For every \( A \in \mathcal{S} \), the vertex \( v_A \) is a branch vertex of \( J \).

Proof. Let \( A \in \mathcal{S} \). Suppose for contradiction that \( \text{bd}(A) \) contains at most two branch vertices of \( H_i \). As \( H_i \) has at least 3 branch vertices (here we use the “2” in \( \max\{2, 84g\} \)), there is some vertex \( w \in V(H_i) \) such that \( w \notin \text{bd}(A) \). As \( V(H_i) \cap IS_i = \emptyset \) and \( A \subseteq IS_i \), we have \( w \notin \text{cl}(A) \). Since every face of \( G \) is incident to at least 3 vertices and \( A \) contains at least one face of \( G \), there must be a vertex \( w' \in V(G) \cap \text{cl}(A) \) that is not a branch vertex of \( H_i \). As \( G \) is 3-connected, there are three internally disjoint paths from \( w \) to \( w' \) in \( G \). All these paths must enter \( \text{cl}(A) \) in a branch vertex of \( H_i \) on \( \text{bd}(A) \). Hence \( \text{bd}(A) \) contains at least three branch vertices of \( H_i \), which is a contradiction.

Claim 2. \( J \) has more than \( 14g \) branch vertices.

Proof. Note first that all branch vertices of \( H_i \) that are not in the boundary of an \( A \in \mathcal{S} \) are also branch vertices of \( J \). Let \( A \in \mathcal{S} \). Then \( v_A \) is a branch vertex of \( J \) by Claim 1, and \( \text{bd}(A) \) contains at most 6 branch vertices of \( H_i \). Thus the number of branch vertices of \( H_i \) is at most 6 times the number of branch vertices of \( J \), and as \( H_i \) has more than \( 84g \) branch vertices, the claim follows.

Claim 3. For every face \( f \in F(J) \), every arcwise connected component of \( f \cap XS_i \) is in \( F(G) \cap F(H_i) \).

Proof. Let \( f \in F(J) \), and let \( f' \) be an arcwise connected component of \( f \cap XS_i \). Then \( V(G) \cap f' = \emptyset \), because \( V(G) \cap XS_i = XV_i = V(J) \cap XS_i \). For every edge \( e \in E(G) \) we have \( e \cap f' = \emptyset \), because if \( e \cap XS_i \neq \emptyset \) then \( e \in XE_i \subseteq E(J) \). Furthermore, \( \text{bd}(f') \subseteq \text{bd}(XS_i) \cup (\text{bd}(f) \cap XS_i) \subseteq H_i \). Hence \( f' \) is an arcwise connected component of \( S \setminus H_i \) with \( f' \cap G = \emptyset \) and hence a face of both \( H_i \) and \( G \).

By Lemma \[9.1.12\], \( J \) has a face whose boundary contains at most 6 branch vertices of \( J \). Let \( f \) be such a face, and let \( f' \) be an arcwise connected component of \( f \cap XS_i \). Such a component exists, because if \( f \subseteq IS_i \cup BS_i \), then \( f \in \mathcal{A} \setminus \mathcal{S} \). But this is a contradiction, because \( \text{bd}(f) \) contains only 6 branch vertices. By Claim 3, \( f' \) is a face of both \( H_i \) and \( G \). Let \( v \in \text{bd}(f') \) be a branch vertex of \( H_i \). Then \( v \in \text{bd}(f) \), because all branch vertices of \( H_i \) are vertices of \( J \). If \( v \) is not a branch vertex of \( J \), then \( v \in \text{bd}(A) \) for some \( A \in \mathcal{S} \) with \( v_A \in \text{bd}(f) \). As \( v_A \) is a branch vertex of \( J \), there are at most 6 such \( A \in \mathcal{S} \) with \( v_A \in \text{bd}(f) \). For each of them, there are at most 6 branch vertices of \( H_i \) in \( \text{bd}(A) \cap \text{bd}(f') \). Altogether, it follows that \( \text{bd}(f') \) contains at most 36 branch vertices.

By Fact \[9.1.17\], \( \text{bd}(f') \) is a chordless and nonseparating cycle of \( G \). Hence it is suitable.

\[ \square \]

**Corollary 9.4.9.** If the graph \( H_{\infty} \) contains no reducing cycle with at most 36 branch vertices, then \( H_{\infty} \) has at most \( \max\{2, 84g\} \) branch vertices.
Proof. Let \( m \in \mathbb{N} \) such that \( H_\infty = H_m \) and suppose that \( H_m \) contains no reducing cycle with at most 36 branch vertices, but has more than \( \max\{2, 84\} \) branch vertices. Then by Lemma 9.4.8, \( H_m \) has a suitable facial cycle \( C \) that bounds a face in \( F(G) \setminus F_m \). This cycle has at most 36 branch vertices and hence is nonreducing. Therefore, the face bounded by \( C \) is in \( F_{m+1} \setminus F_m \), which is a contradiction.

For every \( i \in \mathbb{N} \cup \{\infty\} \), let \( \mathcal{F}_i \) be the set of all boundary cycles of faces in \( F_i \). Observe that the sets \( IV_i, BV_i, XV_i, IE_i, BE_i, XE_i \) can be defined in terms of \( \mathcal{F}_i \) alone with no reference to the embedding of \( G \):

(G) \( IE_i \) is the set of all edges of \( G \) that appear in precisely two cycles in \( \mathcal{F}_i \);

(H) \( BE_i \) is the set of all edges of \( G \) that appear in precisely one cycle in \( \mathcal{F}_i \);

(I) \( XE_i \) is the set of all edges of \( G \) that appear in no cycle in \( \mathcal{F}_i \);

(J) \( BV_i \) is the set of all vertices incident to an edge in \( BE_i \);

(K) \( IV_i \) is the set of all vertices in \( V(G) \setminus BV_i \) that are incident to an edge in \( IE_i \);

(L) \( XV_i = V(G) \setminus (IV_i \cup BV_i) \).

Hence \( H_i \) can also be defined in terms of \( \mathcal{F}_i \) alone.

Lemma 9.4.10. For every \( i \in \mathbb{N} \), the set \( \mathcal{F}_{i+1} \) is the set of all cycles in \( H_i \) that are suitable and nonreducing.

Proof. By Lemma 9.1.15, all suitable and nonreducing cycles are facial cycles of \( G \).

Lemma 9.4.10 is of crucial importance. The sets \( \mathcal{F}_i \) where originally defined as sets of facial cycles of the embedded graph \( G \), and also \( IV_i, \ldots, XE_i \) and \( H_i \) were defined “topologically” by considering the region of \( S \) covered by these faces. Lemma 9.4.10 gives us a definition of the set \( \mathcal{F}_i \) and hence of \( IV_i, \ldots, XE_i \) and \( H_i \) only in terms of the abstract graph \( G \); this definition does not depend on the embedding of \( G \) in \( S \) in any way. Note that it is important here that we work with nonreducing cycles, which are defined in terms of the abstract graph \( G \), instead of noncontractible cycles, which are defined in terms of the embedding of \( G \) in \( S \).

Lemma 9.4.11. Every nonreducing cycle \( C \subseteq H_\infty \) contains at least 3 branch vertices of \( H_\infty \).

Proof. Suppose for contradiction that \( H_\infty \) contains a nonreducing cycle with less than 3 branch vertices of \( H_\infty \). Let \( C \) be such a cycle of minimum length.

Claim 1. Let \( e \in E(H_\infty) \) be an edge incident to at least one vertex that is not a branch vertex of \( H_\infty \). Then \( e \) is incident with one face in \( F_\infty \) and one face in \( F(G) \setminus F_\infty \).

Proof. Let \( v, v' \) be the endvertices of \( e \) and suppose that \( \deg_{H_\infty}(v) \leq 2 \). Let \( f_1, f_2 \) be the two faces of \( G \) incident with \( e \). If both \( f_1 \) and \( f_2 \) were contained in \( F_\infty \), then \( e \) would not be an edge of \( H_\infty \). Hence at least one of the faces, say \( f_1 \), is in \( F(G) \setminus F_\infty \). For \( i = 1, 2 \), let \( C_i \) be the boundary cycle of \( f_i \) and \( e_i \in E(C_i) \setminus \{e\} \) such that \( e_i \) is incident with \( v \). By Fact 9.1.17(2) and because \( e \in E(C_1) \cap E(C_2) \), we have \( e_1 \neq e_2 \). As \( v \) is not a branch vertex of \( H_\infty \), at least one of the edges \( e_1, e_2 \) is not an edge of \( H_\infty \). We have \( e_1 \in E(H_\infty) \), because \( f_1 \in F(G) \setminus F_\infty \). Hence \( e_2 \notin E(H_\infty) \) and thus \( f_2 \in F_\infty \).
Claim 2. \( C \) contains at least two branch vertices of \( H_\infty \).

**Proof.** Suppose for contradiction that \( C \) contains at most one branch vertex of \( H_\infty \). If \( C \) contains exactly one branch vertex, let \( v_0 \) be this branch vertex, and otherwise let \( v_0 \in V(C) \) be arbitrary. Let \( v_0 e_1 v_1 e_2 \ldots e_n v_n = v_0 \) be a simple closed walk around \( C \) starting and ending in \( v_0 \). The edge \( e_1 \) is contained in two faces \( f, f' \in F(G) \). By Claim 1, one of them, say \( f \), is contained in \( F_\infty \) and the other one, \( f' \), is not contained in \( F_\infty \). Let \( C' \) be the boundary cycle of \( f' \) and \( e_2 \) the edge on \( C' \) after \( e_1 \), that is, the unique edge in \( E(C') \setminus \{ e_1 \} \) incident with \( v_1 \). Then \( e_2 \in E(H_\infty) \), because it is incident with a face not in \( F_\infty \). As \( v_1 \) is not a branch vertex of \( H_\infty \), it follows that \( e_2 = e_2' \). Applying the same argument inductively, we see that \( C = C' \).

Hence \( C \) is a facial cycle and therefore chordless and nonseparating by Fact 9.1.17(1). We assumed that \( C \) contains only \( 1 \leq 36 \) branch vertex. Thus \( C \) is a suitable cycle. Furthermore, \( C \) is nonreducing. Thus \( C' = C \in F_\infty \) and \( f' \in F_\infty \). This is a contradiction.

Thus \( C \) contains exactly two branch vertices \( v \) and \( v' \). Our idea to derive a contradiction is as follows. \( C \) must be a separating simple closed curve of \( S \), because it is nonreducing and hence contractible. However, the only connection from one side of \( C \) to the other is through the branch vertices \( v, v' \). This contradicts the 3-connectedness of \( G \). We will carry out the argument in detail now.

Let \( P_1, P_2 \) be the two segments of \( C \) connecting \( v \) and \( v' \).

Claim 3. For \( i = 1, 2 \) there is a face \( f_i \in F(G) \setminus F_\infty \) such that \( P_i \subseteq \text{Bd}(f_i) \).

**Proof.** Let \( i \in \{2\} \). Let \( v_0 e_1 v_1 e_2 \ldots e_n v_n \), where \( v_0 = v \) and \( v_n = v' \), be a simple walk along \( P_i \). If \( n = 1 \), then \( P_i = e_1 \), and as an edge of \( H_\infty \), the edge \( e_1 \) is contained in the boundary of some face \( f \in F(G) \setminus F_\infty \). In the following, suppose that \( n \geq 2 \). Then \( v_1, \ldots, v_{n-1} \) are vertices of degree 2 in \( H_\infty \). Let \( f, f' \in F(G) \) be the two faces incident with \( e_1 \). By Claim 1, one of them, say \( f \), is contained in \( F_\infty \) and the other one, \( f' \), is not contained in \( F_\infty \). Let \( C' \) be the boundary cycle of \( f' \) and \( e_2 \) the edge on \( C' \) after \( e_1 \), that is, the unique edge in \( E(C') \setminus \{ e_1 \} \) incident with \( v_1 \). Then \( e_2 \in E(H_\infty) \). As \( v_1 \) is not a branch vertex of \( H_\infty \), it follows that \( e_2 = e_2' \). Applying the same argument inductively, we see that \( P_i \subseteq C' \), and we let \( f_i := f' \).
The following argument is illustrated by Figure 9.8. As $G$ is a simple graph, at least one of the paths $P_1, P_2$, say, $P_1$, is of length at least 2. Let $u_1 \in V(P_1) \setminus \{v, v'\}$. Choose faces $f_1, f_2$ according to Claim 3. For $i = 1, 2$, let $C_i$ be the boundary cycle of $f_i$, and let $Q_i = C_i \setminus (V(P_i) \setminus \{v, v'\})$ be the segment of $C_i$ that connects $v$ and $v'$ and is different from $P_i$.

**Claim 4.** The path $Q_1$ has length at least 2.

**Proof.** Suppose for contradiction that $Q_1$ has length 1, that is, consists of a single edge. Then $|Q_1| < |P_1|$. Let $C' := Q_1 \cup P_2$. Then $V(C') \subseteq V(C)$, and thus $C'$ is nonreducing. Furthermore, $|C'| < |C|$, which contradicts the minimality of $C$. \\

Let $w_1 \in V(Q_1) \setminus \{v, v'\}$. We can find a simple closed curve $c \in S$ that is homotopic to $C$ and contained in $f_1 \cup f_2 \cup \{v, v'\}$. (We have not explained the term “homotopic” yet. We will define it in Section 15.1.1 but the formal definition is not really needed here. For readers not familiar with homotopic curves, it will be sufficient to assume that $c$ is the boundary of a disk in $S$ if and only if $C$ is. We can find such a curve by following $C$ very closely in the interior of the faces $f_1$ and $f_2$.) As $C$ is a nonreducing cycle, the simple closed curve $C$ is contractible (that is, the boundary of a disk in $S$), and hence $c$ is contractible as well.

**Claim 5.** \{v, v'\} separates $u_1$ from $w_1$.

**Proof.** As $G$ is 2-cell embedded, $f_1$ is an open disk, and hence $d := cl(f_1) = f_1 \cup C_1$ is a closed disk. Let $p$ be the segment of $c$ from $v$ to $v'$ that is contained in $d$.

We first show that $d \setminus c = d \setminus p$ has two arcwise connected components $a, b$ such that $w_1 \in a$ and $u_1 \in b$. We can glue a disk on the boundary of the disk $d$ and obtain a sphere $s \supseteq d$. (This sphere $s$ has nothing to do with the surface $S$, it is just a “virtual” surface we create to be able to apply Fact 9.1.1.) Note that now we have three mutually disjoint simple curves $P_1, Q_1, p \subseteq s$ with the same endpoints $v, v'$. Hence we can apply Fact 9.1.1 (with $g_1 := P_1, g_2 := Q_1, g_3 := p$ and $x_1 := u_1, x_2 := w_1$). Note that the arcwise connected component of $s \setminus (P_1 \cup Q_1 \cup p)$ with boundary $P_1 \cup Q_1$ is precisely $s \setminus d$. Thus there is no simple curve $h \subseteq d$ from $u_1$ to $w_1$ that has an empty intersection with $p$.

As $c$ is contractible in $S$, it is separating, and thus $S \setminus c$ has two arcwise connected components $A, B$ with $bd(A) = bd(B) = c$. The components $a, b$ of $d \setminus c$ cannot be contained in the same component, because both $A$ and $B$ must contain points arbitrarily close to $c$ in $d$. Say, $a \subseteq A$ and $b \subseteq B$. Thus $u_1 \in A$ and $w_1 \in B$.

Now $P \subseteq G$ be a path from $u_1$ to $w_1$. Then $P$ is a simple curve in $S$ from $u_1$ to $w_1$, and we must have $P \cap c \neq \emptyset$. Since we have $c \cap G = \{v, v'\}$, it follows that $V(P) \cap \{v, v'\} \neq \emptyset$. Thus $\{v, v'\}$ separates $u_1$ from $w_1$.

Claim 5 contradicts $G$ being 3-connected. \\

The following consequence of Lemma 9.4.11 is what we need later.

**Corollary 9.4.12.** Suppose that $H_\infty$ contains no reducing cycle with less than 3 branch vertices. Then for all $v, w \in V(H_\infty)$ there is at most one isolated path from $v$ to $w$ in $H_\infty$.

**Proof.** Suppose for contradiction that there are vertices $v, w \in V(H_\infty)$ and two distinct isolated paths $P, Q \subseteq V(H_\infty)$ from $v$ to $w$. Then $P$ and $Q$ are internally disjoint, because otherwise they would not be isolated. Hence $C := P \cup Q$ is a cycle, and this cycle contains at most 2 branch vertices $(v$ and $w$). By the assumption of the corollary, $C$ is nonreducing. This contradicts Lemma 9.4.11.
For every $i \in \mathbb{N} \cup \{\infty\}$ we let
\[
\angle(F_i) := \bigcup_{f \in F_i} \angle(f).
\]

**Lemma 9.4.13.** There are $\text{IFP}$-formulae $\angle_g(x_1, x_2, x_3)$ and $\text{aligned}_g(x_1, x_2, x_3, x_4)$ such that for all vertices $v_1, v_2, v_3, v_4 \in V(G)$ we have
\[
G \models \angle_g[v_1, v_2, v_3] \iff (v_1, v_2, v_3) \in \angle(F_\infty),
\]
\[
G \models \text{aligned}_g[v_1, v_2, v_3, v_4] \iff (v_1, v_2, v_3), (v_2, v_3, v_4) \in \angle(F_\infty)
\]
and $(v_1, v_2, v_3) \not\sim (v_2, v_3, v_4)$.

Of course the formulae $\angle_g(x_1, x_2, x_3)$ and $\text{aligned}_g(x_1, x_2, x_3, x_4)$ of the lemma do not depend on the specific graph $G$, but just on $g$ and the $\text{od}$-scheme $\text{A}_g^{-1}$.

**Proof.** In the first step of the proof, we formalise the inductive step of an inductive definition of the set $\angle(F_{i+1})$ from $\angle(F_i)$.

**Claim 1.** Let $X$ be a ternary relation variable. There are $\text{IFP}$-formulae $\text{iv}(X, x)$, $\text{bv}(X, y)$, $\text{xv}(X, x)$, $\text{ie}(X, x_1, x_2)$, $\text{be}(X, x_1, x_2)$, $\text{xe}(X, x_1, x_2)$, $\text{vert}_H(X, x)$, $\text{edge}_H(X, x_1, x_2)$, $\text{ang}(X, x_1, x_2, x_3)$, and $\text{aln}(X, x_1, x_2, x_3, x_4)$ such that for all $i \in \mathbb{N}$ we have
\[
\begin{align*}
\text{iv}[G, \angle(F_i), x] &= IV_i, & \text{ie}[G, \angle(F_i), x_1, x_2] &= IE_i, \\
\text{bv}[G, \angle(F_i), x] &= BV_i, & \text{be}[G, \angle(F_i), x_1, x_2] &= BE_i, \\
\text{xv}[G, \angle(F_i), x] &= XV_i, & \text{xe}[G, \angle(F_i), x_1, x_2] &= BE_i, \\
\text{vert}_H[G, \angle(F_i), x] &= V(H_i), & \text{edge}_H[G, \angle(F_i), x_1, x_2] &= E(H_i),
\end{align*}
\]
and
\[
\begin{align*}
\text{ang}[G, \angle(F_i), x_1, x_2, x_3] &= \angle(F_{i+1}), \\
\text{aln}[G, \angle(F_i), x_1, x_2, x_3, x_4] &= \{(v_1, v_2, v_3, v_4) \mid (v_1, v_2, v_3), (v_2, v_3, v_4) \in \angle(F_{i+1}) \\
&\quad\quad\quad\quad\quad\quad\quad\text{and } (v_1, v_2, v_3) \not\sim (v_2, v_3, v_4)\}\).
\end{align*}
\]

**Proof.** By Lemma 9.2.1 for every angle $\overline{v}$ of $G$ there is exactly one face $f \in F(G)$ such that $\overline{v}$ is incident to $f$. Hence the angles in $\angle(F_i)$ determine the cycles in $F_i$, and we can use $(G)$–$(L)$ to define the formulae $\text{iv}, \text{bv}, \ldots, \text{xe}$. For example, we let
\[
\begin{align*}
\text{be}(X, x_1, x_2) &:= \exists x_3 \left(X(x_1, x_2, x_3) \land \forall x_3' \left(X(x_1, x_2, x_3') \to x_3 = x_3'\right)\right); \\
\text{bv}(X, x) &:= \exists x' \text{ be}(X, x, x').
\end{align*}
\]
Using these formulae, we can define the vertex set and the edge relation of the graph $H_i$ by
\[
\text{vert}_H(X, x) := \text{bv}(X, x) \lor \text{xv}(X, x),
\]
\[
\text{edge}_H(X, x_1, x_2) := \text{be}(X, x_1, x_2) \lor \text{xe}(X, x_1, x_2).
\]
To define the suitable cycles, we need further auxiliary formulae:

- a formula $\text{branch}(X, x)$ such that $\text{branch}[G, \angle(F_i), x]$ is the set of all branch vertices of $H_i$,
• a formula isopath\((X, x, x', y, y')\) such that for all \(v, v', w, w' \in V(G)\) we have \(G \models \text{isopath}[(\angle(F_i), v, v', w, w')]\) if and only if there is an isolated path \(P\) in \(H_i\) from \(v\) to \(v'\) such that \(ww' \in E(P)\),

• a formula isopath'(\(X, x, x', y_1, y_2)\) such that for all \(v, v', w_1, w_2, w'_2 \in V(G)\) we have \(G \models \text{isopath}'[(\angle(F_i), v, v', w_1, w_2, w'_2)]\) if and only if there is an isolated path \(P\) in \(H_i\) from \(v\) to \(v'\) such that \(w_1w'_1, w_2w'_2 \in E(P)\).

Using the Transduction Lemma (Fact 2.4.6) and the formulae \(\text{vert}H\) and \(\text{edge}H\), it is easy to define such a formulae in IFP.

We can specify a cycle \(C \subseteq H_i\) with \(\ell \geq 1\) branch vertices by fixing its \(\ell\) branch vertices, say, \(v_1, v_3, v_5, \ldots, v_{2\ell-1}\), and for every \(i \in [1, \ell]\) fixing the first vertex \(v_{2i}\) on the isolated path from \(v_{2i-1}\) to \(v_{2i+1}\), where we let \(v_{2\ell+1} := v_1\). If the path from \(v_{2i-1}\) to \(v_{2i+1}\) has length 1, we simply let \(v_{2i} = v_{2i+1}\). We call \(C\) the cycle specified by \(\overline{v} = (v_1, \ldots, v_{2\ell})\). We shall define an IFP-formula cycle\(_\ell\)\([X, \overline{v}, y, y']\) such that for all \(\overline{v} \in V(G)^{2\ell}\), the binary relation cycle\(_\ell\)\([G, \angle(F_i), \overline{v}, y, y']\) is the edge relation of the cycle specified by \(\overline{v}\) if \(\overline{v}\) specifies a cycle in \(H_i\), and cycle\(_\ell\)\([G, \angle(F_i), \overline{v}, y, y'] = \emptyset\) otherwise. We let

\[
\text{cycle}_\ell(X, x_1, x_2, \ldots, x_{2\ell}, y, y') := \bigwedge_{i=1}^{\ell-1} \text{branch}(X, x_{2i-1}) \wedge \bigwedge_{i=1}^{\ell-1} \text{isopath}(X, x_{2i-1}, x_{2i+1}, x_{2i-1}, x_{2i}) \wedge \text{isopath}(X, x_{2\ell-1}, x_1, x_{2\ell-1}, x_{2\ell}) \wedge \bigvee_{i=1}^{\ell-1} \text{isopath}'(X, x_{2i-1}, x_{2i+1}, x_{2i}, x_{2i}, y, y') \vee \text{isopath}'(X, x_{2\ell-1}, x_1, x_{2\ell-1}, x_{2\ell}, y, y').
\]

Next, we define formulae chordless\(_\ell\)\([X, \overline{v}]\) and \(\text{non-sep}_\ell(X, \overline{v})\) such that for all \(\overline{v} \in V(G)^{2\ell}\) specifying a cycle \(C\) in \(H_i\) we have

• \(G \models \text{chordless}_\ell[\angle(F_i), \overline{v}]\) if and only if the cycle \(C\) specified by \(\overline{v}\) is chordless in \(G\);

• \(G \models \text{non-sep}_\ell[\angle(F_i), \overline{v}]\) if and only if the cycle \(C\) specified by \(\overline{v}\) is non-separating in \(G\).

It is easy to define such IFP-formulae (cf. p.201 in the proof of Lemma 9.3.6). Note that the properties of the cycle refer to the graph \(G\) and not to \(H_i\).

Let \(\varphi_{g-1}\) be an IFP-sentence that defines the class \(D_{g-1}\). Such a sentence exists by Lemma 7.1.6 because \(D_{g-1} = OT_{A^g-1}\) (see Assumption 9.4.5). Using this sentence, we can define an IFP-formula non-red\(_\ell\)(\(X, \overline{v}\)) such that for all \(\overline{v} \in V(G)^{2\ell}\) specifying a cycle \(C\) in \(H_i\) we have \(G \models \text{non-red}_\ell[\angle(F_i), \overline{v}]\) if and only if the cycle \(C\) specified by \(\overline{v}\) is nonreducible in \(G\). We let

\[
\text{good-cycle}_\ell(X, \overline{v}) := \text{chordless}_\ell(X, \overline{v}) \wedge \text{non-sep}_\ell(X, \overline{v}) \wedge \text{non-red}_\ell(X, \overline{v})\]

Then the cycles specified by the \(2\ell\)-tuples \(\overline{v} \in \text{good-cycle}_\ell[\angle(F_i), \overline{v}]\) are precisely the suitable and nonreducing cycles with \(\ell\) branch vertices. Recall that \(F_{i+1}\) consists of theses cycles for
\[
\ell = 1, \ldots , 36. \text{ Now we let } \\
\text{ang}(X, x_1, x_2, x_3) := \bigvee_{\ell=1}^{36} \exists \bar{z}(\text{good-cycle}_\ell(X, \bar{z}) \land \text{cycle}_\ell(X, \bar{z}, x_1, x_2) \\
\land \text{cycle}_\ell(X, \bar{z}, x_2, x_3) \land x_1 \neq x_3).
\]

We define the formula \(\text{aln}(X, x_1, x_2, x_3, x_4)\) similarly.

It follows from Claim 1 that the formula

\[
\text{angle}_g(x_1, x_2, x_3) := \text{ifp}(X(x_1, x_2, x_3) \leftarrow \text{ang}(X, x_1, x_2, x_3)) (x_1, x_2, x_3)
\]
defines the set \(\angle(F_\infty)\). To complete the proof of the lemma, we let \(\text{aligned}_g(x_1, x_2, x_3, x_4)\) be the formula obtained from the formula \(\text{aln}(X, x_1, x_2, x_3, x_4)\) by replacing each atom of the form \(X(x_1, x_2, x_3)\) by the formula \(\text{angle}_g(x_1, x_2, x_3)\).

Note that the proof of Lemma 9.4.13 also yields the following corollary.

Corollary 9.4.14. There are \(\text{IFP}\)-formulae \(\text{vert}H(x)\) and \(\text{edge}H(x_1, x_2)\) such that

\[
H_\infty = (\text{vert}H[G, x], \text{edge}H[G, x_1, x_2]).
\]

Proof. In the proof of Lemma 9.4.13 we constructed \(\text{IFP}\)-formulae \(\text{vert}H(X, x)\) and \(\text{edge}H(X, x_1, x_2)\) with a free ternary relation variable \(X\) such that \(\text{vert}H[G, \angle(F_i), x] = V(H_i)\) and \(\text{edge}H[G, \angle(F_i), x_1, x_2] = E(H_i)\). Then \(\text{vert}H[G, \angle(F_\infty), x] = V(H_\infty)\) and \(\text{edge}H[G, \angle(F_\infty), x_1, x_2] = E(H_\infty)\). Thus if we replace each subformula of \(\text{vert}H(X, x)\) and \(\text{edge}H(X, x_1, x_2)\) of the form \(X(z_1, z_2, z_3)\) by \(\text{angle}_g(z_1, z_2, z_3)\), we obtain the desired formulae.

9.4.2 Proof of Lemma 9.4.4

In this subsection, we still make Assumption 9.4.5 (but no longer Assumption 9.4.6). We classify the graphs in \(Z_3 \cap E_S\) into finitely many types and prove the existence of definable ordered treelike decompositions for each type. Then we apply the Union Lemma for Definable Ordered Decompositions 7.1.10 to combine the decompositions.

A graph \(G \in Z_3 \cap E_S\) is of type I if there is a set \(W \subseteq V(G)\) such that \(|W| \leq 3\) and \(G \setminus W \in D_{g-1}\). Let \(\mathcal{Y}_I\) be the class of all graphs \(G\) of type I.

Lemma 9.4.15. The class \(\mathcal{Y}_I\) admits \(\text{IFP}\)-definable ordered tree decompositions.

Proof. This follows directly from Assumption 9.4.5 and the Finite Extension Lemma 7.3.1.

Let \(G \in (Z_3 \cap E_S) \setminus \mathcal{Y}_I\). As \(G \in E_S\), we may assume that that \(G\) is actually embedded in \(S\). Observe that the embedding is polyhedral. To see this, note that the representativity of the embedding is at least 3, because \(G \not\in \mathcal{Y}_I\). As \(G\) is 3-connected, this means that the embedding is polyhedral.

Thus \(G\) satisfies Assumption 9.4.6 and we can define the graph \(H_\infty\) as in the previous subsection. The graph \(G\) is of type II if there is a cycle \(C \subseteq H_\infty\) that contains at most 36 branch vertices of \(H_\infty\) and is a reducing cycle of \(G\). Let \(\mathcal{Y}_{II}\) be the class of all graphs of type II.
Lemma 9.4.16. The class $\mathcal{Y}_{II}$ admits lFP-definable ordered tree decompositions.

Proof. It follows from Corollary 9.4.14 and the definition of type II that there are formulae $\text{cycleV}(x)$, $\text{cycleE}(x, y)$ such that for every graph $G \in \mathcal{Y}_{II}$, the cycle $$(\text{cycleV}[G, x], \text{cycleE}[G, x, y])$$ is a reducing cycle with at most 36 branch vertices of $H_\infty$. Since the class of all cycles admits lFP-definable orders, it follows from the Ordered Extension Lemma 7.3.2 and Assumption 9.4.5 that $\mathcal{Y}_{II}$ admits lFP-definable ordered tree decompositions.

A graph $G \in Z_3 \cap \mathcal{E}_S$ is of type III if it is not of type I or II. We let $\mathcal{Y}_{III}$ be the class of all graphs of type III.

Lemma 9.4.17. The class $\mathcal{Y}_{III}$ admits lFP-definable orders.

Proof. Let $G \in \mathcal{Y}_{III}$. Without loss of generality we may assume that $G$ is polyhedrally embedded in $S$. We define $H_\infty$ as usually. We want to apply the First Angle Lemma (Corollary 9.2.4 to be precise) or the Third Angle Lemma 9.2.6 to the formulae $\text{angle}_g(\pi, y_1, y_2, y_3)$, $\text{align}_g(\pi, y_1, y_2, y_3, y_4)$ and the set $F_\infty$ of faces (as $F_0$ in the Third Angle Lemma). If $F_\infty = F(G)$, we can simply apply the Corollary 9.2.4 to define a linear order on $V(G)$. Suppose that $F_\infty \subset F(G)$. Then the set $W$ of the Third Angle Lemma is $V(H_\infty)$. Thus we have to define a formula $\text{rest-order}(\pi, y_1, y_2)$ such that for some tuple $\overline{\pi} \in G^\pi$, the binary relation $\text{rest-order}(G, \overline{\pi}, y_1, y_2)$ is a linear order of $V(H_\infty)$. By Corollary 9.4.9 the graph $H_\infty$ contains at most $k := \max\{2, 84g\}$ branch vertices, and by Corollary 9.4.12 there is at most one isolated path between any two branch vertices of $H_\infty$. Let $u_1, \ldots, u_{k'}$ for some $k' \leq k$ be an enumeration of all branch vertices of $H_\infty$, and let $$\overline{\pi} := (u_1, u_2, \ldots, u_{k'}, u_{k'}, \ldots, u_{k'})$$ where $u_{k'}$ appears $k-k'$ times.

Let $U := \overline{\pi}$, and let $\leq_U$ be the linear order on $U$ defined by $u_i \leq_U u_j \iff i \leq j$. Note that all vertices in $v \in W \setminus U$ appear on an isolated path with endvertices in $U$. Let $p(v) \leq_U q(v)$ be the two endvertices of this isolated path. Let $\leq$ be the linear order on $W$ defined as follows. For all $v, v' \in W$,

$$v \leq v' \iff \begin{cases} \quad v, v' \in U \text{ and } v \leq_U v', \\ \quad \text{or } v \in U \text{ and } v' \not\in U, \\ \quad \text{or } v, v' \not\in U \text{ and } p(v) <_U p(v'), \\ \quad \text{or } v, v' \not\in U \text{ and } p(v) = p(v') \text{ and } q(v) <_U q(v'), \\ \quad \text{or } v, v' \not\in U \text{ and } p(v) = p(v') \text{ and } q(v) = q(v'), \\ \quad \text{and } v \text{ appears before } v' \text{ on the isolated path from } p(v) \text{ to } q(v), \\ \quad \text{or } v = v'. \end{cases}$$

This defines a linear order on $W$, because there is at most one isolated path between any two branch vertices of $H_\infty$. It is straightforward to define this linear order in lFP. Hence an application of the Third Angle Lemma 9.2.6 completes the proof.

Proof of Lemma 9.4.4. As $(\mathcal{E}_S \cap Z_3) = \mathcal{Y}_I \cup \mathcal{Y}_{II} \cup \mathcal{Y}_{III}$, the lemma follows immediately from the previous three lemmas and the Union Lemma 7.1.10.

M. Grohe, Definable Graph Structure Theory
Remark 9.4.18. For \( g = 0 \), the classes \( \mathcal{Y}_I \) and \( \mathcal{Y}_{II} \) are empty. Hence Lemma 9.4.17 implies Theorem 9.3.1.
Part II

Definable Decompositions of Graphs with Excluded Minors
Chapter 10

Quasi-4-Connected Components

We have seen in the previous chapter that 3-connectedness is a very useful assumption when dealing with graphs embeddable in a surface. Definable treelike decompositions of graphs into their 3-connected components allow us to make this assumption. When dealing with general classes of graphs with excluded minors it would be even more useful to assume 4-connectedness, because 3-separators play a special role in the structure theory for such classes, and the structure is significantly simpler if no 3-separators are present. Unfortunately, there is no canonical decomposition of graphs into “4-connected components” (see, however, [18]). In this chapter we develop a substitute: a canonical decomposition of graphs into pieces that are quasi-4-connected, that is, only have irrelevant 3-separators, where a 3-separator is irrelevant if it only separates a single vertex from the rest of the graph. Examples of quasi-4-connected graphs are hexagonal grids (see Figure 10.1). We prove that all graphs have an IFP-definable treelike decomposition into torsos that become quasi-4-connected after contracting the edges of an IFP-definable matching. The statement of this result already indicates that the theory of quasi-4-connected components is fairly complicated and technical. But it will serve its purpose of simplifying applications of the structure theory for graphs with excluded minors in later chapters. Some of the results of this chapter have appeared, in a somewhat different form, in [50].

10.1 Hinges

For every subgraph $A$ of a graph $G$, let

$$\hat{A}^G := G[V(A) \cup N(A)] \cup K[N(A)].$$

The graphs $\hat{A}$ play an important role in our decomposition theory, because essentially the torsos of a decomposition can be written as intersections of such graphs. In general, the graphs $\hat{A}$ for $A \subseteq G$ are no subgraphs, not even minors of $G$. Only if $|N(A)| = 1$, the graph $\hat{A}$ is always a subgraph. More interestingly, if $G$ is 2-connected and $|N(A)| = 2$ then, by Lemma 8.2.2, $\hat{A}$ is a topological minor of $G$ that has a faithful image. This is what we used in the proof of the 3CC Decomposition Lemma 8.3.1 to guarantee that the 3-connected components of a graph are topological subgraphs. Unfortunately, for $k \geq 3$ it is no longer the case that if $G$ is $k$-connected and $|N(A)| \leq k$ the graph $\hat{A}$ is a minor of $G$. 
Chapter 10. Quasi-4-Connected Components

Example 10.1.1. Let $G$ be a hexagonal grid of radius at least 2 (see Figure 10.1 and Figure 14.1). Let $v$ be an arbitrary vertex of $G$ and $A := G \setminus \{v\} \cup N(v)$. Then $G$ is 3-connected, and $A$ is connected with $|N(A)| = |N(v)| = 3$. Yet $\hat{A}$ is not a minor of $G$. This follows from the 3-Hinge Lemma 10.1.5 below (also see Example 10.1.4).

Definition 10.1.2. A $k$-hinge of a graph $G$ is a $k$-separator $S$ of $G$ satisfying the following two conditions.

(i) There are at least two connected components $A$ of $G \setminus S$ with $N(A) = S$.

(ii) For every connected component $A$ of $G \setminus S$ with $N(A) = S$ it holds that $\hat{A} \preceq G$, and $\hat{A}$ has a faithful image in $G$.

Example 10.1.3. All 1-separators of a graph and all 2-separators of a 2-connected graph are hinges.

Example 10.1.4. A hexagonal grid of radius at least 2 (see Figure 10.1 and Example 10.1.1) is 3-connected, but not 4-connected. Its 3-separators are precisely the sets $N(v)$ for the vertices $v$. None of these 3-separators is a hinge.

10.1.1 3-Hinges in 3-Connected Graphs

In the following, we will only be interested in 3-hinges of 3-connected graphs. The following lemma gives a very useful characterisation of such hinges.

Lemma 10.1.5 (3-Hinge Lemma). Let $G$ be a 3-connected graph and $S$ a 3-separator of $G$. Then the following statements are equivalent:

(i) $S$ is a 3-hinge.

(ii) For every connected component $A$ of $G \setminus S$ it holds that $\hat{A} \preceq G$.

(iii) $S$ is not an independent set or $G \setminus S$ has more than two connected components or $G \setminus S$ has two connected components of order greater than one.

Note that in (ii) we are not requiring $\hat{A}$ to have a faithful image in $G$. 

M. Grohe, Definable Graph Structure Theory
10.1. Hinges

Proof of the 3-Hinge Lemma 10.1.5. Let $S = \{s_1, s_2, s_3\}$.

The implication (i) $\implies$ (ii) is trivial.

To prove (ii) $\implies$ (iii), suppose for contradiction that (ii) holds and (iii) does not hold. Then $S$ is an independent set and $G \setminus S$ has exactly two connected components $A, A'$, and one of these components has order 1. Say, $V(A') = \{v\}$. By (ii), the graph $\hat{A} = (G \setminus \{v\}) + \{s_1s_2, s_2s_3, s_1s_3\}$ is a minor of $G$. Note that $|A| = |G| - 1$ and $|E(\hat{A})| = |E(G)|$, because $S$ is an independent set. Let $(Y_w)_{w \in V(\hat{A})}$ be an image of $\hat{A}$ in $G$. If there is a $w \in V(\hat{A})$ such that $|Y_w| \geq 2$, then at least one edge appears in $G[Y_w]$, and there are not enough edges left such that for all $f = ww' \in E(\hat{A})$ there is an edge $e = vv' \in E(G)$ with $v \in Y_w$ and $v' \in Y_{w'}$. Thus $|Y_w| = 1$ for all $w \in V(\hat{A})$. But then there is a $v' \in V(G) \setminus \bigcup_{w \in V(\hat{A})} Y_w$, and as $G$ is connected, $v'$ is incident with some edge. Thus again there are not enough edges to witness that $(Y_w)_{w \in V(\hat{A})}$ is an image of $\hat{A}$. This is a contradiction.

It remains to prove (iii) $\implies$ (i). Suppose that (iii) holds. As $S$ is a 3-separator, $G \setminus S$ has at least two connected components, and as $G$ is 3-connected, for all connected components $A$ of $G \setminus S$ it holds that $N(A) = S$. Thus we only have to prove that for every connected component $A$ of $G \setminus S$ the graph $\hat{A}$ has a faithful image in $G$.

Let $A$ be a connected component of $G \setminus S$. Suppose first that $S$ is not an independent set. Say, $s_1s_2 \in E(G)$. Let $A' \neq A$ be another connected component of $G \setminus S$. We define a faithful image $(Y_e)_{e \in V(\hat{A})}$ of $\hat{A}$ in $G$ as follows. For all $v \in V(\hat{A}) \setminus \{s_3\}$, we let $Y_v := \{v\}$, and we let $Y_{s_3} := V(A') \cup \{s_3\}$.

Suppose next that $G \setminus S$ has more than two connected components, say, $A, A', A''$. Then $\hat{A}$ has a faithful image in $G$ as well, because by contracting $A''$ we can obtain an edge between two vertices in $S$, and then we can apply the same argument as above.

Finally, suppose that $G \setminus S$ has a connected component $A' \neq A$ with $|A'| \geq 2$. Let $w \in V(A')$. As $G$ is 3-connected, there are internally disjoint paths $P_i$, for $i \in [3]$, from $w$ to $s_i$. At least one of these paths, say, $P_1$, can be chosen to have length at least 2. To see this, suppose that $P_1, P_2, P_3$ have length 1. Let $w' \in N(A')(w)$; such a $w'$ exists because $|A'| \geq 2$. Then there is a path $Q$ from $w'$ to $S$ in $G \setminus \{w\}$. Without loss of generality we may assume that $s_1$ is an endvertex of $Q$. Then we can replace $P_1$ by the path $w'Qs_1$ of length greater than 1. In the following, we assume that $P_1$ has length at least 2. Then $V(P_1) \setminus \{s_1, w\} \neq \emptyset$. Let $Q$ be a path from $V(P_1) \setminus \{s_1, w\}$ to $V(P_2) \cup V(P_3) \setminus \{w\}$ in $G \setminus \{s_1, w\}$. Such a path exists because $G$ is 3-connected. Say, $w_1 \in V(P_1)$ and $w_2 \in V(P_2)$ are the endpoints of $Q$. By contracting the piece of $P_1$ from $w_1$ to $s_1$ and the piece $P_2$ from $w_2$ to $s_2$, and contracting $Q$ to a single edge, we obtain an edge between $s_1$ and $s_2$. Now we can argue as above to prove that $\hat{A}$ has a faithful image in $G$. □

An important consequence of the previous lemma is that 3-hinges of 3-connected graphs are definable in IFP.

Corollary 10.1.6. There is an IFP-formula $\text{hinge}(x_1, x_2, x_3)$ such that for every 3-connected graph $G$ and every tuple $(v_1, v_2, v_3) \in V(G)^3$ we have

$$G \models \text{hinge}(v_1, v_2, v_3) \iff \{v_1, v_2, v_3\} \text{ is a 3-hinge of } G.$$ 

Proof. Condition (iii) of the previous lemma is IFP-definable. □
10.1.2 3-Hinges Near a Clique

**Definition 10.1.7.** Let $G$ be a graph and $W \subseteq V(G)$. A set $X \subseteq V(G)$ is **inseparable** from $W$ if there is no $S \subseteq V(G)$ such that:

(i) $S$ separates $X$ from $W$ in $G$.

(ii) $|S| \leq |X|$.

(iii) $W \not\subseteq S$ and $X \not\subseteq S$.

If $G$ is a 3-connected graph and $W$ a nonseparating clique of order at least 3 in $G$, then by $\mathcal{H}(G,W)$ we denote the set of all 3-hinges of $G$ that are inseparable from $W$. Since $W$ is a clique, for each $S \in \mathcal{H}(G,W)$ there is a unique connected component $A$ of $G \setminus S$ with $V(A) \cap W \neq \emptyset$. We denote the vertex set of this component by $X(S)$, and we let $Y(S) := V(G) \setminus (S \cup X(S))$.

**Example 10.1.8.** Let $G$ be the graph in Figure 10.2. Let $W := \{w_0,w_1,w_2\}$. Let $S_1 := \{w_0,w_1,s_1\}$ and $S_2 := \{w_0,w_2,s_2\}$. Then $\mathcal{H}(G,W) = \{S_1,S_2\}$. Moreover, we have $X(S_1) = \{w_2\}$, $Y(S_1) = \{s_0,s_2\}$, $X(S_2) = \{w_1\}$, $Y(S_2) = \{s_0,s_1\}$.

The following lemma shows that to establish that a 3-hinge $S$ is contained in $\mathcal{H}(G,W)$ it suffices to prove that there is no 3-hinge separating $S$ from $W$.

**Lemma 10.1.9.** Let $G$ be a 3-connected graph and $W \subseteq V(G)$ a clique of order $|W| \geq 3$ in $G$. Let $S$ be a 3-hinge of $G$ and $S'$ a 3-separator that separates $S$ from $W$. Then $S'$ is a 3-hinge.

**Proof.** Suppose for contradiction that $S'$ is not a hinge. Then $S \neq S'$. Let $X$ be the vertex set of the connected component of $G \setminus S'$ that contains $W \setminus S'$. Such a component exists because $W$ is a clique. By the 3-Hinge Lemma 10.1.5, $S'$ is an independent set. Thus we have $|W \cap S'| \leq 1$ and $|X| \geq 2$. Let $Y := V(G) \setminus (X \cup S')$. Then by the 3-Hinge Lemma 10.1.5 we have $|Y| = 1$. Moreover, $S \subseteq S' \cup Y$, because $S'$ separates $S$ from $W$. As $S \neq S'$, this implies $|S \cap Y| = |Y| = 1$ and $|S \cap S'| = 2$. Let $s$ be the unique vertex in $Y = S \cap Y$. Let $A, A'$ be distinct connected components of $G \setminus S$ with $N(A) = N(A') = S$. Let $a \in N(s) \cap V(A)$ and $a' \in N(s) \cap V(A')$. Then $a, a' \in Y \cup S'$. It follows that $|Y \cup S'| \geq 5$ and thus $|S'| \geq 4$, which is a contradiction.

**Definition 10.1.10.** Let $G$ be a graph and $W \subseteq V(G)$. A **split extension** of $W$ in $G$ is a vertex $x \in V(G) \setminus W$ such that for every connected component $A$ of $G \setminus (W \cup \{x\})$ it holds that $|N(A)| \leq |W|$.

Note that if $W$ is a 3-clique in a 3-connected graph $G$ and $x$ is a split extension of $W$, then for every connected component $A$ of $G \setminus (W \cup \{x\})$ the set $N(A)$ is a 3-hinge, because it contains at least two vertices of the clique $W$ and thus is not an independent set. By the 3-Hinge Lemma 10.1.5, this implies that it is a hinge.

**Example 10.1.11.** Let $G$ be the graph in Figure 10.2. Let $W := \{w_0,w_1,w_2\}$. Then $s_0, s_1, s_2$ are split extensions of $W$.  

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M. Grohe, *Definable Graph Structure Theory*
10.1. Hinges

Lemma 10.1.12. Let $G$ be a 3-connected graph and $W$ a nonseparating 3-clique in $G$ that has no split extension. Let $S_1, S_2 \in \mathcal{S}(G,W)$ with $S_1 \neq S_2$. For $i = 1, 2$, let $X_i := X(S_i)$ and $Y_i := Y(S_i)$.

Then $Y_1 \cap Y_2 = \emptyset$, and one of the following two conditions is satisfied:

(i) $Y_1 \cap S_2 = S_1 \cap Y_2 = \emptyset$;

(ii) $S_1 \cap S_2 = \emptyset$ and $|Y_1 \cap S_2| = |S_1 \cap Y_2| = 1$, and for the unique vertices $s_1 \in S_1 \cap Y_2$ and $s_2 \in Y_1 \cap S_2$ it holds that $s_1 s_2 \in E(G)$. Furthermore, for $i = 1, 2$ either $Y_i \cap X_{3-i} \neq \emptyset$ or there is an edge between the two vertices in $S_i \setminus \{s_i\}$.

The following example shows that the lemma is false without the assumption that there be no split extensions of $W$.

Example 10.1.13. Let $G$ be the graph of Figure 10.2 (again) and $W := \{w_0, w_1, w_2\}$ (cf. Examples 10.1.8 and 10.1.11). Then for the hinges $S_1 := \{w_0, w_1, s_1\}$ and $S_2 := \{w_0, w_2, s_2\}$ in $\mathcal{S}(G,W)$ we have $Y(S_1) \cap Y(S_2) = \{s_0\} \neq \emptyset$.

The following example shows that case (ii) of the lemma is possible.

Example 10.1.14. Let $G$ be the graph in Figure 10.3. Let $W := \{w_0, w_1, w_2\}$. Observe that $W$ has no split extension in $G$, because for every vertex $v \in V(G) \setminus W$ the graph $A := G \setminus (W \cup \{v\})$ is connected, and it holds that $N(A) = W \cup \{v\}$. 

Preliminary Version
Let \( S_1 := \{ w_1, s'_1, s_1 \} \) and \( S_2 := \{ w_2, s'_2, s_2 \} \). Then \( S_1, S_2 \in \mathcal{S}(G, W) \), and we have \( X(S_1) = \{ w_0, w_2, s'_2 \} \), \( Y_1 := Y(S_1) = \{ s_2 \} \), \( X(S_2) = \{ w_0, w_1, s'_1 \} \), \( Y_2 := Y(S_2) = \{ s_1 \} \). Then we have \( Y_1 \cap Y_2 = \emptyset \), \( Y_1 \cap S_2 = \{ s_2 \} \), \( Y_2 \cap S_1 = \{ s_1 \} \). Moreover, \( s_1 s_2, w_1 s'_1, w_2 s'_2 \in E(G) \).

**Proof.** Claim 1.

\[
|S_{1X}| \geq 1, \quad |S_{2X}| \geq 1. \tag{10.1.1}
\]

**Case 1:** \( W \subseteq S_1 \cup S_2 \).

Then either \( |W \cap S_1| \geq 2 \) or \( |W \cap S_2| \geq 2 \). Without loss of generality we assume that \( |W \cap S_1| \geq 2 \). Figures 10.4(b) and (c) show the only two configurations that are consistent with all the constraints we have derived or imposed so far.

I claim that in both configurations the unique vertex in \( S_{1Y} \) is a split extension of \( W \). To see this, let \( A \) be a connected component of \( G \setminus (W \cup S_{1Y}) \). Suppose for contradiction that \( N(A) = W \cup S_{1Y} \). Then there is a path \( P \) with internal vertices in \( A \) from \( S_{1X} \) to \( S_{1Y} \). This path must intersect \( S_2 \), which is only possible in \( S_{2Y} \). Hence \( S_{2Y} \cap V(A) \neq \emptyset \). But then there is a path from \( S_{2Y} \) to \( S_{2X} \) which does not intersect \( S_1 \), which is impossible. Thus \( N(A) \subseteq W \cup S_{1Y} \). This proves that the unique vertex in \( S_{1Y} \) is a split extension of \( W \), which contradicts our assumption that \( W \) have no split extension.

**Case 2:** \( W \not\subseteq S_1 \cup S_2 \).

\( S := S_+ \cup S_{1X} \cup S_{2X} \) separates \( S_1 \) from \( W \), and in this case we have \( W \not\subseteq S \) and

**Figure 10.4.**
$S_1 \not\subseteq S$. By \eqref{10.1.2} and \eqref{10.1.3} we have $|S| = 3$. This contradicts $S_1 \in \mathcal{S}(G,W)$ being inseparable from $W$. \hfill \square

In the following, let us assume that (i) does not hold. Without loss of generality we assume that $Y_2 \cap S_1 \neq \emptyset$. We shall prove (ii). Let $s_1 \in Y_2 \cap S_1$. As $G$ is 3-connected, we have $N(Y_1) = S_1$, and thus there is an $s_2 \in Y_1$ such that $s_1s_2 \in E(G)$. Since $s_1 \in Y_2$ we have $s_2 \in S_2 \cup Y_2$, and as we have already established $Y_1 \cap Y_2 = \emptyset$, it follows that $s_2 \in S_2$.

**Claim 2.** $|S_1| + |S_2| + |S_3| \geq 4$.

**Proof.** Suppose for contradiction that $|S_1| + |S_2| + |S_3| < 4$. Let $S := S_1 \cup S_2 \cup S_3$. If $W \subseteq S$, then we argue as in Case 1 of the proof of Claim 1 to show that $s_1$ is a split extension of $W$. Otherwise, $S$ separates $W$ from $S_1$. In both cases, we have a contradiction. \hfill \square

It is easy to see that the (in)equalities $|S_1| \geq 1$, $|S_2| \geq 1$, $|S_3| \geq 4$, $|S_1| \geq 1$, $|S_2| \geq 1$, $|S_3| \geq 3$, $|S_2| + |S_3| \geq 4$, $|S_1| + |S_2| + |S_3| \geq 4$ have the unique solution $|S_1| = |S_2| = 2$, $|S_3| = 4$, $|S_1| = |S_2| = 1$.

It remains to prove that for $i = 1,2$ either $Y_i \cap X_{3-i} \neq \emptyset$ or there is an edge between the two vertices in $S_i \setminus \{s_i\}$. Consider the case $i = 1$. Let $S_1 = \{s_1, s_1', s_1''\}$. As $S_1$ is a 3-hinge, by the 3-Hinge Lemma \ref{10.1.5} either $|Y_1| \geq 2$ or $S_1$ is not an independent set. If $|Y_1| \geq 2$, then $Y_1 \cap X_2 \neq \emptyset$, because $Y_1 \cap Y_2 = \emptyset$ and $Y_1 \cap S_2 = 1$. If $S_1$ is not an independent set, then $s_1's_1'' \in E(G)$, because $S_2$ separates $s_1$ from $\{s_1', s_1''\}$ and thus $s_1's_1', s_1's_1'' \notin E(G)$. \hspace{1cm} \square

**Remark 10.1.15.** It is worth noting that all assertions of Lemma \ref{10.1.12} except the last sentence of (ii) ("Furthermore, for $i = 1,2$ either $Y_i \cap X_{3-i} \neq \emptyset$ or there is an edge between the two vertices in $S_i \setminus \{s_i\}$") also hold if $S_1$ and $S_2$ are merely 3-separators that are inseparable from $W$ and not 3-hinges. However, the additional assertion for 3-hinges will be crucial in the next subsection (in the proof of the Crossedge Independence Lemma \ref{10.1.18}). In the terminology of the next subsection, the crossedges of the 3-separators that are inseparable from $W$ do not necessarily form a 3-matching. \hfill \square

### 10.1.3 Crossedges

Throughout this subsection, we make the following assumptions:

**Assumption 10.1.16.** $G$ is a 3-connected graph and $W$ a nonseparating 3-clique in $G$ that has no split extension.

If two 3-hinges $S_1, S_2 \in \mathcal{S}(G,W)$ satisfy Lemma \ref{10.1.12} (iii), then we say that they cross. Let $S_1, S_2 \in \mathcal{S}(G,W)$ be two crossing 3-hinges. Then by Lemma \ref{10.1.12} (ii), $S_1 \cap S_2 = \emptyset$ and $|Y(S_1) \cap S_2| = |S_1 \cap Y(S_2)| = 1$. For $i = 1,2$, let $s_i$ be the unique vertex in $S_i \cap Y(S_{3-i})$. Then $s_1s_2 \in E(G)$. We call the edge $s_1s_2$ the crossedge of $S_1, S_2$. We let $E_x(G,W)$ be the set of all crossedges of pairs of crossing 3-hinges in $\mathcal{S}(G,W)$.

**Lemma 10.1.17.** Let $e = s_1s_2 \in E_x(G,W)$. Then $N^G(s_1) \cap N^G(s_2) = \emptyset$.

**Proof.** Let $e$ be the crossedge of $S_1, S_2 \in \mathcal{S}(G,W)$. For $i = 1,2$, let $Y_i := Y(S_i)$ and suppose that $s_i \in S_i$. Then by Lemma \ref{10.1.12} (ii) we have $S_1 \cap S_2 = \emptyset$, and $Y_1 \cap Y_2 = \emptyset$, and $S_i \cap Y_{3-i} = \{s_i\}$. From $s_i \in Y_{3-i}$ it follows that $N(s_i) \subseteq Y_{3-i} \cup S_{3-i}$. Thus

$$N(s_1) \cap N(s_2) \subseteq ((S_2 \cup Y_2) \cap (S_1 \cup Y_1)) \setminus \{s_1, s_2\} = \emptyset.$$ \hspace{1cm} \square

Preliminary Version
The following crucial lemma implies that the crossedges form a matching. This will allow us to deal with them more or less independently.

**Lemma 10.1.18 (Crossedge Independence Lemma).** Let $S_1, S_2, S'_2 \in \mathcal{S}(G, W)$ be pairwise distinct. Suppose that both $S_1$, $S_2$ and $S_1, S'_2$ cross, and let $e, e'$ be the respective crossedges. Then $e \cap e' = \emptyset$.

**Proof.** Let $e = s_1s_2$ and $e' = s'_1s'_2$ with $s_1, s'_1 \in S_1$. Let $X_1 := X(S_1)$, $Y_1 := Y(S_1)$, $X_2 := X(S_2)$, $Y_2 := Y(S_2)$, and $Y'_2 := Y(S'_2)$. By Lemma 10.1.12 we have

$$Y_1 \cap Y_2 = Y_1 \cap Y'_2 = Y_2 \cap Y'_2 = \emptyset$$  \hspace{1cm}  (10.4)
$$S_1 \cap Y_2 = \{s_1\} \text{ and } S_1 \cap Y'_2 = \{s'_1\}$$  \hspace{1cm}  (10.5)
$$S_2 \cap Y_1 = \{s_2\} \text{ and } S_2 \cap Y'_1 = \{s'_2\}$$  \hspace{1cm}  (10.6)
$$S_1 \cap S_2 = S_1 \cap S'_2 = \emptyset.$$  \hspace{1cm}  (10.7)

Observe that $s_1 \neq s'_1$, because $s_1 \in Y_2$ and $s'_1 \in Y'_2$ and $Y_2 \cap Y'_2 = \emptyset$ by (10.4). Furthermore, $s_1 \neq s'_2$, because $s_1 \in S_2$ and $s'_2 \in Y_1$ an $S_2 \cap Y_2 = \emptyset$, and symmetrically $s'_1 \neq s_2$. Hence to prove that $e \cap e' = \emptyset$, it only remains to prove that

$$s_2 \neq s'_2.$$  \hspace{1cm}  (10.8)

Suppose for contradiction that $s_2 = s'_2$. Then $S_2 \cap S'_2 \neq \emptyset$, and hence by Lemma 10.1.12 the 3-hinges $S_2, S'_2$ are not crossing. Thus $S_2 \cap Y'_2 = \emptyset$ and $Y_2 \cap S'_2 = \emptyset$.

We have $s_1 \in S_1$ and hence $s_1 \notin S_2 \cup S'_2$ by (10.7). As $s_1 \in Y_2$ by (10.5), we have $s_1 \in Y_2 \setminus (S_2 \cup S'_2)$. By symmetry, $s'_1 \in Y'_2 \setminus (S_2 \cup S'_2)$. It follows that $s_1s'_1 \notin E(G)$, because $Y_2$ and $Y'_2$ are disjoint and $N^G(Y_2) \subseteq S_2 \cup S'_2$ and $N^G(Y'_2) \subseteq S_2 \cup S'_2$. Let $s''_1$ be the unique element in $S_1 \setminus \{s_1, s'_1\}$. Then by (10.5) and (10.7) we have $s''_1 \in X_2 \cap X'_2$. As $S_2$ separates $X_2$ from $Y_2$, we have $s''_1s_1 \notin E(G)$, and symmetrically $s''_1s'_1 \notin E(G)$. Hence $S_1$ is an independent set.

As $S_1$ is a 3-hinge, it follows from the 3-Hinge Lemma 10.1.5 that $|Y_1| \geq 2$. We have $Y_1 \cap Y_2 = \emptyset$ and $Y_1 \cap S_2 = \{s_2\}$. Hence $Y_1 \cap X_2 = \emptyset$. Let $A$ be a connected component of $G[Y_1 \cap X_2]$. As $G$ is 3-connected, we have $|N(A)| \geq 3$, and as $N(A) \subseteq N(Y_1 \cap X_2) \subseteq (S_1 \cap X_2) \cup (S_2 \cap X_2) \cup (Y_1 \cap S_2) = \{s'_1, s'_1, s_2\}$, it follows that $N(A) = \{s'_1, s'_1, s_2\}$. As $S'_2$ separates $s''_1 \in X'_2$ from $s'_1 \in Y'_2$, it holds that $V(A) \cap S'_2 \neq \emptyset$. As $V(A) \subseteq Y_1$ and $Y_1 \cap S'_2 = \{s'_2\}$, we have $s'_2 \in V(A)$. Thus $s'_2 \neq s_2 \in N(A)$, which contradicts our assumption $s_2 = s'_2$ and completes the proof of (10.8).

**Corollary 10.1.19.** The edges in $E_\times(G, W)$ are mutually disjoint.

**Proof.** Let $e, e' \in E_\times(G, W)$. Let $S_1, S_2 \in \mathcal{S}(G, W)$ be crossing 3-hinges and $s_1 \in S_1, s_2 \in S_2$ such that $e = s_1s_2$. Similarly, let $S'_1, S'_2 \in \mathcal{S}(G, W)$ be crossing 3-hinges and $s'_1 \in S'_1, s'_2 \in S'_2$ such that $e' = s'_1s'_2$. Suppose for contradiction that $s_1 = s'_1$. Then $S_1 \cap Y(s'_2) \supseteq \{s_1\} \neq \emptyset$. Hence $S_1$ and $S'_2$ cross, and the crossedge contains $s_1$. This contradicts the Crossedge Independence Lemma 10.1.18.

We now prove a sequence of lemmas that will enable us to contract all crossedges in an arbitrary order. For the rest of this subsection, we make the following assumption (in addition to Assumption 10.1.16):

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M. Grohe, *Definable Graph Structure Theory*
Lemma 10.1.12 that $s$ is 3-connected, for all vertices $s \in S$, Assumption 10.1.20.

Case 1: $G'$ is 3-connected.

Figure 10.5. The setup of assumption [10.1.20] the clique $W$ may have a nonempty intersection with $S_1$ and $S_2$.

Assumption 10.1.20. $S_1, S_2 \in \mathcal{S}(G, W)$ are crossing 3-hinges with crossedge $e = s_1s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$.

To simplify the notation, we let $G' := G-e$, and we let $s' \in V(G')$ be the vertex corresponding to the edge $e$ (so $V(G') = (V(G) \setminus \{s_1, s_2\}) \cup \{s'\}$). For $i = 1, 2$ we let $X_i := X(S_i)$ and $Y_i := Y(S_i)$. Furthermore, we assume that $S_i = \{s_i, s'_i, s''_i\}$. Then it follows from Lemma [10.1.12] that $S_1 \cap X_{3-i} = \{s'_i, s''_i\}$. Figure 10.5 illustrates the situation.

Lemma 10.1.21. $G'$ is 3-connected.

Proof. Suppose for contradiction that $G'$ is not 3-connected and let $\{t, t'\}$ be a separator of $G'$. Then $s' \in \{t, t'\}$, say, $s' = t'$, because $G$ is 3-connected. It follows that $\{t, s_1, s_2\}$ is a separator of $G$. In the following case distinction, we shall prove that $G \setminus \{t, s_1, s_2\}$ is connected, which will obviously be a contradiction.

Case 1: $t \in X_1$.

As $G$ is 3-connected, for all vertices $x \in (X_1 \cup S_1) \setminus \{t, s_1\}$ there is a path $P_x$ from $x$ to a vertex $s_x \in S_1 \setminus \{s_1\}$ with all internal vertices in $X_1$. Let $s_y$ be the unique vertex in $S_1 \setminus \{s_1, s_x\}$. I claim that for every vertex $y \in (Y_1 \setminus \{s_2\}) \cup \{s_y\}$ there is a path $Q_y$ from $y$ to $s_x$ with all internal vertices in $Y_1 \setminus \{s_2\}$. This will show that $G \setminus \{t, s_1, s_2\}$ is connected. Suppose first that $y \in Y_1 \setminus \{s_x\}$. Let $Q_y$ be a path from $y$ to $s_x$ in $G \setminus \{s_y, s_2\}$. Then $s_1 \not\in V(Q_y)$, because $(S_1 \cap X_2) \cup (S_1 \cap S_2) \cup (S_2 \cap Y_1) = \{s_x, s_y, s_2\}$ separates $y \in Y_1 \cap X_2$ from $s_1 \in S_1 \cap Y_2$, and therefore the first vertex on $Q_y$ that is not in $Y_1$ must be $s_x$. To deal with $y = s_y$, remember that by Lemma [10.1.12] either there is an edge from $s_y$ to $s_x$, or $Y_1 \cap X_2 \neq \emptyset$. In the latter case, take an arbitrary $y' \in Y_1 \cap X_2 = Y_1 \setminus \{s_2\}$, and let $Q_{y'}$ be a path from $y'$ to $s_x$ with all internal vertices in $Y_1 \setminus \{s_2\}$. By a similar reasoning as above, there is also a path $Q'_{y'}$ from $y'$ to $s_y$ with all internal vertices in $Y_1 \setminus \{s_2\}$. Then $Q_y \cup Q'_{y'}$ contains a path from $s_y$ to $s_x$ with all internal vertices in $Y_1 \setminus \{s_2\}$.

Case 2: $t \in S_1$.

Let $s$ be the vertex in $S_1 \setminus \{s_1, t\}$. As $G$ is 3-connected, for every vertex $x \in X_1$ there is a path $P_x \subseteq G \setminus \{t, s_1\}$ from $x$ to $s$ such that $V(P_x) \setminus \{s\} \subseteq X_1$. Furthermore, for every $y \in Y_1$ there is a path $Q_y \subseteq G \setminus \{s_2, t\}$ from $y$ to $s$ with all internal vertices in $Y(S) \setminus \{s_2\}$. Thus $G \setminus \{s_1, s_2, t\}$ is connected.
Case 3: \( t \in Y_1 \).

We argue similarly to Case 1, but with the roles of \( X_1 \) and \( Y_1 \) switched. For all vertices \( y \in Y_1 \setminus \{t, s_2\} \) there is a path \( P_y \subseteq G \setminus \{t, s_2\} \) from \( y \) to a vertex \( s_y \in S_1 \setminus \{s_1\} \) such that \( V(P_y) \setminus \{s_y\} \subseteq Y_1 \). Let \( s_x \) be the unique vertex in \( S_1 \setminus \{s_1, s_y\} \). Then for every vertex \( x \in X_1 \cup \{s_x\} \) there is a path \( Q_x \) from \( x \) to \( s_y \) with all internal vertices in \( X_1 \). This proves that \( G \setminus \{s_1, s_2, t\} \) is connected. 

\[ \square \]

**Lemma 10.1.22.** \( W \subseteq V(G') \), and \( W \) has no split extension in \( G' \).

**Proof.** As \( W \subseteq (X_1 \cup S_1) \cap (X_2 \cup S_2) \) and \( s_i \in Y_{3-i} \) for \( i = 1, 2 \) it holds that \( W \subseteq V(G) \setminus \{s_1, s_2\} \subseteq V(G') \). Suppose for contradiction that \( W \) has a split extension \( v \) in \( G' \).

**Case 1:** \( v \neq s' \).

We shall prove that \( v \) is also a split extension of \( W \) in \( G \), which will contradict our assumption that \( W \) have no split extension in \( G \). Let \( A \) be a connected component of \( G \setminus (W \cup \{v\}) \). If \( e \in E(A) \) the let \( A' := A/e \); otherwise, let \( A' := A \). Then \( A' \) is a connected component of \( G' \setminus (W \cup \{v\}) \) with \( N^G(A') = N^G(A) \). Thus \(|N^G(A)| = |N^G(A')| \leq |W|\), because \( v \) is a split extension of \( W \) in \( G' \).

**Case 2:** \( v = s' \).

In this case, we shall prove that \( s_1 \) is a split extension of \( W \) in \( G \). Let \( A \) be a connected component of \( G \setminus (W \cup \{s_1\}) \).

**Case 2a:** \( s_2 \not\in V(A) \).

Then \( A \) is also connected components of \( G' \setminus (W \cup \{s'\}) \), and we have

\[ N^G(A) = \begin{cases} (N^G(A) \setminus \{s'\}) \cup \{s_1\} & \text{if } s_1 \in N^G(A), \\ N^G(A) & \text{otherwise}. \end{cases} \]

Thus \(|N^G(A)| = |N^G(A')| \leq |W|\).

**Case 2b:** \( s_2 \in V(A) \).

We shall prove that \( W \not\subseteq N^G(A) \). As \( N^G(A) \subseteq W \cup \{s_1\} \), this will imply \(|N^G(A)| \leq |W|\). Suppose for contradiction that \( W \subseteq N^G(A) \). Then \( V(A) \cap S_1 \neq \emptyset \), because \( W \cap X_1 \neq \emptyset \) and \( V(A) \cap Y_1 \supseteq \{s_2\} \neq \emptyset \). Say, \( s'_1 \in V(A) \). As \( G \) is 3-connected, every vertex in \( Y_1 \cap X_2 \) has a path to \( s'_1 \) with every internal vertex in \( Y_1 \cap X_2 \). This implies that \( Y_1 \cap X_2 \subseteq V(A) \). Let \( A' = A \setminus \{s_2\} \). Then \( V(A') \setminus Y_1 = V(A) \setminus Y_1 \), and \( V(A') \cap Y_1 = V(A) \cap Y_1 = Y_1 \cap X_2 = Y_1 \cap X_2 \). It follows that \( A' \) is connected and thus a connected component of \( G' \setminus (W \cup \{s'\}) \). Thus \(|N^G(A')| \leq |W|\), and as \( s' \in N^G(A') \), it follows that \( W \not\subseteq N^G(A') \). But \( N^G(A') \cap W = N^G(A) \cap W \), and thus \( W \not\subseteq N^G(A) \). 

For a set \( S' \subseteq V(G') \) with \( W \not\subseteq S' \), we let \( X'(S') \) be the vertex set of the connected component of \( G' \setminus S' \) that contains \( W \setminus S' \), and we let \( Y'(S') := V(G') \setminus (S' \cup X'(S')) \).

**Lemma 10.1.23.** For \( i = 1, 2 \), if \( Y_i \cap X_{3-i} \neq \emptyset \) then \( S'_i := (S_i \setminus s_i) \cup \{s'\} \in \mathcal{F}(G', W) \). Furthermore, \( X'(S'_i) = X_i \) and \( Y'(S'_i) = Y_i \cap X_{3-i} = Y(S_i) \setminus \{s_{3-i}\} \).

**Proof.** By symmetry, it suffices to give the proof for \( i = 1 \). Suppose that \( Y_1 \cap X_2 \neq \emptyset \). Then \( S'_1 \) separates the nonempty set \( W \setminus S_1 \) from the nonempty set \( Y_1 \cap X_2 \). Furthermore, \( S'_1 \) is
inseparable from $W$ because if a $3$-separator $S'$ of $G'$ with $W, S' \not\subseteq S'$ separates $S'$ from $W$ then either $S'$ or $(S' \setminus \{s'\}) \cup \{s_1\}$ is a $3$-separator of $G$ that separates $S_1$ from $W$.

Next, we shall prove that $S'_1$ is a $3$-hinge of $G'$. $S'_1$ separates $X_1$ from $Y_1 \cap X_2$. Note that $s'_2, s'_3 \in X_1$ and thus $|X_1| \geq 2$. Hence if $|Y_1 \cap X_2| \geq 2$, then $S'_1$ is a $3$-hinge by the $3$-Hinge Lemma [10.1.5]. In the following, we assume that $|Y_1 \cap X_2| = 1$. Let $y$ be the unique vertex in $Y_1 \cap X_2$. If there is an edge from $s_2$ to $s'_1$ or $s'_3$ in $G$, then there is an edge from $s'$ to $s'_1$ or $s'_3$ in $G'$, and $S'_1$ is not an independent set. Again by the $3$-Hinge Lemma [10.1.5] this implies that $S'_1$ is a $3$-hinge. Let us assume that there is no edge from $s_2$ to $s'_1$ or $s'_3$ in $G$. Then $N^G(s_2) \subseteq \{y, s_1\}$, which is impossible because $G$ is $3$-connected.

We have $X'(S'_1) = X_1$ and $Y'(S'_1) = Y_1 \cap X_2$, because $S'_1$ separates $X_1$ from $Y_1 \cap X_2$ and $X_1 \cup (Y_1 \cap X_2) \cup S'_1 = V(G')$ and $X_1$ is connected and $W \cap X_1 \neq \emptyset$. Since $Y_1 \cap S_2 = \{s_2\}$ and $Y_1 \cap Y_2 = \emptyset$, we have $Y_1 \cap X_2 = Y_1 \setminus \{s_2\}$.

\[ \square \]

**Lemma 10.1.24.** Let $S \in \mathcal{H}(G, W) \setminus \{S_1, S_2\}$. Then $s_1, s_2 \in X(S)$ and $S \in \mathcal{H}(G', W)$. Furthermore, $X'(S) = (X(S) \setminus \{s_1, s_2\}) \cup \{s'\}$ and $Y'(S) = Y(S)$.

**Proof.** We first observe that $s_1, s_2 \in X(S)$. Indeed, if $s_1 \notin Y(S)$, then $S$ and $S_1$ are crossing and the crossof of $S$ and $S_1$ has a nonempty intersection with the crossof of $S_1$ and $S_2$. This contradicts the Crossedge Independence Lemma [10.1.18]. Thus $s_1 \notin Y(S)$ and $s_2 \notin Y(S)$ by symmetry. Similarly, if $s_1 \in S$, then $S \cap Y_2 \neq \emptyset$. Thus $S$ and $S_2$ are crossing, and $s_1$ is a vertex in the crossof. Again, this contradicts the Crossedge Independence Lemma [10.1.18]. Thus $s_1 \notin S$ and $s_2 \notin S$ by symmetry. It follows directly from $s_1, s_2 \in X(S)$ that $X'(S) = (X(S) \setminus \{s_1, s_2\}) \cup \{s'\}$ and $Y'(S) = Y(S)$.

**Claim 1.** $S$ is a $3$-hinge of $G'$.

**Proof.** Note first that $S$ is a $3$-separator of $G'$, because it is a $3$-separator of $G$. If $S$ is not an independent set or $G' \setminus S$ has more than two connected components then $S$ is a $3$-hinge by the $3$-Hinge Lemma [10.1.5]. Suppose that $S$ is an independent set and $G' \setminus S$ has exactly two connected components. Then the vertex sets of these components are $Y(S)$ and $X'(S) = (X(S) \setminus \{s_1, s_2\}) \cup \{s'\}$. We have $|Y(S)| \geq 2$ because $S$ is a $3$-hinge of $G$. Furthermore, $|X'(S)| \geq |X(S) \cap W| \geq 2$, because $S$ is an independent set. Thus $S$ is a $3$-hinge by the $3$-Hinge Lemma [10.1.5].

It remains to prove that $S$ is inseparable from $W$ in $G'$. Suppose for contradiction that there is a $T' = \{t_1, t_2, t_3\} \subseteq V(G')$ such that $W, S \not\subseteq T'$ and $T'$ separates $S$ from $W$ in $G'$. Then $s' \in T'$, because otherwise $T'$ separates $S$ from $W$ in $G$, which contradicts $S \in \mathcal{H}(G, W)$.

Say, $t_3 = s'$. Let $T := \{t_1, t_2, s_1, s_2\}$. Then $T$ separates $S$ from $W$ in $G$.

**Claim 2.** For $i = 1, 2$ we have $S_i \not\subseteq T$, and $T$ does not separate $S_i$ from $W$ in $G$.

**Proof.** Suppose for contradiction that $S_i \subseteq T$. Then $T = S_i \cup \{s_{3-i}\} \subseteq S_i \cup Y_i$. Hence $S \subseteq S_i \cup Y_i$, because $T$ separates $S$ from $W$. But then $S_i$ separates $S$ from $W$ as well, which contradicts $S \in \mathcal{H}(G, W)$.

If $T$ separates $S_i$ from $W$, then $\{t_1, t_2, s_i\}$ separates $S_i$ from $W$ as well, which contradicts $S_i \in \mathcal{H}(G, W)$.

To derive the desired contradiction (to our assumption that $T'$ separates $S$ from $W$ in $G'$), we distinguish between several cases:

**Case 1:** $\{t_1, t_2\} \cap Y_1 \neq \emptyset$ and $\{t_1, t_2\} \cap Y_2 \neq \emptyset$.

Without loss of generality we assume that $t_1 \in Y_1$ and $t_2 \in Y_2$. Let $Q, Q', Q''$ be
mutually disjoint paths from $W$ to $s_1, s'_1, s''_1$, respectively. Then all internal vertices $Q, Q', Q''$ are in $X_1$. Thus $t_1 \notin V(Q \cup Q' \cup Q'')$.

Furthermore, we have $t_2 \notin V(Q' \cup Q'')$. To see this, suppose that $t_2 \in V(Q')$. Since $W \cup \{s'_1\} \subseteq X_2 \cup S_2$ and $t_2 \in Y_2$, the path $Q'$ goes from $X_2 \cup S_2$ to $Y_2$ and back. Thus $|V(Q') \cap S_2| \geq 2$, because $S_2$ separates $X_2$ from $Y_2$. As $V(Q') \subseteq X_1 \cup \{s'_1\}$ and thus $s_2 \notin V(Q')$, it follows that $\{s'_2, s''_2\} \subseteq V(Q')$. We also have $V(Q) \cap \{s'_2, s''_2\} \neq \emptyset$, because $W \subseteq X_2 \cup S_2$ and $s_1 \in Y_2$. Hence $Q \cap Q' \neq \emptyset$. This is a contradiction. It follows that $Q', Q'' \subseteq G \setminus T$.

**Case 1a:** $(S \setminus T) \cap S_1 \neq \emptyset$.
Then either $s'_1 \in S$ or $s''_1 \in S$, and $Q'$ or $Q''$ is path from $S$ to $W$ in $G \setminus T$. This is a contradiction, because $T$ separates $S$ from $W$.

**Case 1b:** $(S \setminus T) \cap Y_1 \neq \emptyset$.
Let $s \in (S \setminus T) \cap Y_1$. Let $P, P', P''$ be internally disjoint paths from $s$ to $s_1, s'_1, s''_1$, respectively. Then all internal vertices of $P, P', P''$ are in $Y_1$. Hence $s_2 \in V(P)$, because $s \in Y_1 \setminus T \subseteq Y_1 \cap X_2$ and the only way to reach $s_1$ from within $Y_1$ is through $s_2$. Moreover, $t_1$ can only be contained in one of the two paths $P', P''$. Say, $t_1 \notin P'$. Then $P' \subseteq G \setminus T$, and thus $P' \cup Q'$ is a path from $S$ to $W$ in $G \setminus T$. This is a contradiction.

**Case 1c:** $(S \setminus T) \cap X_1 \neq \emptyset$.
Let $s \in (S \setminus T) \cap X_1$. Let $P, P', P''$ be internally disjoint paths from $s$ to $s_1, s'_1, s''_1$, respectively. Then all internal vertices of $P, P', P''$ are in $X_1$. As $s_1 \in V(P)$ and $t_1, s_2 \in Y_1$, we have $V(P' \cup P'') \subseteq T \subseteq \{t_2\}$. Hence one of the paths $P', P''$, say $P''$, has an empty intersection with $T$. Then $P' \cup Q'$ contains a path from $S$ to $W$ in $G \setminus T$. Again, this is a contradiction.

**Case 2:** $\{t_1, t_2\} \cap Y_1 = \emptyset$.
We shall prove that $\{t_1, t_2, s_1\}$ separates $S$ from $W$, which contradicts $S \in \mathcal{S}(G, W)$. Let $P \subseteq G \setminus \{t_1, t_2\}$ be a path from a vertex $s \in S \setminus T$ to $W \setminus T$. Suppose for contradiction that $s_1 \notin V(P)$. Then $s_2 \in V(P)$ because $T$ separates $S$ from $W$. Let $y$ be the predecessor of $s_2$ on the path $P$. Then $y \neq s_1$ and thus $y \in (S_1 \cup Y_1) \cap X_2$.

By Claim 3 there is a path $Q \subseteq G \setminus T$ from a vertex $w \in W$ to a vertex $s_w \in S_1$.

**Claim 3.** There is a path $R$ from $y$ to $s_w$ with all internal vertices in $Y_1 \cap X_2$.

**Proof.** The claim is trivial if $y = s_w$ or $ys_w \in E(G)$. If $y \in S_1 \setminus \{s_w\}$ then $\{y, s_w\} = \{s'_1, s''_1\}$. Thus if $ys_w \notin E(G)$, then by Lemma 10.1.12 we have $X_2 \cap Y_1 \neq \emptyset$. Let $A$ be a connected component of $G[X_2 \cap Y_1]$. Then $N(A) = \{y, s_w, s_2\}$, and thus we can find the desired path $R$ with internal vertices in $A$. Finally, if $y \in X_2 \cap Y_1$, we let $A$ be the connected component of $G[X_2 \cap Y_1]$ that contains $y$, and again we can find the desired path $R$ with internal vertices in $A$.

Then $sPy \cup R \cup Q \subseteq G \setminus T$ contains a path from $S$ to $W$ in $G \setminus T$, which is a contradiction.

**Case 3:** $\{t_1, t_2\} \cap Y_2 = \emptyset$.

Symmetric to Case 2. □

As in Lemma 10.1.23 for $i = 1, 2$ we let $S'_i := (S_i \setminus \{s_i\}) \cup \{s'_i\}$. We let $X'_i := X'(S'_i)$ and $Y'_i := Y'(S'_i)$. By Lemma 10.1.23 we have $X'_i = X_i$ and $Y'_i = Y_i \cap X_{3-i}$.

M. Grohe, Definable Graph Structure Theory
Lemma 10.1.25. Let \( S' \in \mathcal{S}(G', W) \). Then either \( s' \not\in S' \) and \( S' = S'_1 \) or \( S' = S'_2 \).

Proof. Suppose first that \( s' \not\in S' \). Then clearly, \( S' \) is also a 3-hinge of \( G \), and it is inseparable from \( W \) because any separator of \( G \) also yields a separator of \( G' \) of at most the same size. In the following, we assume that \( s' \in S' \).

Let \( y \in N^{G'}(s') \cap Y'(S') \). Then either \( y \in N^{G}(s_1) \setminus \{s_2\} \) or \( y \in N^{G}(s_2) \setminus \{s_1\} \). Without loss of generality we assume \( y \in N^{G}(s_1) \setminus \{s_2\} \subseteq (X_1 \cup Y_2) \cup (S_2 \setminus \{s_2\}) \). We shall prove that \( S' = S'_2 \). Suppose for contradiction that this is not the case. By Lemma 10.1.23 we have \( S'_2 \in \mathcal{S}(G', W) \) and \( Y'_2 = (X_1 \cap Y_2) \). By Lemma 10.1.12 we have \( Y'_2 \cap Y'(S') = \emptyset \) and therefore \( y \not\in (X_1 \cap Y_2) \). Hence \( y \in (S_2 \setminus \{s_1\}) \subseteq S'_2 \). This means that \( S' \) and \( S'_2 \) are crossing, which contradicts Lemma 10.1.26 because \( S'_2 \cap S' \supseteq \{s'\} \neq \emptyset \).

Lemma 10.1.26. \( E_\chi(G', W) = E_\chi(G, W) \setminus \{s_1s_2\} \).

Proof. To prove that \( E_\chi(G', W) \subseteq E_\chi(G, W) \setminus \{s_1s_2\} \), let \( s_as_b \in E_\chi(G', W) \) be the crossedge of hinges \( S'_a, S'_b \in \mathcal{S}(G', W) \) in \( G' \), where \( s_a \in S'_a \) and \( s_b \in S'_b \). Then by Lemma 10.1.25 for \( x \in \{a, b\} \), either \( S'_x = S'_1 \) or \( S'_x = S'_2 \) or \( s' \not\in S'_x \) and \( S'_x \in \mathcal{S}(G, W) \). Let \( S_x := S_i \) if \( S_x = S'_i \) for \( i = 1 \) or \( i = 2 \), and let \( S_x := S'_x \) otherwise. Suppose for contradiction that \( s' \not\in \{s_a, s_b\} \). Say, \( s' \in s_a \in S'_a \). Then \( s' \in Y'(S'_a) \) and thus \( s' \not\in X'(S'_a) \). This contradicts Lemma 10.1.24.

Hence \( s' \not\in \{s_a, s_b\} \), which implies that \( s_a \in S'_a \) and \( s_b \in S'_b \). It follows from Lemmas 10.1.23 and 10.1.24 that \( s_a \in Y(S_b) \) and \( s_b \in Y(S_a) \). Thus \( S_a \) and \( S_b \) cross with crossedge \( s_as_b \), which implies that \( s_as_b \in E_\chi(G, W) \setminus \{s_1s_2\} \).

To prove the converse inclusion, let \( e \in E_\chi(G, W) \setminus \{s_1s_2\} \) be the crossedge of hinges \( S_a, S_b \in \mathcal{S}(G, W) \). Note that \( \{S_a, S_b\} \neq \{S_1, S_2\} \) and thus \( e \cap \{s_1, s_2\} = \emptyset \). Hence \( e \in E(G') \). For \( x \in \{a, b\} \), let \( S'_x = S'_i \) if \( S_x = S_i \) for \( i = 1 \) or \( i = 2 \), and let \( S'_x = S_x \) otherwise. If \( S_x = S_i \) then \( Y_i \cap X_{3-i} = Y_i \setminus \{s_3-i\} \neq \emptyset \), because the set contains an endvertex of \( e \). Hence \( S'_x \in \mathcal{S}(G', W) \) by Lemma 10.1.23. If \( S'_x = S_i \) then \( S'_x \in \mathcal{S}(G', W) \) by Lemma 10.1.24. Moreover, \( e \) is the crossedge of \( S'_a \) and \( S'_b \) in \( G' \). Thus \( e \in E_\chi(G', W) \).

To summarise the findings of this subsection: Lemmas 10.1.21 and 10.1.26 say that if we contract the crossedge \( s_1s_2 \) then Figure 10.5 becomes Figure 10.6. That is, in the resulting graph \( G' = G/s_1s_2 \) the 3-hinges \( S_1 \) and \( S_2 \) become \( S'_1 \) and \( S'_2 \), which may not be 3-hinges because \( Y'_1 \) or \( Y'_2 \) may be empty, and all other 3-hinges in \( \mathcal{S}(G, W) \) remain unaffected and become 3-hinges in \( \mathcal{S}(G', W) \). Note that \( S'_1 \) and \( S'_2 \) do not cross.

10.1.4 Contracting the Crossedges

Throughout this subsection, we make the following assumption:

Assumption 10.1.27. \( G \) is a 3-connected graph, and \( W \subset V(G) \) is a nonseparating 3-clique in \( G \) that has no split extension.

We let
\[
G^* := G/E_\chi(G, W),
\]
be the graph obtained from \( G \) by contracting all crossedges of 3-hinges in \( \mathcal{S}(G, W) \). Moreover, we assume that \( E_\chi(G, W) = \{e_1, \ldots, e_m\} \). For all \( i \in [m] \), we let \( x_i \) be the vertex of \( G^* \) corresponding to the contracted edge \( e_i \). Observe that
\[
V(G) \setminus \bigcup_{i=1}^{m} e_i = V(G^*) \setminus \{x_1, \ldots, x_m\}.
\]

Preliminary Version
Chapter 10. Quasi-4-Connected Components

Lemma 10.1.28.  
1. $G^*$ is a 3-connected graph.
2. $W \subseteq V(G^*)$, and $W$ has no split extension in $G^*$.
3. There are no crossing 3-hinges in $\delta(G^*, W)$.

Proof. For all $i \in [0, m]$, we let $G^i := G/\{e_1, \ldots, e_i\}$. By induction on $i$, we prove:

1. $G^i$ is a 3-connected graph.
2. $W \subseteq V(G^i)$, and $W$ has no split extension in $G^i$.
3. $E_x(G^i, W) = \{e_{i+1}, \ldots, e_m\}$.

Here (i) follows from Lemma 10.1.21, (ii) follows from Lemma 10.1.22, and (iii) follows from Lemma 10.1.26. As $G^* = G^m$, this proves the lemma.

For every set $S \subseteq V(G^*)$ with $W \not\subseteq S$, we let $X^*(S)$ be the vertex set of the connected component of $G^* \setminus S$ that contains $W \setminus S$, and we let $Y^*(S) := V(G^*) \setminus (S \cup X^*(S))$. We let

$$J^* := J^*(G, W) := \left(G^* \setminus \bigcup_{S \in \delta(G^*, W)} Y^*(S)\right) \cup \bigcup_{S \in \delta(G^*, W)} K[S].$$

(10.1.10)

Observe that $W \subseteq V(J^*)$.

Lemma 10.1.29. $J^*$ is a minor of $G^*$ that has a faithful image in $G^*$.

Proof. For every $S \in \delta(G^*, W)$, let $\hat{A}_S := G^*[X^*(S) \cup S] \cup K[S]$. As $S$ is a hinge of $G^*$ the graph $\hat{A}_S$ is a minor of $G^*$ that has a faithful image $(Z_{S,v})_{v \in X^*(S) \cup S}$ in $G^*$. As the image is faithful, for all $v \in X^*(S)$ we have $Z_{S,v} = \{v\}$ and for all $v \in S$ we have $Z_{S,v} = \{v\} \cup I'_{S,v}$ for some set $I'_{S,v} \subseteq Y^*(S)$.

Let $S, S' \in \delta(G^*, W)$ with $S \neq S'$. Then $S$ and $S'$ do not cross, and thus $Y^*(S) \cap Y^*(S') = Y^*(S) \cap Y^*(S') = \emptyset$. Hence for all $v, v' \in V(\hat{A}_S) \cap V(\hat{A}_{S'})$ we have $Z_{S,v} \cap Z_{S',v'} = \{v\}$ if $v = v'$ and $Z_{S,v} \cap Z_{S',v'} = \emptyset$ otherwise.

Note that $V(J^*) = \bigcap_{S \in \delta(G^*, W)} V(\hat{A}_S)$. For all $v \in V(J^*)$ we let

$$Z_v := \bigcup_{S \in \delta(G^*, W)} Z_{S,v}.$$
It is easy to see that \((Z_v)_{v\in V(J^*)}\) is a faithful image of \(J^*\) in \(G^*\). \qedhere

\textbf{Corollary 10.1.30.} \(J^*\) is a minor of \(G\).

The graph \(J^*\) may have 3-separators and even 3-hinges, but it turns out that all its 3-separators are irrelevant in the following sense:

\textbf{Definition 10.1.31.} A separator \(S\) of a graph \(G\) is irrelevant if \(G \setminus S\) has exactly two connected components, one of which has order 1.

Note that by the 3-Hinge Lemma\(^{10.1.5}\) all 3-hinges of a 3-connected graph are irrelevant separators. The converse does not hold, because there may be irrelevant 3-separators that are independent sets and hence are not 3-hinges.

\textbf{Lemma 10.1.32.} \(J^*\) is 3-connected and only has irrelevant 3-separators.

\textbf{Proof.} Let \(S^*\) be a separator of \(J^*\) of order \(|S^*|\leq 3\).

\textbf{Claim 1.} Let \(v_1, v_2 \in V(J^*)\) such that \(S^*\) separates \(v_1\) from \(v_2\) in \(J^*\). Then \(S^*\) separates \(v_1\) from \(v_2\) in \(G^*\).

\textbf{Proof.} For \(i = 1, 2\), let \(A_i^*\) be the connected component of \(v_i\) in \(J^* \setminus S^*\). Suppose for contradiction that there is a path \(P \subseteq G^*\) from a vertex \(w_1 \in V(A_1^*)\) to a vertex \(w_2 \in V(A_2^*)\) with all internal vertices in \(V(G^*) \setminus V(J^*)\). Then there is a connected component \(A\) of \(V(G^*) \setminus V(J^*)\) such that all internal vertices of \(P\) are in \(V(A)\). It follows from the definition of \(J^*\) and the fact that the \(Y^*(S)\) for \(S \in \mathcal{S}(G^*, W)\) are mutually disjoint that there is an \(S \in \mathcal{S}(G^*, W)\) such that \(A\) is a connected component of \(G^*[Y^*(S)]\). Then \(w_1, w_2 \in N(A) \subseteq S\).

As \(K[S] \subseteq J^*\), this implies \(w_1 w_2 \in E(J^*)\). This is a contradiction, because \(w_1\) and \(w_2\) come from distinct connected components of \(J^* \setminus S^*\).

Claim 1 implies that \(S^*\) is a separator of \(G^*\). As \(G^*\) is 3-connected, it follows that \(|S^*| = 3\).

Suppose for contradiction that \(S^*\) is not irrelevant. Then \(J^* \setminus S^*\) either has at least three connected components or two connected components that both have order greater than one. It follows from Claim 1 that \(G^* \setminus S^*\) either has at least three connected components or two connected components that both have order greater than one. Hence \(S^*\) is a 3-hinge of \(G^*\).

\textbf{Claim 2.} \(S^* \in \mathcal{S}(G^*, W)\).

\textbf{Proof.} We need to prove that \(S^*\) is inseparable from \(W\) in \(G^*\). Suppose for contradiction that \(S'\) with \(S^*, W \nsubseteq S'\) is a 3-separator of \(G^*\) that separates \(S^*\) from \(W\). Then by Lemma\(^{10.1.9}\) \(S'\) is a 3-hinge of \(G^*\). Without loss of generality, we may assume that \(S' \in \mathcal{S}(G^*, W)\). As \(S'\) separates \(S^*\) from \(W\), we have \(S^* \cap Y^*(S') \neq \emptyset\) and thus \(S^* \subseteq V(J^*)\). This is a contradiction.

Now let \(v, v' \in V(J^*)\) such that \(S^*\) separates \(v\) from \(v'\) in \(J^*\) and thus in \(G^*\). Then at most one of these two vertices is in \(X^*(S^*)\) and thus at least one is in \(Y^*(S^*)\). Say, \(v \in Y^*(S^*)\). Then \(v \notin V(J^*)\), which is a contradiction. \qedhere

Now we define another graph \(J\) which may be viewed as the graph obtained from \(J^*\) by “uncontracting” all the crosseges (this view on \(J\) will be formally established in Lemma\(^{10.1.37}\)).

We define \(J\) by

\[ V(J) := \left( V(G) \setminus \bigcup_{S \in \mathcal{S}(G,W)} Y(S) \right) \cup \bigcup_{S \in \mathcal{S}(G,W)} S \] (10.1.11)
and

\[ J := J(G, W) := G[V(J)] \cup \bigcup_{A \text{ connected component of } G \setminus J} K[N(A)]. \]  

(10.1.12)

Lemma 10.1.33. Let \( A \) be a connected component of \( G \setminus V(J) \). Then there is exactly one \( S_A \in \delta(G, W) \) such that \( V(A) \subseteq Y(S_A) \setminus \bigcup_{S \in \delta(G, W)} S \).

Furthermore, \( |N(A)| = 3 \) and either \( N(A) = S_A \) and there are no hinges \( S \in \delta(G, W) \) crossing \( S_A \), or there are \( a \in [3] \) and distinct hinges \( S_1, \ldots, S_a \in \delta(G, W) \) crossing \( S_A \) with crossedges \( s_1s_1', \ldots, s_d's_d \), where \( s_1, \ldots, s_d \in S_A \), and

\[ N(A) = \{s_1', \ldots, s_d'\} \cup (S_A \setminus \{s_1, \ldots, s_d\}) , \]

and there are no hinges \( S \in \delta(G, W) \setminus \{S_1, \ldots, S_a\} \) crossing \( S_A \).

We call \( S_A \) the entrance hinge of \( A \) in \( J \) and \( d = |N(A) \setminus S| \) the distance between \( A \) and \( S_A \).

Proof of Lemma 10.1.33 We have

\[ V(A) \subseteq V(G) \setminus V(J) = \bigcup_{S \in \delta(G, W)} \left( Y(S) \setminus \bigcup_{S' \in \delta(G, W)} S' \right) \]

Since by Lemma 10.1.12 the sets \( Y(S) \setminus \bigcup_{S' \in \delta(G, W)} S' \) for \( S \in \delta(G, W) \) are mutually disjoint and not connected by edges of \( G \), there is a unique \( S_A \in \delta(G, W) \) such that

\[ V(A) \subseteq Y(S_A) \setminus \bigcup_{S \in \delta(G, W)} S = Y(S_A) \setminus \bigcup_{S \in \delta(G, W)} (S \cap Y(S_A)). \]

Note that there are at most three hinges \( S \in \delta(G, W) \) with \( S \cap Y(S_A) \neq \emptyset \), because each such hinge crosses \( S_A \), and the crossedges must be disjoint. Suppose that \( S_A = \{s_1, s_2, s_3\} \). For \( i = 1, 2, 3 \), if there is a hinge \( S \in \delta(G, W) \) that crosses \( S_A \) with a crossedge that has \( s_i \) as an endvertex, we let \( S_i \) be this hinge and \( s_i's_i' \) the crossedge. If there is no such hinge, we leave \( S_i \) and \( s_i' \) undefined.

Note that if \( s_i' \) is defined then it is the only neighbour of \( s_i \) in \( Y(S_A) \). (This follows from Lemma 10.1.12) Hence if \( s_i \in N(A) \) we have \( s_i' \in V(A) \), which contradicts \( V(A) \cap S_i = \emptyset \). This means that either \( s_i' \) is undefined or \( s_i \notin N(A) \). As \( |N(A)| \geq 3 \), this implies the assertions of the lemma.

Corollary 10.1.34. Let \( A \) be a connected component of \( G \setminus V(J) \) and \( e \in E_x(G, W) \) a crossedge. Then at most one endvertex of \( e \) is in \( N(A) \).

Corollary 10.1.35. Let \( S, S' \in \delta(G, W) \) be crossing hinges with crossedge \( ss' \), where \( s \in S \) and \( s' \in S' \). Then \( N^J(s') \subseteq Y(S) \cup S \).

Proof. As \( s' \in Y(S) \) we have \( N^G(s') \subseteq Y(S) \cup S \). Suppose there is an edge \( s'v \in E(J) \setminus E(G) \). Then there is a connected component \( A \) of \( G \setminus J \) such that \( s', v \in N(A) \). As \( s' \in Y(S) \), the entrance hinge of \( A \) is \( S \). Thus \( N(A) \subseteq Y(S) \cup S \).

M. Grohe, *Definable Graph Structure Theory*
Lemma 10.1.36. \( G \setminus J = G^* \setminus J^* \). 

Proof. For every \( v \in V(G) \), if \( v \) is an endvertex of a crossedge \( e_i \) for some \( i \in [m] \), let \( v^* := x_i \) (the vertex of \( G^* \) corresponding to the contracted edge \( e_i \)), and otherwise, let \( v^* := v \). We first prove that

\[
V(G) \setminus V(J) = V(G^*) \setminus V(J^*). \quad (10.1.13)
\]

To prove the inclusion \( \subseteq \), let \( v \in V(G) \setminus V(J) \). Then \( v \in Y(S) \setminus \bigcup_{S' \in \mathcal{H}(G,W)} S' \) for some \( S \in \mathcal{H}(G,W) \). Let \( S^* := \{ s^* \mid s \in S \} \). It follows from Lemmas 10.1.23 and 10.1.24 that \( S^* \in \mathcal{H}(G^*,W) \) and \( v \in Y^*(S^*) \). Thus \( v \in V(G^*) \setminus V(J^*) \).

To prove the converse inclusion \( \supseteq \), let \( v \in V(G^*) \setminus V(J^*) \). Then \( v \in Y^*(S^*) \) for some \( S^* \in \mathcal{H}(G^*,W) \). By Lemma 10.1.25, there is an \( S \in \mathcal{H}(G,W) \) such that \( S^* = \{ s^* \mid s \in S \} \), and by Lemmas 10.1.23 and 10.1.24 we have \( Y^*(S^*) \subseteq Y(S) \setminus \bigcup_{S' \in \mathcal{H}(G,W)} S' \). Hence \( v \in V(G) \setminus V(J) \). This completes the proof of \( (10.1.13) \).

It follows that

\[
G^* \setminus J^* = G^*[V(G^*) \setminus V(J^*)] = G^*[V(G) \setminus V(J)] = G[V(G) \setminus V(J)] = G \setminus J. \quad \square
\]

Lemma 10.1.37. \( J/E_x(G,W) = J^* \).

Proof. It follows from \( (10.1.9) \) and Lemma 10.1.36 that

\[
V(J) \setminus \bigcup_{i=1}^{m} e_i = V(J^*) \setminus \{ x_1, \ldots, x_m \}. \quad (10.1.14)
\]

This implies

\[
V(J/E_x(G,W)) = V(J^*). \quad (10.1.15)
\]

As in the proof of the previous lemma, for every \( v \in V(G) \) we let \( v^* := x_i \) if \( v \) is an endvertex of \( e_i \) for some \( i \in [m] \) and \( v^* := v \) otherwise.

Claim 1. For all \( vw \in E(J) \) with \( v^* \neq w^* \) we have \( v^* w^* \in E(J^*) \).

Proof. Let \( vw \in E(J) \) with \( v^* \neq w^* \). If \( vw \in E(G) \) then the claim follows from \( (10.1.15) \). Otherwise, there is a connected component \( A \) of \( G \setminus J \) such that \( v,w \in N(A) \). Let \( S \) be the entrance hinge of \( A \) in \( J \) and \( S^* := \{ s^* \mid s \in S \} \). Then by Lemma 10.1.33 we have \( v^*, \ w^* \in S^* \). Furthermore, \( V(A) \subseteq Y(S^*) \) and thus \( Y(S^*) \neq \emptyset \), and it follows from Lemmas 10.1.23 and 10.1.24 that \( S^* \in \mathcal{H}(G^*,W) \). Hence \( v^* w^* \in E(J^*) \).

Claim 2. For all \( v_s w_s \in E(J^*) \) there are \( v,w \in E(J) \) such that \( v^* = v_s \) and \( w^* = w_s \) and \( vw \in E(J) \).

Proof. Let \( v_s w_s \in E(J) \). If \( v_s, w_s \in E(G^*) \), then the claim follows from \( (10.1.15) \). Otherwise, there is a hinge \( S^* \in \mathcal{H}(G^*,W) \) such that \( v_s, w_s \in S^* \). By Lemma 10.1.25 there is an \( S \in \mathcal{H}(G,W) \) such that \( S^* = \{ s^* \mid s \in S \} \), and by Lemmas 10.1.23 and 10.1.24 we have \( Y^*(S^*) = Y(S) \setminus \bigcup_{S' \in \mathcal{H}(G,W)} S' \). As \( S^* \) is a hinge of \( G^* \), there is a path \( P^* \subseteq G^* \) from \( v_s \) to \( w_s \) with all internal vertices in \( Y^*(S^*) \). Since \( v_s w_s \not\in E(G^*) \), the length of this path is at least 2. Note that \( P^* \setminus \{ v_s, w_s \} \) is a path in \( G^* \setminus J^* = G \setminus J \). Thus there are vertices \( v,w \in V(G) \) such that \( v^* = v_s \) and \( w^* = w_s \) and a path \( P \subseteq G \) from \( v \) to \( w \) with \( P \setminus \{ v,w \} = P^* \setminus \{ v_s, w_s \} \). Let \( A \) be the connected component of \( G \setminus J \) that contains \( P \setminus \{ v,w \} \). Then \( v,w \in N(A) \) and thus \( vw \in E(J) \).

Claims 1 and 2 imply \( E(J/E_x(G,W)) = E(J^*) \) and thus complete the proof of the lemma. \( \square \)
For later reference, we prove a number of additional, fairly technical lemmas.

**Lemma 10.1.38.** For all connected components $A$ of $G \setminus J$ the set $N(A)$ is a 3-hinge in $G$.

*Proof.* Let $A$ be a connected component of $G \setminus V(J)$ and $S := N(A)$ and suppose for contradiction that $S$ is not a hinge. Then by the 3-Hinge Lemma 10.1.5, $S$ is an independent set. Thus $|V(J) \setminus S| \geq 2$, because $V(J)$ contains the clique $W$ and $|W \cap S| \leq 1$. Again by the 3-Hinge Lemma 10.1.5 it follows that $|V(A)| = 1$ and there is no connected component $A' \neq A$ of $G \setminus V(J)$ with $N(A') = S$.

Let $S_A$ be the entrance hinge of $A$ in $J$ and $d$ the distance between $A$ and $S_A$. Then $d \geq 1$, because $S_A$ is a hinge and $S$ is not. Suppose that $S_A = \{s_1, s_2, s_3\}$ such that for $i = 1, \ldots, d$ there is a hinge $S_i \in \mathcal{H}(G, W)$ crossing $S_A$ with crossedge $s_i' s_i$. Then $S = \{s_1', s_2', s_3', s_4, s_5, \ldots, s_7\}$.

Since $G$ is 3-connected, $\deg(s_1') \geq 3$. Let $v, w \in N(s_1') \setminus \{s_1\}$. Since $|V(A)| = 1$, at most one of the vertices $v, w$ is in $V(A)$. Say, $v \notin V(A)$.

We have $v \notin S$, because $S$ is an independent set. Suppose for contradiction that $v \in S \setminus S_A$. Then $v = s_i$ for $i = 2$ or $i = 3$, because $v \notin \{s_1\}$ by definition. As $s_i = v \notin S$, we have $i \leq d$ and $s_i' \in S$. We have $s_i' \in Y(S_A) \cap X(S_i)$ and $s_i \in Y(S_i)$ and $s_i' s_i = s_i v \in E(G)$. This is impossible, because $S_i$ separates $X(S_i)$ from $Y(S_i)$.

Hence $v \notin S \cup S_A$. Let $A'$ be the connected component of $G \setminus (S \cup S_A)$ with $v \in V(A')$. Then $V(A') \subseteq Y(S_A) \cap X(S_1)$, because $s_1' \in N(A') \cap Y(S)$ and $Y(S) \setminus \{s_1'\} = Y(S) \cap X(S_1)$. Moreover, $A \cap A' = \emptyset$, because $v \in V(A') \setminus V(A)$. Furthermore, $N(A') \neq S$ by our assumption that $S$ is not a hinge. As $N(A') \geq 3$ and $N(A') \subseteq S \cup S_A$, at least one of the five vertices $s_1, s_2, s_3, s_2', s_3'$ (as far as they are defined) is in $N(A') \setminus S$. We have $s_1 \notin N(A')$, because $s_1 \in Y(S_1)$ and $V(A') \subseteq X(S_1)$. We have $s_2 \notin N(A') \setminus S$, because if $s_2$ is defined then it is in $S$ by Lemma 10.1.33. Similarly, we have $s_3 \notin N(A') \setminus S$. Thus either $s_2 \in N(A') \setminus S$ or $s_3 \in N(A') \setminus S$.

Suppose that $s_2 \in N(A') \setminus S$. Then $s_2'$ is defined and in $S$. We have $s_2' \in Y(S) \cap X(S_2)$ and $s_2 \in S \cap Y(S_2)$ and $s_1', s_2' \in N(A')$. As $S_2$ separates $X(S_2)$ from $Y(S_2)$, we must have $V(A) \cap S_2 \neq \emptyset$. As $V(A) \subseteq Y(S)$ and $S_2 \cap Y(S) = \{s_2'\}$, we have $s_2' \in A'$. But $A'$ is a connected component of $G \setminus (S \cup S_A)$ and $s_2' \in S$. This is a contradiction. Similarly, the remaining assumption $s_3 \in N(A') \setminus S$ leads to a contradiction.

**Lemma 10.1.39.** $J$ is a minor of $G$ that has a faithful image in $G$.

*Proof.* Let $A_1, \ldots, A_m$ be the connected components of $G \setminus J$. For each $i \in [m]$, let $S_i' := N(A_i)$, and let $S_i$ be the entrance hinge of $A_i$ in $J$. Let $H_i$ be the connected component of $G \setminus S_i'$ that contains $W \setminus S_i'$. Note that $V(H_i) = X(S_i) \cup (S_i \setminus S_i')$. This implies $V(J) \subseteq V(H_i) \cup S_i'$. Because

$$V(J) \subseteq X(S_i) \cup S_i \cup \bigcup_{S \in \mathcal{H}(G, W)} (S \cap Y(S_i)) = X(S_i) \cup S_i \cup S_i'$$

by Lemma 10.1.33.

As $S_i'$ is a 3-hinge of $G$ by Lemma 10.1.38, the graph $\tilde{H}_i = G[V(H_i) \cup S_i'] \cup K[S_i']$ has a faithful image in $G$. We can easily combine these faithful images to a faithful image of $J$ in $G$ (similarly to the proof of Lemma 10.1.29).

**Lemma 10.1.40.** $J$ is 3-connected, and for all connected components $A$ of $G \setminus J$ the graph $J \setminus N(A)$ is connected.
Proof. It is easy to see that $J$ is 3-connected, because for every 3-connected graph $H$, every 3-separator $S$ of $H$, and every connected component $A$ of $H \setminus S$ the graph $\overline{A}^H$ is 3-connected.

Now let $A$ be a connected component of $G \setminus J$ and $S := N(A)$. Let $S_A$ be the entrance hinge of $A$ in $J$ and $X := X(S_A)$. Then $N(X)$ is connected, and as $S_A = N(X)$ the graph $G[X \cup (S_A \setminus S)]$ is connected as well. Furthermore, it follows from Lemma 10.1.33 that $V(J) \setminus S \subseteq X \cup (S_A \setminus S)$.

Now let $v, w \in V(J) \setminus S$. Then there is a path $P$ from $v$ to $w$ in the connected graph $G[X \cup (S_A \setminus S)]$. It follows from the definition of $J$ that we can turn $P$ in to a path $P' \subseteq J \setminus S$ from $v$ to $w$. This implies that $J \setminus S$ is connected. □

Lemma 10.1.41. Let $ss' \in E_x(G, W)$ be a crossedge. Then $|N^J(s)| = 3$. Furthermore, if $|N^J(s') \setminus \{s'\}| = 3$, and if $|N^J(s') \setminus \{s\}| = 3$ and that the two vertices in $N^J(s') \setminus \{s\}$ are the endvertices of an edge in $E(J) \setminus E_x(G, W)$. Let $d \in [3]$ be the number of hinges in $S(G, W)$ crossing $S$. Let $S_i$ for $i \in [2, d]$ be these hinges (we have already defined $S_1$ before), and let $s_i s_i'$ with $s_i \in S$ be the corresponding crossedges. Denote the elements of $S \setminus \{s_1, \ldots, s_d\}$ by $s_{d+1}, \ldots, s_3$.

Case 1: $Y(S) = \{s'_1, \ldots, s'_d\}$.

In this case, there is no connected component $A$ of $G \setminus J$ with entrance hinge $S$. Thus for $i = 1, \ldots, d$ the vertex $s'_i$ is not contained in $N^G(A)$ for any connected component $A$ of $G \setminus J$, and therefore no edge in $E(J) \setminus E(G)$ is incident with $s'_i$. That is, we have $N^J(s'_i) = N^G(s'_i)$.

Case 1a: $d = 1$.

Then $X(S_1) \cap Y(S) = \emptyset$, and by the 3-Hinge Lemma 10.1.5 we have $N^J(s'_1) = N^G(s'_1) = \{s_1, s_2, s_3\}$ and $s_2 s_3 \in E(G)$ and thus $s_2 s_3 \in E(J)$. Moreover, we have $s_2 s_3 \not\in E_x(G, W)$, because if $s_2 s_3$ was a crossedge, this would contradict Lemma 10.1.24 applied to $S$ and the two hinges with crossedge $s_2 s_3$.

Case 1b: $d = 2$.

Then $X(S_1) \cap Y(S) = \{s'_2\}$ and $X(S_2) \cap Y(S) = \{s'_1\}$. Since $s_1 \in Y(S_1)$ and $s_2 \in X(S_1)$ we have $s_1 s_2 \not\in E(G)$, and from $N^G(s'_2) \geq 3$ and $N^G(s'_2) \subseteq Y(S) \cup S$ it follows that $N^J(s'_2) = N^G(s'_2) = \{s'_1, s_2, s_3\}$. In particular, $s_2 s_3 \in E(J)$. Similarly, we have $N^J(s'_1) = \{s_1, s'_2, s_3\}$. Thus $|N^J(s'_1)| = 3$. We have already seen that $s_2 s_3 \in E(J)$, and we have $s'_2 s'_3 \not\in E_x(G, W)$, because no edge in $E_x(G, W)$ has an endvertex with $s'_2 s'_3 \not\in E_x(G, W)$ in common.

Case 1c: $d = 3$.

Then $X(S_1) \cap Y(S) = \{s'_2, s'_3\}$ and $X(S_2) \cap Y(S) = \{s'_1, s'_3\}$ and $X(S_3) \cap Y(S) = \{s'_1, s'_2\}$. Then $s_i s'_j \not\in E(G)$ for any $i \neq j$, and we get $N^J(s'_1) = N^G(s'_1) = \{s_1, s'_2, s'_3\}$ and $N^J(s'_2) = \{s'_1, s_2, s'_3\}$. Thus $|N^J(s'_1)| = 3$ and $s'_2 s'_3 \in E(J)$. We have $s'_2 s'_3 \not\in E_x(G, W)$, because no edge in $E_x(G, W)$ has an endvertex with $s'_2 s'_3 \not\in E_x(G, W)$ in common.
Case 2: \( Y(S) \supset \{ s_1', \ldots, s_d' \} \).
Then \( Y(S) \setminus \bigcup_{S' \in \Theta(G,W)} S' = Y(S) \setminus \{ s_1', \ldots, s_d' \} \neq \emptyset \). Let \( A \) be a connected component of \( G[Y(S) \setminus \bigcup_{S' \in \Theta(G,W)} S'] \). Then \( A \) is a connected component of \( G \setminus J \) with entrance hinge \( S \), and by Lemma 10.1.33 we have \( S' := N^G(A) = \{ s_1', \ldots, s_d', s_{d+1}', \ldots, s_3 \} \). Then \( S' \) is a clique in \( J \). Moreover, for \( i = 1, \ldots, d \), all edges of \( E(J) \setminus E(G) \) that are incident with \( s_i' \) are edges between vertices in \( S' \), because by Lemma 10.1.33 for every connected component \( A' \) with entrance hinge \( S \) we have \( N^G(A') = S' \) as well.

Now we argue very similarly to Case 1 (with a case distinction depending on \( d \)) and prove that

\[
N^J(s_1') = \begin{cases} \{ s_1, s_2, s_3 \} & \text{if } d = 1 \\ \{ s_1, s_2', s_3 \} & \text{if } d = 2 \\ \{ s_1, s_2', s_3' \} & \text{if } d = 3 \end{cases} = \{ s_1 \} \cup (S' \setminus \{ s_1' \}).
\]

This implies that \( |N^J(s_1')| = 3 \), and as \( S' \) is a clique in \( J \) it also implies that the two vertices in \( N^J(s_1') \setminus \{ s_1 \} \) are adjacent in \( J \). As in Case 1, it implies that the edge between these two vertices is not in \( E_\times(G,W) \).

**Lemma 10.1.42.** Let \( e = s_1s_2 \in E_\times(G,W) \) be a crossedge. Then \( N^J(s_1) \cap N^J(s_2) = \emptyset \), and there is no edge \( e' \in E_\times(G,W) \setminus \{ e \} \) such that \( e' \cap N^J(s_1) \neq \emptyset \) and \( e' \cap N^J(s_2) \neq \emptyset \).

**Proof.** Let \( S_1, S_2 \) be the two crossing 3-hinges with \( s_i \in S_i \) for \( i = 1, 2 \). By Corollary 10.1.35 for \( i = 1, 2 \) we have

\[
N^J(s_i) \subseteq S_{3-i} \cup Y(S_{3-i}),
\]

and thus \( N^J(s_1) \cap N^J(s_2) = \emptyset \). Suppose for contradiction that there is a crossedge \( e' = s_1's_2' \in E_\times(G,W) \setminus \{ e \} \) such that \( s_1' \in N^J(s_1) \) and \( s_2' \in N^J(s_2) \). Then by (10.1.16) for \( i = 1, 2 \) we have

\[
s_i' \in S_{3-i} \cup Y(S_{3-i}).
\]

Let \( S_1', S_2' \in \Theta(G,W) \) such that \( e' \) is the crossedge of \( S_1', S_2' \) and \( s_i' \in S_i' \) for \( i = 1, 2 \). Then

\[
s_i' \in S_i' \cap Y(S_{3-i}).
\]

Since \( Y(S_2') \cap Y(S_2) = \emptyset \), by (10.1.17) and (10.1.18) we have \( s_i' \in S_2 \cap Y(S_2') \). Thus \( S_2 \) and \( S_2' \) cross, and \( s_i' \) is an endvertex of the crossedge. As the crossedges are disjoint, this implies \( S_2 = S_1' \). Similarly, we derive \( S_1 = S_2' \). Thus \( e = e' \), which is a contradiction.

The final lemma shows that the main objects defined in this section are actually IFP-definable:

**Lemma 10.1.43.** There are IFP-formulae \( \text{hinge}(z_1, z_2, z_3, y_1, y_2, y_3) \) and \( \text{cross-edge}(z_1, z_2, z_3, y_1, y_2) \) and IFP-graph transductions \( \Theta(z_1, z_2, z_3) \) and \( \Theta^*(z_1, z_2, z_3) \) such that for every 3-connected graph \( G \) and for every nonseparating 3-clique \( W = \{ w_1, w_2, w_3 \} \) of \( G \) that has no split extension in \( G \) we have:

(1) \( G \models \text{hinge}[w_1, w_2, w_3, v_1, v_2, v_3] \iff \{ v_1, v_2, v_3 \} \in \Theta(G,W); \)

(2) \( G \models \text{cross-edge}[w_1, w_2, w_3, v_1, v_2] \iff v_1v_2 \in E_\times(G,W); \)

(3) \( \Theta[G, w_1, w_2, w_3] = J(G,W) \) and \( \Theta^*[G, w_1, w_2, w_3] = J^*(G,W) \).

**Proof.** This follows easily from Corollary 10.1.6.

M. Grohe, *Definable Graph Structure Theory*
10.2 Decomposition into Quasi-4-Connected Components

Recall that a separator $S$ of a graph $G$ is irrelevant if $G \setminus S$ has precisely two connected components, one of which has order 1.

**Definition 10.2.1.** A graph is quasi-4-connected if either it is a complete graph of order at most 3, or it is 3-connected and all its 3-separators are irrelevant.

We denote the class of all quasi-4-connected graphs by $\mathcal{QZ}_4^*$. Note that $\mathcal{QZ}_4^* \subseteq \mathcal{Z}_3^*$.

**Example 10.2.2.** Hexagonal grids (see Figure 10.1 and Figure 14.1) are quasi-4-connected, but not 4-connected.

**Example 10.2.3.** Every 3-connected graph of order at most 5 is quasi-4-connected.

The goal of this section is to prove the following lemma, which may be viewed as the main result of this chapter. To cover all applications, the lemma is long and technical. The core statement is that all graphs admit a definable treelike decomposition into torsos that are minors of the graph and almost quasi-4-connected, in the sense that they can be turned into quasi-4-connected graphs by contracting a definable matching.

**Lemma 10.2.4 (Q4C Decomposition Lemma).** There is a $d$-scheme $\Lambda_{q4c}$ and a formula $\mu(\bar{x}, y, z)$ such that for all graphs $G$ the decomposition $\Delta_{q4c} := (D_{q4c}, \sigma_{q4c}, \alpha_{q4c}) := \Lambda_{q4c}[G]$ has the following properties.

(i) $\Delta_{q4c}$ is a treelike decomposition of $G$.

(ii) The adhesion of $\Delta_{q4c}$ is at most 3.

(iii) The decomposition $\Delta_{q4c}$ is tight.

(iv) For all $t \in V(D_{q4c})$ the torso $\tau_{q4c}(t)$ is a minor of $G$ that has a faithful image in $G$.

(v) For all $t \in V(D_{q4c})$ the torso $\tau_{q4c}(t)$ is in $\mathcal{Z}_3^*$, that is, either 3-connected or a complete graph of order at most 3.

For $t := \tau \in V(D_{q4c})$ we let $J_t := \tau_{q4c}(t)$ and

$$M_t := \{vw \in \binom{V(G)}{2} \mid G \models \mu(\bar{v}, v, w)\}.$$

Then $M_t \subseteq E(J_t)$ and the following holds.

(vi) $J_t/M_t \in \mathcal{QZ}_4^*$.

(vii) $M_t$ is a matching in $J_t$.

(viii) For every edge $e \in M_t$ and every connected component $A$ of $G \setminus V(J_t)$ it holds that $|N^G(A) \cap e| \leq 1$.

(ix) For every edge $e = vw \in M_t$ it holds that $|N^{J_t}(v)| = 3$. Furthermore, if $N^{J_t}(v) \setminus \{w\} = \{w_1, w_2\}$ then $w_1w_2 \in E(J_t) \setminus M_t$.

(x) For every edge $e = vw \in M_t$ it holds that $N^{J_t}(v) \cap N^{J_t}(w) = \emptyset$, and there is no edge $e' \in M_t \setminus \{e\}$ that has one endvertex in $N^{J_t}(v)$ and one endvertex in $N^{J_t}(w)$.
We call the graphs \( J_t/M_t \) the quasi-4-connected components of \( G \). For every quasi-4-connected component \( J_t^* := J_t/M_t \), we call \( J_t \) the torso and \( M_t \) the matching associated with \( J_t^* \), and we call \( t \) an index of \( J_t^* \).

To proof Q4C Decomposition Lemma is similar to the proof of the 3CC Decomposition Lemma [8.3.1]. We start by proving a decomposition lemma for 3-connected graphs. Combining this lemma with the 3CC Decomposition Lemma by means of the Decomposition Lifting Lemma [8.3.2] we obtain a decomposition of arbitrary graphs into “almost” quasi-4-connected torsos. In the last step of the proof we show that these torsos have all the properties listed in the Q4C Decomposition Lemma.

10.2.1 Decomposition of 3-Connected Graphs

**Lemma 10.2.5.** There is a 4-dimensional d-scheme \( \Lambda \) such that for all 3-connected graphs \( G \) that are not quasi-4-connected the decomposition \( \Delta := (D, \sigma, \alpha) := \Lambda[G] \) is treelike, and for every node \( \tau = (v_1, v_2, v_3, v_4) \in V(D) \) the following conditions are satisfied.

1. \( \{v_1, v_2, v_3\} \) is a 3-separator of \( G \).
2. Either \( v_4 \in \{v_1, v_2, v_3\} \) and \( \sigma(\tau) = \emptyset \) or \( v_4 \notin \{v_1, v_2, v_3\} \) and \( \sigma(\tau) = \{v_1, v_2, v_3\} \).
3. \( \alpha(\tau) \) is the vertex set of the connected component of \( G \setminus \sigma(\tau) \) that contains \( v_4 \).
4. Either \( \tau(\tau) = K[\tau] \) or \( \sigma(\tau) = \{v_1, v_2, v_3\} \) has no split extension in the graph \( H := G[\gamma(\tau)] \cup K[\sigma(\tau)] \) and \( \tau(\tau) = J(H, \sigma(\tau)) \) (defined as in (10.1.11) and (10.1.12)).
5. \( \tau(\tau) \) is a minor of \( G \) that has a faithful image.

The proof of this lemma is very similar to the proof of Lemma [8.3.4] the decomposition of a 2-connected graph into its 3-connected components.

**Proof.** To explain the definition of \( \Lambda \), we fix a 3-connected graph \( G \) that is not quasi-4-connected. We shall define a treelike decomposition \( \Delta = (D, \sigma, \alpha) \) of \( G \) satisfying (i)–(v) for every node \( \tau \in V(D) \). It will be straightforward to define a 4-dimensional d-scheme \( \Lambda \), of course not depending on the specific graph \( G \), such that \( \Delta = \Lambda[G] \).

For each tuple \( \tau = (v_1, v_2, v_3, v_4) \in V(G)^4 \), we define a set \( S_\tau \) and two graphs \( A_\tau, H_\tau \subseteq G \). We let

\[
S_\tau := \begin{cases} 
\emptyset & \text{if } v_4 \in \{v_1, v_2, v_3\}, \\
\{v_1, v_2, v_3\} & \text{otherwise.}
\end{cases}
\]

We let \( A_\tau \) be the connected component of \( G \setminus S_\tau \) that contains \( v_4 \), and we let

\[
H_\tau := \tilde{A}_\tau = G[V(A_\tau) \cup S_\tau] \cup K[S_\tau].
\]

Here we use that \( S_\tau = N(A_\tau) \), because \( G \) is 3-connected. Observe that the graph \( H_\tau \) is 3-connected and the 3-clique \( S_\tau \) is not a separator of \( H_\tau \). If \( S_\tau \) has no split extension in \( H_\tau \), we let

\[
J_\tau := J(H_\tau, S_\tau)
\]
(defined as in (10.1.11) and (10.1.12)).

The decomposition \( \Delta \) will have four kinds of nodes: r-nodes (root nodes), s-nodes (split-extension nodes), c-nodes (component nodes), and b-nodes (bounded nodes). All nodes will be tuples \( \tau = (v_1, \ldots, v_4) \in V(G)^4 \) such that

M. Grohe, *Definable Graph Structure Theory*
10.2. Decomposition into Quasi-4-Connected Components

(A) \( \{v_1, v_2, v_3\} \) is a 3-separator of \( G \).

Let \( \tau = (v_1, \ldots, v_4) \in V(G)^4 \) be a tuple satisfying [A]

(B) \( \tau \) is an \( r \)-node if \( v_4 \in \{v_1, v_2, v_3\} \) and \( \{v_1, v_2, v_3\} \) is a 3-hinge of \( G \).

(C) \( \tau \) is an \( b \)-node if \( v_4 \notin \{v_1, v_2, v_3\} \) and \( V(A_\tau) = \{v_4\} \).

(D) \( \tau \) is an \( s \)-node if \( v_4 \notin \{v_1, v_2, v_3\} \) and \( V(A_\tau) \neq \{v_4\} \) and \( S_\tau = \{v_1, v_2, v_3\} \) is a 3-hinge of \( G \) and \( v_4 \) is a split extension of \( S_\tau \) in \( H_\tau \).

(E) \( \tau \) is a \( c \)-node if \( v_4 \notin \{v_1, v_2, v_3\} \) and \( V(A_\tau) \neq \{v_4\} \) and \( S_\tau = \{v_1, v_2, v_3\} \) is a 3-hinge of \( G \) and there is no split extension of \( S_\tau \) in \( H_\tau \).

We let \( V_r, V_s, V_c, V_b \) be the sets of \( r \)-nodes, \( s \)-nodes, \( c \)-nodes, and \( b \)-nodes respectively, and\( V(D) := V_r \cup V_s \cup V_c \cup V_b \). Note that \( V_r \neq \emptyset \), because \( G \) has at least one 3-separator \( S \) that is not irrelevant. Recall that each such separator is a 3-hinge.

**Claim 1.** For all \( \tau \in V(D) \) the graph \( H_\tau \) is a minor of \( G \) that has a faithful image.

**Proof.** For \( \tau \in V_r \) we have \( H_\tau = G \). For \( \tau \in V_s \cup V_c \) the set \( S_\tau \) is a 3-hinge of \( G \), and thus the claim follows from the definition of hinges. For \( b \)-nodes \( \tau \in V_b \), the separator \( S_\tau \) is not necessarily a hinge, but we have \( |G \setminus H_\tau| \geq 2 \), because otherwise \( G \) would be quasi-4-connected (by Example [10.2.3]). Then it follows as in the proof of the 3-Hinge Lemma [10.1.5] that \( H_\tau \) is a minor of \( G \) that has a faithful image.

**Claim 2.** Let \( A \subseteq G \) be a connected subgraph such that \( |N(A)| \leq 3 \) and either \( |A| = 1 \) or \( N(A) \) is a hinge. Then there is a node \( \tau \in V(D) \) such that \( A_\tau = A \) and \( S_\tau = N(A) \).

**Proof.** As \( G \) is 3-connected, we have \( |N(A)| = 3 \). Let \( S := \{v_1, v_2, v_3\} := N(A) \) and \( H := \tilde{A} \). If \( V(A) = \{v_4\} \) for some \( v_4 \in V(G) \), then \( \tau := (v_1, \ldots, v_4) \in V_b \). Otherwise, if \( S \) is a 3-hinge that has a split extension \( v_4 \) in \( H \), then \( (v_1, \ldots, v_4) \in V_b \). If \( S \) is a 3-hinge that has no split extension in \( H \), then \( \tau := (v_1, \ldots, v_4) \in V_c \) for every \( v_4 \in V(A) \). In all three cases, \( A_\tau = A \) and \( S_\tau = S \).

To define the edge relation \( E(D) \), let \( \tau, \bar{\tau} \in V(D) \). Then \( \tau \bar{\tau} \in E(D) \) if one of the following conditions is satisfied:

(F) \( \tau \in V_r \) and \( \bar{\tau} \in V_s \cup V_c \cup V_b \) and \( S_{\bar{\tau}} = \bar{\tau} \);

(G) \( \tau \in V_s \) and \( \bar{\tau} \in V_s \cup V_c \cup V_b \) and \( A_{\bar{\tau}} \) is a connected component of \( H_{\bar{\tau}} \setminus \bar{\tau} \);

(H) \( \tau \in V_c \) and \( \bar{\tau} \in V_s \cup V_c \cup V_b \) and \( A_{\bar{\tau}} \) is a connected component of \( H_{\bar{\tau}} \setminus J_{\bar{\tau}} \).

We define \( \sigma, \alpha : V(D) \to 2^{V(G)} \) as follows:

(I) for all \( \tau \in V(D) \) we let \( \sigma(\tau) := S_\tau \) and \( \alpha(\tau) := V(A_\tau) \).

This completes the definition of the decomposition \( \Delta \).

Using Corollary [10.1.6] and Lemma [10.1.43] it is straightforward to construct a \( d \)-scheme \( \Lambda \) (not depending on the specific graph \( G \)) such that \( \Delta = \Lambda(G) \).

**Claim 3.** \( \Delta \) is a strict treelike decomposition.

**Proof.** It is immediate from the definitions that \( \Delta \) satisfies [TL.2]
To prove (TL.3s), let $\overline{v} = (v_1, \ldots, v_4)$, $\overline{w} = (w_1, \ldots, w_4) \in V(D)$ such that $\overline{vw} \in E(D)$. If $\overline{v}$ is an r-node, then $\alpha(\overline{v}) = \gamma(\overline{v}) = V(G)$. Thus $\gamma(\overline{w}) \subseteq \gamma(\overline{v})$ and $\alpha(\overline{w}) \subseteq \alpha(\overline{v})$. Moreover, we have $\sigma(\overline{w}) = S_{\overline{w}}$ and thus $\alpha(\overline{w}) \subseteq \alpha(\overline{v})$.

Suppose next that $\overline{v}$ is an s-node. Then $A_{\overline{v}}$ is the connected component of $H_\overline{v} \setminus \tilde{v}$ that contains $w_4$. Then $\gamma(\overline{w}) = S_{\overline{w}} \cup V(A_{\overline{v}}) \subseteq V(H_{\overline{v}}) = \gamma(\overline{v})$ and $\alpha(\overline{w}) = V(A_{\overline{w}}) \subseteq V(H_{\overline{v}}) \setminus \tilde{v} \subseteq V(H_{\overline{v}}) \setminus S_{\overline{v}} = V(A_{\overline{v}}) = \alpha(\overline{v})$. Finally, suppose that $\overline{v}$ is a c-node. Then $\gamma(\overline{w}) = S_{\overline{w}} \cup V(A_{\overline{v}}) \subseteq V(H_{\overline{v}}) = \gamma(\overline{v})$ and $\alpha(\overline{w}) = V(A_{\overline{w}}) \subseteq V(A_{\overline{v}}) = \alpha(\overline{v})$, because $A_{\overline{w}}$ is a connected component of $H_{\overline{v}} \setminus J_{\overline{w}}$ and $S_{\overline{w}} \subseteq V(J_{\overline{w}})$.

To prove (TL.4), let $\overline{v} \in V(D)$ and $\overline{w}_1, \overline{w}_2 \in N(D)(\overline{v})$. For $i = 1, 2$, let $S_i := S_{\overline{w}_i}$ and $A_i := A_{\overline{w}_i}$. If $A_1 = A_2$ then $S_1 = N(A_1) = N(A_2) = S_2$ and thus $\overline{w}_1 \parallel \overline{w}_2$. So suppose that $A_1 \neq A_2$. If $\overline{v}$ is an r-node, then both $A_1, A_2$ are connected components of $G \setminus \tilde{v}$, and thus we have $V(A_1) \cap V(A_2) = \emptyset$ and $V(A_i) \cap S_i \subseteq V(A_i) \cap \tilde{v} = \emptyset$, which implies $\overline{w}_1 \parallel \overline{w}_2$. If $\overline{v}$ is a c-node, then both $A_1$ and $A_2$ are connected components of $H_{\overline{v}} \setminus J_{\overline{w}}$, and we have $V(A_1) \cap V(A_2) = \emptyset$ and $V(A_i) \cap S_i \subseteq V(A_i) \cap V(J_{\overline{w}}) = \emptyset$. Again, this implies $\overline{w}_1 \parallel \overline{w}_2$.

(TL.5) holds because $V_r \neq \emptyset$.

It is immediate from the definitions that $\Delta$ satisfies (i)–(iii). To prove (iv), let $\overline{v} = (v_1, v_2, v_3, v_4) \in V(D)$.

Case 1: $\overline{v}$ is an r-node.

Then $\alpha(\overline{v}) = V(G)$. Let $S := \{v_1, v_2, v_3\}$. As $S$ is a 3-hinge of $G$, by Claim 2 for every connected component $A'$ of $G \setminus S$ there is a node $\overline{v}' \in N_+(\overline{v})$ such that $\sigma(\overline{v}') = S_{\overline{v}} = S$ and $\alpha(\overline{v}') = V(A_{\overline{v}'}) = V(A')$. This implies $\beta(\overline{v}) = S = \tilde{v}$.

To prove that $\tau(\overline{v}) = K[S]$, choose an arbitrary $\overline{v}' \in N_+(\overline{v})$. (Such a $\overline{v}'$ exists, because $|G| \geq 4$ and thus $G \setminus S \neq \emptyset$.) Then $\sigma(\overline{v}') = S$, and as $K[\sigma(\overline{v}')] \subseteq \tau(\overline{v})$, it follows that $\tau(\overline{v}) = K[S]$.

Case 2: $\overline{v}$ is an b-node.

Let $S := S_{\overline{v}} = \{v_1, v_2, v_3\}$ and $A := A_{\overline{v}}$. Then $N_+(\overline{v}) = \emptyset$ and $V(A) = \{v_4\}$ and hence $\beta(\overline{v}) = \gamma(\overline{v}) = \tilde{v}$. As $G$ is 3-connected we have $N(v_4) = S$. Furthermore, $S$ is a separator of $G$, and thus there is a connected component $A' \neq A$ of $G \setminus S$. Then $N(A') = S$, and as $V(A') \subseteq V(G) \setminus (S \cup V(A)) = V(G) \setminus \gamma(\overline{v})$ it follows that $K[S] \subseteq \tau(\overline{v})$. Hence $\tau(\overline{v}) = K[S]$.

Case 3: $\overline{v}$ is an s-node.

Let $S := S_{\overline{v}} = \{v_1, v_2, v_3\}$ and $A := A_{\overline{v}}$ and $H := H_{\overline{v}}$. Then $v_4$ is a split extension of $S$ in $H$.

We first prove that $\beta(\overline{v}) = \tilde{v}$. We have $\gamma(\overline{v}) = V(H)$, and for every $\overline{v}' \in N_+(\overline{v})$, the set $\alpha(\overline{v}') = V(A_{\overline{v}'})$ is the vertex set of a connected component of $H \setminus \tilde{v}$. Thus $\overline{v} \subseteq \beta(\overline{v})$. To prove the converse inclusion $\tilde{v} \subseteq \beta(\overline{v})$, let $A'$ be a connected component of $H \setminus \tilde{v}$. We shall prove that there is a $\overline{v}' \in N_+(\overline{v})$ such that $\alpha(\overline{v}') = A_{\overline{v}'} = A'$. Let $S' := N^G(A')$ and $H' := \tilde{A}'$. As $O^G(H) \subseteq N^G(A) \subseteq \tilde{v}$, the graph $A'$ is also a connected component of $G \setminus \tilde{v}$, and we have $S' = N^H(A')$. We have $|S'| \leq |S| \leq 3$, because $v_4$ is a split extension of $S$ in $H$. Thus by Claim 2, it suffices to prove that either $|A'| = 1$ or $S'$ is a hinge of $G$. Suppose that $|A'| \geq 2$. As $S$ is a 3-hinge of $G$, we have $|G \setminus H| \geq 1$ and thus $|G \setminus H'| \geq |G \setminus H| + |\tilde{v}| \geq 2$. Thus by the 3-Hinge Lemma 10.1.5, $S'$ is a 3-hinge.

To see that $\tau(\overline{v}) = K[\overline{v}]$, note first that for distinct $i, j \in [3]$ we have $v_i, v_j \in \sigma(\overline{v})$ and thus $v_i v_j \in E(\tau(\overline{v}))$. To see that $v_i v_4 \in E(\tau(\overline{v}))$ for all $i \in [3]$, let $i, j, k \in [3]$ be distinct.

M. Grohe, Definable Graph Structure Theory
As \( G \) is 3-connected, there is a path from \( v_4 \) to \( v_i \) in \( G \setminus \{ v_j, v_k \} \) and thus either an edge from \( v_4 \) to \( v_i \) in \( G \) or a connected component \( A' \) of \( G \setminus \tilde{v} \) with \( v_4, v_i \in N^G(A') \). In the latter case, there is a \( \tau' \in N^D(\tilde{v}) \) such that \( A' = A_{\tau'} \), which implies that \( v_i, v_4 \in \sigma(\tau) \) and thus \( v_i v_4 \in E(\tau(\tau)) \).

**Case 4: \( \tau \) is a c-node.**

Let \( A := A_{\tau} \) and \( H := H_{\tau} \) and \( J := J_{\tau} \).

We shall prove that \( \beta(\tau) = V(J) \). We have \( \gamma(\tau) = V(H) \), and for every \( \tau' \in N^D(\tau) \), the set \( \alpha(\tau') = V(A_{\tau'}) \) is the vertex set of a connected component of \( H \setminus J \). Thus \( V(J) \subseteq \beta(\tau) \). To prove the converse inclusion \( V(J) \supseteq \beta(\tau) \), let \( A' \) be a connected component of \( H \setminus J \). We shall prove that there is a \( \tau' \in N^D(\tau) \) such that \( \alpha(\tau') = V(A_{\tau'}) = V(A') \).

Let \( S' := N^G(A') \) and \( H' := H \). As \( \partial^G(H) \subseteq N^G(A) = S \subseteq V(J) \), the graph \( A' \) is also a connected component of \( G \setminus \tilde{v} \), and we have \( S' = N^H(A') \). By Lemma 10.1.38, \( S' \) is a 3-hinge of \( H \). Thus \( |S'| \leq 3 \). If \( |A'| > 1 \), then (arguing as in Case 2) it is also a 3-hinge of \( G \). Hence it follows from Claim 2 that there is a \( \tau' \in N^D(\tau) \) such that \( A_{\tau'} = A' \).

Hence \( \beta(\tau) = V(J) \), and it follows from the definition of \( J \) that \( \tau(\tau) = J \).

To prove \( \sigma(\tau) \), we need the following claim.

**Claim 4.** Let \( \tau \in V_r \cup V_s \cup V_c \) and \( \tau \in N^D(\tilde{v}) \). Then \( \sigma(\tau) \) is a hinge of \( H_{\tau} \).

**Proof.** If \( \tau \in V_r \), then \( \sigma(\tau) = \{ v_1, v_2, v_3 \} \), which is a hinge of \( G = H_{\tau} \) by \( \text{(B)} \). If \( \tau \in V_s \), then \( S_{\tau} \subseteq \tilde{v} \) and thus \( |S_{\tau} \cap \tilde{v}| \geq 2 \). As \( S_r \) is a clique of \( H_{\tau} \), it follows that \( S_{\tau} \) is not an independent set and thus a 3-hinge by the 3-Hinge Lemma 10.1.35. If \( \tau \in V_c \), then \( A_{\tau} \) is a connected component of \( H_{\tau} \setminus J(H_{\tau}, S_{\tau}) \). Thus \( \sigma(\tau) = N^H(\tau(A_{\tau})) \) is a 3-hinge of \( H_{\tau} \) by Lemma 10.1.38.

Let \( \tau \in V(D) \). To prove \( \sigma(\tau) \), by Claim 1, it suffices to prove that \( \tau(\tau) \) is a minor of \( H_{\tau} \) that has a faithful image. This is trivial if \( \tau \in V_b \), because then \( \tau(\tau) = H_{\tau} \). So suppose that \( \tau \in V_r \cup V_s \cup V_c \). Let \( N^D(\tau) = \{ \tilde{v}_1, \ldots, \tilde{v}_m \} \), and for all \( i \in [m] \), let \( A_i := G[\alpha(\tilde{v}_i)] \) and \( S_i := N^G(A_i) \). Let \( H_0 := H_{\tau} \) and \( H_i := (H_{i-1} \setminus A_i) \cup K[S_i] \) for all \( i \in [m] \). A straightforward induction using Claim 4 shows that for every \( i \in [0, m] \) the graph \( H_i \) is a minor of \( H_{\tau} \) that has a faithful image. As \( H_m = \tau(\tau) \), this implies \( \sigma(\tau) \). \( \square \)

### 10.2.2 Lifting the Decomposition to Arbitrary Graphs

Let \( \Lambda_{3cc} \) be the \( d \)-scheme of the 3CC Decomposition Lemma 8.3.1. Recall that a 3-connected component of a graph \( G \) is defined to be a torso of the decomposition \( \Lambda_{3cc}[G] \) and that a 3-connected component is proper if its order is at least 4.

**Lemma 10.2.6.** There is a \( d \)-scheme \( \Lambda_{4c} \) such that for all graphs \( G \) the decomposition \( \Delta_{4c} := (D_{4c}, \sigma_{4c}, \alpha_{4c}) := \Lambda_{4c}[G] \) has the following properties:

(i) \( \Delta_{4c} \) is a treelike decomposition of \( G \).

(ii) The adhesion of \( \Delta_{4c} \) is at most 3.

(iii) The decomposition \( \Delta_{4c} \) is tight.

(iv) For all \( t \in V(D_{4c}) \) the torso \( \tau_{4c}(t) \) is a minor of \( G \) that has a faithful image in \( G \).

(v) For all \( t \in V(D_{4c}) \), the torso \( \tau_{4c}(t) \) satisfies one of the following statements.
a. \( \tau_{q4c}(t) \) is a clique of order at most 4.

b. \( \tau_{q4c}(t) \) is quasi-4-connected.

c. There is a proper 3-connected component \( H \) of \( G \), a 3-hinge \( S \) of \( H \), and a connected component \( A \) of \( H \setminus S \) such that \( S \) has no split extension in the graph \( \overrightarrow{A}^H \)

and \( \tau_{q4c}(t) = J(\overrightarrow{A}^H, S) \).

Depending on which statement is satisfied, we call \( t \) a node of type a, b, or c.

**Proof.** The proof follows the proof of the 3CC Decomposition Lemma 8.3.1. To explain the proof, we fix a graph \( G \). Let \( \Lambda^1 := \Lambda_{3cc} \) and \( \Delta^1 = (D^1, \sigma^1, \alpha^1) := \Lambda^1[G] \). For every \( t_1 \in V(D^1) \), let \( H_{t_1} := \tau^1(t_1) \). Then

(A) \( \Delta^1 \) is treelike;

(B) the adhesion of \( \Delta^1 \) is at most 2;

(C) the decomposition \( \Delta^1 \) is tight;

(D) for all \( t_1 \in V(D^1) \) the torso \( H_{t_1} \) is a minor of \( G \) that has a faithful image in \( G \);

(E) for all \( t_1 \in V(D^1) \) it holds that \( H_{t_1} \in Z_3^* \).

By the Normalisation Lemma for Definable Treelike Decompositions 5.2.3 we may assume without loss of generality that the decomposition \( \Delta^1 \) is strict and normal.

Let \( (\Lambda^2) \) be a 4-dimensional d-scheme with \( \mathcal{T}(\Lambda^2) \supseteq Z_3^* \) such that for all graph \( G \in Z_3^* \), if \( G \) is 3-connected, but not quasi-4-connected, then \( (\Lambda^2)[G] \) has properties \( i \)–\( v \) of Lemma 10.2.5 and if \( G \) is quasi-4-connected, then \( (\Lambda^2)[G] \) is the trivial 4-dimensional decomposition of \( G \) (see Example 5.1.6). Let \( \Lambda^2(\overline{v}_1) \) be a parametrised d-scheme such that for every \( \overline{v}_1 \in V(D^1) \) the scheme \( \Lambda^2(\overline{v}_1) \) defines the decomposition \( (\Lambda^2)[H_{\overline{v}_1}] \) of the torso \( H_{\overline{v}_1} \) within \((G, \overline{v}_1)\). Let \( \Delta^2_{\overline{v}_1} := (D^2_{\overline{v}_1}, \sigma^2_{\overline{v}_1}, \alpha^2_{\overline{v}_1}) := \Lambda^2[G, \overline{v}_1] \), viewed as a decomposition of \( H_{\overline{v}_1} \). Again by the Normalisation Lemma, without loss of generality we may assume that \( \Delta^2_{\overline{v}_1} \) is strict and normal. For all \( \overline{v}_1 \in V(D^1) \), \( \overline{v}_2 \in V(D^2_{\overline{v}_1}) \), let \( J_{\overline{v}_1\overline{v}_2} := \tau^2_{\overline{v}_1}(\overline{v}_2) \).

**Claim 1.** For all \( t_1 \in V(D^1) \) the following conditions are satisfied.

(F) \( \Delta^2_{t_1} \) is treelike.

(G) The adhesion of \( \Delta^2_{t_1} \) is at most 3.

(H) The decomposition \( \Delta^2_{t_1} \) is tight.

(I) For all \( t_2 \in V(D^2_{t_1}) \), the torso \( J_{t_1t_2} \) is a minor of \( G \) that has a faithful image.

(J) For all \( t_2 \in V(D^2_{t_1}) \), the torso \( J_{t_1t_2} \) satisfies one of the following statements:

a. \( J_{t_1t_2} \) is a clique of order at most 4;

b. \( J_{t_1t_2} \) is quasi-4-connected and \( \Delta^2_{t_1} \) is trivial;

c. \( H_{t_1} \) is a proper 3-connected component of \( G \) and there is a 3-hinge \( S \) of \( H_{t_1} \) and a connected component \( A \) of \( H_{t_1} \setminus S \) such that \( S \) has no split extension in the graph \( \overrightarrow{A}^{H_{t_1}} \) and \( J_{t_1t_2} = J(\overrightarrow{A}^{H_{t_1}}, S) \).
Proof. (F) (G) follow directly from Lemma 10.2.5. To prove (H) note that if the decomposition $\Delta_{i1}^2$ is nontrivial then the graph $H_{i1}$ is 3-connected, which implies that for every separator $S$ of order 3 and every connected component $A$ of $H_{i1} \setminus S$ it holds that $S = N^{H_{i1}}(A) = \partial H_{i1}(V(A) \cup S)$. Now (H) follows from Lemma 10.2.5(i) and (ii) (I) follows from (E) and Lemma 10.2.5(iv) and the fact that the adhesion of $\Delta^1$ is at most 2 by (B) and that $\Delta_{i1}^2$ is trivial unless $H_{i1}$ is 3-connected, but not quasi-4-connected.

We apply the Decomposition Lifting Lemma 5.6.2 to $\Lambda^1$ and $\Lambda^2(\mathcal{L})$. Let $\Lambda$ be the resulting $d$-scheme and $\Delta := \Lambda[G]$. To simplify the notation, in the following we denote nodes of $\Delta$ by $t, u$ and variants like $t'$, nodes of $\Delta^1$ by $t_1, u_1$ and variants, and nodes of decompositions $\Delta_{i1}^2$ by $t_2, u_2$ and variants. By the Decomposition Lifting Lemma, the decomposition $\Delta$ has the following properties.

(K) $\Delta$ is treelike.

(L) $V(D) \subseteq \left\{ t_1t_2 \mid t_1 \in V(D^1), t_2 \in V(D_{i1}^2) \right\}$.

(M) For $t \in V(D)$, we have $\tau(t) = J_t$.

Remember that for $t = t_1t_2$ we have $J_t = J_{t_1t_2} = \tau_{i1}^2(t_2)$.

(N) For $t = t_1t_2 \in V(D)$,

a. either $\sigma(t) = \sigma_{i1}^2(t_2)$ and $\alpha(t) \cap \beta^1(t_1) = \alpha_{i1}^2(t_2)$

b. or there is a $u_2 \in N^2_{t, t_1}(t_2)$ such that $\sigma(t) = \sigma_{t_1}^2(u_2)$ and $\alpha(t) \cap \beta^1(t_1) = (\beta^1(t_1) \setminus \gamma^2_{t_1}(u_2))$

c. or $\sigma(t) = \sigma^1(t_1)$ and $\alpha(t) = \alpha^1(t_1)$.

Furthermore, cases a. and b. only occur if $\Delta_{i1}^2$ is nontrivial.

$\Delta$ has almost all properties required by the lemma. Assertion (i) holds by (K), Assertion (ii) follows from (B), (G), (N). Assertion (iv) follows from (I) and (M). Assertion (v) follows from (J) and (M).

Unfortunately, the decomposition $\Delta$ is not necessarily tight; its components $\alpha(t)$ may be disconnected. To fix this, we will apply Lemma 5.3.2. The following two claims show that $\Delta$ satisfies the assumptions of that lemma.

Claim 2. For all $t \in V(D)$ it holds that $\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))$.

Proof. Let $t = t_1t_2 \in V(D)$, and let $H := H_{i1}$. By (N), we are in one of the following three cases.

Case a: $\sigma(t) = \sigma_{i1}^2(t_2)$ and $\alpha(t) \cap \beta^1(t_1) = \alpha_{i1}^2(t_2)$.

Then $\sigma(t) = N^H(\alpha(t) \cap V(H)) = \partial^H(\gamma(t) \cap V(H))$, because $\Delta_{i1}^2$ is tight.

To prove $\sigma(t) = N^G(\alpha(t))$, by (TL.2) we only need to prove $\sigma(t) \subseteq N^G(\alpha(t))$. Let $v \in \sigma(t)$. Let $w \in \alpha(t) \cap V(H)$ such that $vw \in E(H)$. If $vw \in E(G)$, we have $v \in N^G(\alpha(t))$. So suppose that $vw \not\in E(G)$. Then either $v, w \in \sigma^1(t_1)$, or there is a $u_1 \in N^1(t_1)$ such that $v, w \in \sigma^1(u_1)$. Let $S := \sigma^1(t_1)$ if $v, w \in \sigma^1(t_1)$ and $S := \sigma^1(u_1)$ for some $u_1 \in N^1(t_1)$ with $v, w \in \sigma^1(u_1)$ otherwise. As $H$ is a minor of $G$ that has a
faithful image (by \([D]\)), there is a path \(P \subseteq G\) from \(v\) to \(w\) with all internal vertices in 
\(V(G) \setminus V(H)\). As \(w \in \alpha(t)\) and as \(P\) has no internal vertex in 
\(\sigma(t) = \sigma^2_t(t_2) \subseteq V(H)\), all internal vertices of \(P\) are in \(\alpha(t)\). In particular, the neighbour of \(v\) on \(P\) is in \(\alpha(t)\), which proves that \(v \in N^G(\alpha(t))\).

To prove that \(\sigma(t) = \partial^G(\gamma(t))\), again by \([TL.2]\) it suffices to prove that \(\sigma(t) \subseteq \partial^G(\gamma(t))\). Let again \(v \in \sigma(t)\). We argue similarly as above. Let \(w \in V(H) \setminus \gamma(t)\) such that 
\(vw \in E(H)\). If \(vw \in E(G)\), we have \(v \in \partial^G(\gamma(t))\). So suppose that \(vw \notin E(G)\). Then, 
as above, there is a path \(P \subseteq G\) from \(v\) to \(w\) with all internal vertices in 
\(V(G) \setminus V(H) \supseteq V(G) \setminus \sigma(t)\). As \(w \in V(G) \setminus \gamma(t)\), the neighbour of \(v\) on \(P\) is in \(V(G) \setminus \gamma(t)\) as well. Thus \(\partial^G(\gamma(t))\).

Case b: \(\sigma(t) = \sigma^2_t(u_2)\) and \(\alpha(t) \cap \beta^1(t_1) = \beta^1(t_1) \setminus \sigma^2_t(u_2)\) for some \(u_2 \in N^2_t(t_1)\).

Then \(\sigma(t) = \partial^H(\gamma^2_t(u_2)) = \partial^H(\beta^1(t_1) \setminus \gamma^2_t(u_2)) = N^H(\alpha(t) \cap \beta^1(t_1))\) and similarly 
\(\sigma(t) = N^H(\alpha^2_t(u_2)) = \partial^H(\beta^1(t_1) \setminus \alpha^2_t(u_2)) = \partial^H(\gamma(t) \cap \beta^1(t_1))\). Now we argue as in 
Case a to prove that \(\sigma(t_1) = N^G(\alpha(t)) = \partial^G(\gamma(t))\).

Case c: \(\sigma(t) = \sigma^1(t_1)\) and \(\alpha(t) = \alpha^1(t_1)\).

Then \(\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))\), because \(\Delta^1\) is tight. \(\square\)

Claim 3. For all \(t \in V(D)\) the graph \(J_t \setminus \sigma(t)\) is either empty or connected, and if 
\(V(J_t) \setminus \sigma(t) \neq \emptyset\) it holds that 
\(N^H(V(J_t) \setminus \sigma(t)) = \sigma(t)\).

Proof. Let \(t = t_1t_2 \in V(D)\). By \([J]\) and \([M]\) we are in one of the following three cases.

Case a: \(J_t\) is a clique of order at most 4.

Then the claim is trivial.

Case b: \(J_t\) is quasi-4-connected and \(\Delta^2_{t_1t_2}\) is trivial.

Then \(\sigma(t) = \emptyset\) or \(\sigma(t) = \sigma^1(t_1)\) and thus \(|\sigma(t)| \leq 2\). Then the claim follows, because 
\(\sigma(t)\) is not a separator of \(J_t\).

Case c: \(H := H_{t_1}\) is a proper 3-connected component of \(G\) and there is a 3-hinge \(S\) of \(H\) and 
a connected component \(A\) of \(H \setminus S\) such that \(S\) has no split extension in the graph \(\tilde{A}^H\) and 
\(J_t = J(\tilde{A}^H, S)\).

In this case, the claim follows from Lemma \(10.1.40\). \(\square\)

We can apply Lemma \(5.3.2\) to transform \(\Delta\) to the desired decomposition \(\Delta_{q4c}\). \(\square\)

10.2.3 Proof of the Q4C Decomposition Lemma \(10.2.4\)

We let \(\Lambda_{q4c}\) be the \(d\)-scheme obtained by Lemma \(10.2.6\). Let \(G\) be a graph and \(\Delta_{q4c} := (D_{q4c}, \sigma_{q4c}, \alpha_{q4c}) := \Lambda_{q4c}[G]\). Assertions \(i\) \((iv)\) of the Q4C Decomposition Lemma are identical with the corresponding assertions of Lemma \(10.2.6\). Assertion \(v\) of the Q4C Decomposition Lemma follows easily from assertion \(v\) of Lemma \(10.2.6\) and, for nodes of type c, 
Lemma \(10.1.40\).

To define the matching \(M_t\), let \(t \in V(D_{q4c})\). If \(t\) is a node of type a or b, we let \(M_t := \emptyset\). Then assertion \(vi\) of Lemma \(10.2.4\) is immediate from Lemma \(10.2.6\) \(v\) a and b, and \(vii\) \(\emptyset\) are trivial. So suppose that \(t\) is a node of type c. Let \(H\) be a proper 3-connected component of \(G\), \(S\) a 3-hinge of \(H\), and \(A\) a connected component of \(H \setminus S\) such that \(\tau_{q4c}(t) = \)}
10.3 The Q4C Lifting Lemma

A consequence of the Q4C Decomposition Lemma is a lifting lemma that allows us to lift definable ordered treelike decompositions from the quasi-4-connected components of a graph to the whole graph. We will apply the lemma in the next chapter to prove that $K_5$-free graphs admit definable ordered treelike decompositions. For later applications we need a generalisation of the lemma, the Q4C Completion Lemma 12.6.2.

**Lemma 10.3.1 (Q4C Lifting Lemma).** Let $C$ be a minor ideal, and let $C^*$ be the class of all quasi-4-connected graphs in $C$. Suppose that $C^*$ admits IFP-definable ordered treelike decompositions. Then $C$ admits IFP-definable ordered treelike decompositions.

Besides the Q4C Decomposition Lemma 10.2.4, the proof of the lifting lemma uses the following extension lemma. It allows us to extend decompositions from the quasi-4-connected components of a graph to the torsos associated with them.

**Lemma 10.3.2.** For every parametrised od-scheme $\Lambda^*$ there is a parametrised od-scheme $\Lambda(\overline{\varpi})$, where the length of $\overline{\varpi}$ matches the dimension of the d-scheme $\Lambda_{q4c}$, such that the following holds.

Let $G$ be a graph and $J^*_\varpi$ a quasi-4-connected component of $G$ with index $\varpi$, and let $J_\varpi$ be the torso associated with $J^*_\varpi$. Suppose that $\Lambda^*[J^*_\varpi]$ is an ordered treelike decomposition of $J^*_\varpi$. Then $\Lambda(\overline{\varpi})$ defines an ordered treelike decomposition of $J_\varpi$ within $(G, \varpi)$.

In the proof of this lemma, we use a notation that will be useful on several occasions later in this book. For that reason, we lay out this notation in the following proviso.

**Proviso 10.3.3.** Suppose that $J^*$ is a quasi-4-connected component of a graph $G$, and $J$ and $M$ are the torso and matching associated with $J^*$.

For every edge $e \in M$ we let $v_e$ be the vertex of $J^*$ corresponding to $e$. We let

$$V_0 := V(J^*) \setminus \{v_e \mid e \in M\} = V(J) \setminus \bigcup M$$

For every $w \in V(J)$, we let

$$w^* := \begin{cases} w & \text{if } w \in V_0, \\ v_e & \text{if } w \text{ is an endvertex of an edge } e \in M. \end{cases}$$

For a set $W \subseteq V(J)$, we let

$$W^* := \{w^* \mid w \in W\} = (W \cap V_0) \cup \{v_e \mid e \cap W \neq \emptyset\}.$$ 

and for an induced subgraph $H$ of $J$ we let

$$H^* := J^*[V(H)^*].$$

Preliminary Version
Finally, for every set $W \subseteq V(J^*)$ we let
\[
W^\circ := (W \cap V_0) \cup \{v, w \mid e = vw \in M \text{ with } v_e \in W\}.
\]

Note that the vertices $v_e$ for $e \in M$ are mutually distinct, because $M$ is a matching. Furthermore, $V(J^*) = V(J)^*$ and thus indeed $J^* = J^*[V(J)^*]$. Thus the new notation is consistent with the $J/J^*$-notation used so far. Observe that $(W^\circ)^* = W$ for all $W \subseteq V(J^*)$ and $(W^\circ)^* \supseteq W$ for all $W \subseteq V(J)$.

**Proof of Lemma 10.3.3** To explain the definition of $\Lambda(\pi)$, we fix a graph $G$. We let $J^*$ be a quasi-4-connected component of $G$ and $J, M$ the torso and matching associated with $J^*$. Of course the definition of $\Lambda$ will not depend on our choice of $G$ or $J^*$. We use the notation introduced in Proviso 10.3.3. We let $\nu_j \in V(\Lambda_{q4c}(G))$ be an index of $J^*$. We assume that $\Delta^*:=(D^*,\sigma^*,\alpha^*,\leq^*) := \Lambda^*[J^*]$ is an ordered treelike decomposition of $J^*$. By Lemma 7.1.14 we may assume without loss of generality that $\Delta^*$ is tight.

We define a decomposition $\Delta' = (D',\sigma',\alpha')$ of $J$ as follows. We let $D' := D^*$, and for all $t \in V(D)$ we let $\sigma'(t) := (\sigma^*(t))^\circ$ and $\alpha'(t) := (\alpha^*(t))^\circ$.

**Claim 1.** $\Delta'$ is a treelike decomposition of $J$.

**Proof.** Straightforward.

**Claim 2.** There is a parameterised d-scheme $\Lambda'(\pi)$ (not depending on the graph $G$ or the index $\nu_j$ of $J$) such that $\Delta' = \Lambda'[G,\nu_j]$.

**Proof.** It is straightforward to define $\Lambda(\pi)$ from the od-scheme $\Lambda^*$ defining $\Delta^*$.

However, it is not obvious how to define linear orders on the bags of $\Delta'$ in a canonical way in order to expand the decomposition $\Delta'$ to an ordered decomposition. To be able to do so, we have to modify the decomposition by adding some nodes. While trying to order the bags of $\Delta'$, we shall define a new ordered decomposition $\Delta = (D,\sigma,\alpha,\leq)$ of $H$. Fortunately, we will be able to define $\Delta$ by modifying $\Delta'$ locally at every node. We proceed in a bottom up manner starting at the leaves of $D'$.

Let $t \in V(D')$. The linear order $\leq_t^\circ$ of $\beta^*(t)$ induces a partial order $\leq_t'$ of $\beta'(t)$ where for each $e = ab \in M$ with $v_e \in \beta^*(t)$, the vertex $v_e$ is replaced by $a$ and $b$. To extend this partial order to a linear order, we have to decide whether to put $a$ or $b$ first. We make a case distinction. In all except for one case, there is a canonical choice whether to put $a$ or $b$ first.

**Case 1:** $N^J(v_e) \cap \beta^*(t) \neq \emptyset$.

Let $w \in N^J(v_e) \cap \beta^*(t)$ be minimal with respect to $\leq_t^\circ$. By the Q4C Decomposition Lemma 10.2.4(x) we have $N^J(a) \cap N^J(b) = \emptyset$. Thus if $w \in V_0$, then $w$ is adjacent to either $a$ or $b$ in $J$, but not both. If $w = v_f$ for some $f \in M$, then again by the Q4C Decomposition Lemma 10.2.4(x) either $a$ or $b$, but not both, are adjacent to an endvertex of $f$. If $w$ or one of the vertices it comes from is adjacent to $a$, then we let $a <_t b$; otherwise we let $b <_t a$.

**Case 2:** $N^J(v_e) \cap \beta^*(t) = \emptyset$.

By the Q4C Decomposition Lemma 10.2.4(ix) we have $|N^J(a)| = 3$, say, $N^J(a) = \{b, a_1, a_2\}$, and $a_1a_2 \in E(J) \setminus M$, which implies $a_1^t \neq a_2^*$ and $a_1a_2^* \in E(J^*)$. Similarly,
We make a few more observations before we proceed with the definition of the new decomposition \( \Delta \). Let \( e = ab \) be t-critical and \( a_1, a_2, b_1, b_2, u_a, u_b \) as above. Then \( v_e \not\in \sigma^*(t) \) by the tightness of \( \Delta^* \), because \( N^J(v_e) \subseteq \sigma^*(u_a) \cup \sigma^*(u_b) \subseteq \sigma^*(u) \). Furthermore, we have \( v_e \in \sigma^*(u_a) = \sigma^*(u_b) \) by \([\text{TL.2}]\). There may be nodes \( u \in N^D_+(t) \setminus \{u_a\} \) with \( a_1, a_2 \in \sigma^*(u) \). However, by \([\text{TL.4}]\) for each such node \( u \) we have \( \sigma^*(u) = \sigma^*(u_a) \) and thus \( \sigma^*(u) = \sigma^*(u_a) \), because \( \Delta^* \) is tight and thus \( \sigma^*(u) = N^J(\sigma^*(u)) \). Of course the analogous statement holds for \( b \).

Note that if there are no \( t \)-critical edges, by the previous case distinction we know for all \( e = ab \in M \) whether \( a <_t b \) or \( b <_t a \). Combined with the partial order \( \leq_t \), this yields a linear order \( \leq_t \) of \( \beta'(t) \).

Otherwise, let \( e_1, \ldots, e_m \) be an enumeration of all \( t \)-critical edges. (We can actually include the case \( m = 0 \) that there are no \( t \)-critical edges here.) For all \( i \in [m] \), let \( v_i = v_{e_i} \), and let \( a_i, b_i \) be the endvertices of \( e_i \). Let \( u_{i,a}, u_{i,b} \in N^D_+(t) \) such that \( (N^J(b_i))^* \subseteq \sigma^*(u_{i,a}) \) and \( (N^J(b_i))^* \subseteq \sigma^*(u_{i,b}) \). Without loss of generality we may assume that \( v_1 <^*_t v_2 <^*_t \ldots <^*_t v_m \). We define a graph \( H \) with vertex set \( V(H) := [m] \) and edge set

\[
E(H) := \{ ij \in \binom{[m]}{2} \mid \sigma^*(u_{i,a}) = \sigma^*(u_{j,a}) \text{ or } \sigma^*(u_{i,a}) = \sigma^*(u_{j,b}) \text{ or } \\
\sigma^*(u_{i,b}) = \sigma^*(u_{j,a}) \text{ or } \sigma^*(u_{i,b}) = \sigma^*(u_{j,b}) \}.
\]

We let \( I_1, \ldots, I_\ell \) be the connected components of \( H \). (If \( m = 0 \) and thus \( H = \emptyset \) we let \( \ell = 0 \).) For every \( j \in [\ell] \), we let \( U_j \) be the set of all \( u \in N^D_+(t) \) such that there is an \( i \in V(I_j) \) with

Preliminary Version
\[ \alpha^*(u) = \alpha^*(u_{i,a}) \text{ or } \alpha^*(u) = \alpha^*(u_{i,b}). \] Furthermore, we let \( U_0 := N_{\Delta'}^D(t) \setminus \bigcup_{j=1}^\ell U_j \). (Note that if \( m = 0 \) then \( U_0 = N_{\Delta'}^D(t) \).) Observe that for all \( j \in [\ell] \), all \( i \in V(I_j) \), and all \( u \in U_j \) it holds that \( \sigma^*(u) = \sigma^*(u_{i,a}) = \sigma^*(u_{i,b}) \). Furthermore, \( v_i \in \sigma^*(u) \) and \( v_k \notin \sigma^*(u) \) for any \( k \in [m] \setminus V(I_j) \). The latter follows from the tightness of \( \Delta^* \).

We are now ready to define the modified decomposition \( \Delta \) in the vicinity of \( t \). Our inductive construction of \( \Delta \) will imply that \( u \in V(D) \) for all \( u \in N_{\Delta'}^D(t) \). We add \( t \) and fresh vertices \( t_{j,a} \) and \( t_{j,b} \) for all \( j \in [\ell] \) to \( V(D) \). We let \( N_{\Delta'}^J(t) := U_0 \cup \{ t_{j,a}, t_{j,b} \mid j \in [\ell] \} \). For every \( j \in [\ell] \), we let \( N_{\Delta'}^D(t_{j,a}) := N_{\Delta'}^D(t_{j,b}) := U_j \). We let
\[
\sigma(t) := \sigma'(t) \quad \text{and} \quad \alpha(t) := \alpha'(t).
\]
For each \( j \in [\ell] \), we let
\[
\sigma(t_{j,a}) := \sigma(t_{j,b}) := \sigma'(u) \setminus \{ a_i, b_i \mid i \in V(I_j) \}
\]
for some (and hence for all) \( u \in U_j \). We let
\[
\alpha(t_{j,a}) := \alpha(t_{j,b}) := \{ a_i, b_i \mid i \in V(I_j) \} \cup \bigcup_{u \in U_j} \alpha'(u).
\]
This completes our definition of the decomposition \((D, \sigma, \alpha)\). Observe that
\[
\gamma(t_{j,a}) = \bigcup_{u \in U_j} \gamma'(u).
\]

**Claim 3.** \((D, \sigma, \alpha)\) is a treelike decomposition of \( J \).

**Proof.** \( [\text{TL.1}] \) is obvious from the construction of \( D \) from the acyclic directed graph \( D' \).

To prove \( [\text{TL.2}] \) let \( x \in V(D) \). If \( x = t \in V(D') \) then the assertion follows from \( [\text{TL.2}] \) for \( \Delta' \). So suppose that \( x \in \{ t_{j,a}, t_{j,b} \} \) for some \( t \in V(D') \) and \( j \in [\ell] \). (Here \( \ell \) depends on \( t \).) Using the same notation as above, we have \( \sigma(x) = \sigma'(u) \setminus \{ a_i, b_i \mid i \in V(I_j) \} \) for some \( u \in U_j \) and \( \alpha(x) = \bigcup_{u' \in U_j} \alpha'(u') \cup \{ a_i, b_i \mid i \in V(I_j) \} \). Remember that for all \( u' \in U_j \) we have \( \sigma'(u') = \sigma'(u) \) and thus \( \sigma'(u') \cap \sigma'(u) = \emptyset \). This implies \( \sigma(x) \cap \alpha(x) = \emptyset \). We have \( N^J(\alpha(x)) \subseteq \sigma(x) \) because for all \( i \in V(I_j) \) it holds that \( N^J(a_i) \cup N^J(b_i) \subseteq \bigcup_{u' \in U_j} \alpha'(u') \).

To prove \( [\text{TL.3}] \) let \( x \in V(D) \) and \( y \in N_{\Delta'}^D(x) \).

**Case A:** \( x = t \in V(D') \).

If \( y = u \in U_0 \), then \( \gamma(y) = \gamma'(u) \subseteq \gamma'(t) = \gamma(x) \) by \( [\text{TL.3}] \) for \( \Delta' \) and similarly \( \alpha(y) \subseteq \alpha(x) \). Suppose that \( y = t_{j,a} \) or \( y = t_{j,b} \) for some \( j \in [\ell] \). Then
\[
\alpha(y) = \{ a_i, b_i \mid i \in V(I_j) \} \cup \bigcup_{u \in U_j} \alpha'(u).
\]
Since for all \( i \in [m] \) we have \( v_i \in \beta^*(t) \setminus \sigma^*(t) \subseteq \alpha^*(t) \), we have \( a_i, b_i \in \alpha'(t) = \alpha(x) \). By \( [\text{TL.3}] \) for \( \Delta' \) we have \( \alpha'(u) \subseteq \alpha'(t) = \alpha(x) \) for all \( u' \in U_j \subseteq N_{\Delta'}^D(t) \). Hence \( \alpha(y) \subseteq \alpha(x) \). Furthermore, for some \( u \in U_j \) we have
\[
\sigma(y) \subseteq \sigma'(u) \subseteq \gamma'(u) \subseteq \gamma'(t) = \gamma(x),
\]
which implies \( \gamma(y) \subseteq \gamma(x) \).

M. Grohe, Definable Graph Structure Theory
10.3. The Q4C Lifting Lemma

Case B: \( x = t_{j,a} \) or \( x = t_{j,b} \) for some \( t \in V(D') \) and \( j \in [\ell] \).

Then

\[
\alpha(x) = \{ a_i, b_i \mid i \in V(I_j) \} \cup \bigcup_{u \in U_j} \alpha'(u)
\]

and

\[
\gamma(x) = \bigcup_{u \in U_j} \gamma'(u).
\]

Furthermore, we have \( y \in N^D_+(x) = U_j \), say, \( y = u \). Hence \( \alpha(y) = \alpha'(u) \subseteq \alpha(x) \) and similarly \( \gamma(y) \subseteq \gamma(x) \).

To prove (TL.4) let \( x \in V(D) \) and \( y_1, y_2 \in N^D_+(x) \).

Case A: \( x = t \in V(D') \).

Case A1: \( y_1, y_2 \in U_0 \subseteq N^D_+(t) \)

Then the assertion follows from (TL.4) for \( \Delta' \).

Case A2: \( y_1 \in U_0 \) and \( y_2 \in \{ t_{j,a}, t_{j,b} \} \) for some \( j \in [\ell] \).

Say, \( y_1 = u_1 \). Then \( \gamma(y_1) = \gamma'(u_1) \) and

\[
\gamma(y_2) = \bigcup_{u \in U_j} \gamma'(u).
\]

Remember that for all \( u \in U_j \) we have \( \alpha'(u_1) \neq \alpha'(u) \) and thus \( \gamma'(u_1) \cap \gamma'(u) = \sigma'(u_1) \cap \sigma'(u) \) by (TL.4) for \( \Delta' \). Hence

\[
\gamma(y_1) \cap \gamma(y_2) = \gamma'(u_1) \cap \bigcup_{u \in U_j} \gamma'(u) = \sigma'(u_1) \cap \sigma'(u) = \sigma(y_1) \cap \sigma(y_2)
\]

for some and hence for all \( u \in U_j \).

Case A3: \( y_1 \in \{ t_{j,a}, t_{j,b} \} \) for some \( j \in [\ell] \) and \( y_2 \in U_0 \).

Symmetric to Case A2.

Case A4: \( y_1 \in \{ t_{j_1,a}, t_{j_1,b} \} \) and \( y_2 \in \{ t_{j_2,a}, t_{j_2,b} \} \) for some \( j_1, j_2 \in [\ell] \) with \( j_1 \neq j_2 \).

Then for \( h = 1, 2 \) we have

\[
\gamma(y_h) = \bigcup_{u \in U_{j_h}} \gamma'(u).
\]

For all \( u_1 \in U_{j_1} \) and \( u_2 \in U_{j_2} \) we have \( \alpha'(u_1) \neq \alpha'(u_2) \) and thus \( \gamma'(u_1) \cap \gamma'(u_2) = \sigma'(u_1) \cap \sigma'(u_2) \). Furthermore, for all \( i \in I_{j_1} \) and \( u_2 \in U_{j_2} \) we have \( a_i, b_i \notin \sigma'(u_2) \) and similarly for all \( u_1 \in U_{j_1} \) and \( i \in I_{j_2} \) we have \( a_i, b_i \notin \sigma'(u_1) \). This implies \( \sigma'(u_1) \cap \sigma'(u_2) = \sigma(u_1) \cap \sigma(u_2) \) for all \( u_1 \in U_1 \) and \( u_2 \in U_2 \). Hence

\[
\gamma(y_1) \cap \gamma(y_2) = \bigcup_{u \in U_{j_1}} \gamma'(u) \cap \bigcup_{u \in U_{j_2}} \gamma'(u) = \sigma'(u_1) \cap \sigma'(u_2) = \sigma(y_1) \cap \sigma(y_2)
\]

for some and hence for all \( u_1 \in U_{j_1}, u_2 \in U_{j_2} \).

Case A5: \( y_1 \in \{ t_{j,a}, t_{j,b} \} \) and \( y_2 \in \{ t_{j,a}, t_{j,b} \} \) for some \( j \in [\ell] \).

Then \( y_1 \parallel y_2 \).
Case B: $x = t_{j,a}$ or $x = t_{j,b}$ for some $t \in V(D')$ and $j \in [\ell]$. Then $y_1, y_2 \in U_j \subseteq N_+^c(t)$, and the assertion follows from (TL.4) for $\Delta'$.

Finally, (TL.5) follows immediately from (TL.5) for $\Delta'$.

Claim 4. For all $t \in V(D')$ we have

$$\beta(t) = \beta'(t) \setminus \{a_1, b_1, a_2, b_2, \ldots, a_m, b_m\}.$$ 

Furthermore, for all $j \in [\ell]$ we have

$$\beta(t_{j,a}) = \beta(t_{j,b}) = \sigma'(u)$$

for some and hence for all $u \in U_j$. (Here $\ell, m, a_1, b_1, \ldots, a_m, b_m, U_j$ are defined as above, depending on $t$.)

Proof. This follows immediately from the definitions. 

The most difficult step in the definition of our ordered treelike decomposition $\Delta = (D, \sigma, \alpha, \leq)$ is the definition of the linear orders $\leq_x$ for $x \in V(D)$. We define $\leq_x$ for $x = t \in V(D')$ first. This is easy, because we have removed the endvertices $a_i, b_i$ of all $t$-critical edges $e_i = a_i b_i$ from $\beta(t)$. In the Case 1–2d above we have already decided whether $a < t b$ or $b < t a$ for all non-critical edges $e = ab \in M \setminus \{e_1, \ldots, e_m\}$. Using this, we can extend the restriction of the partial order $\leq'_t$ to $\beta(t)$ to a total linear order.

Let us now define $\leq_x$ for $x = t_{j,a}$ and $x = t_{j,b}$, for some $t \in V(D')$ and $j \in [\ell]$. The crucial observation is that while we still have no canonical way of deciding whether $a < t b$ or $b < t a$ for the $t$-critical indices $i \in V(I_j)$, once we have made the choice for one $i \in V(I_j)$, we have a canonical way of extending it to all other $i' \in V(I_j)$. So we only have to make one arbitrary choice. We introduced two nodes $t_{j,a}$ and $t_{j,b}$, which so far are indistinguishable, to have one for both choices of a linear order. Recall that the natural order of the indices $i \in V(I_j) \subseteq [m]$ coincides with order $\leq'_t$ on the $v_i$. Let $i_{\min}(j) := \min(V(I_j))$. We traverse the connected graph $I_j$ by a depth first search starting at $i_{\min}(j)$ that, at each node, uses the natural order on $V(I_j)$ to decide which outgoing edge to traverse next. Let $i_1 = i_{\min}(j), \ldots, i_p$ be the resulting enumeration of $V(I_j)$. To simplify the notation, let us assume that $i_k = k$ for all $k \in [p]$ (hence $V(I_j) = [p]$).

a. We inductively define a linear order $\leq_a$ on $\{a_k, b_k \mid k \in [p]\}$ as follows: We let $a_1 \leq_a b_1$. If $\leq_a$ is defined on $\{a_1, b_1, \ldots, a_k, b_k\}$, we extend it to $a_{k+1}, b_{k+1}$ by first letting $a_i, b_i < a_{k+1}, b_{k+1}$ for all $i \in [k]$. To decide whether $a_{k+1} < b_{k+1}$ or $b_{k+1} < a_{k+1}$, consider the edge $\{i, k+1\} \in E(I_j)$ used to enter $k+1$ in the depth first search. By the definition of $E(H)$, either $\alpha^*(u_{i,a}) = \alpha^*(u_{k+1,a})$ or $\alpha^*(u_{i,a}) = \alpha^*(u_{k+1,b})$ or $\alpha^*(u_{i,b}) = \alpha^*(u_{k+1,a})$ or $\alpha^*(u_{i,b}) = \alpha^*(u_{k+1,b})$. These cases are not mutually exclusive, but not all combinations are possible because $\alpha^*(u_{i,a}) \neq \alpha^*(u_{i,b})$ and $\alpha^*(u_{k+1,a}) \neq \alpha^*(u_{k+1,b})$.

Suppose first that $a_i < b_i$. If $\alpha^*(u_{i,a}) = \alpha^*(u_{k+1,a})$ or $\alpha^*(u_{i,b}) = \alpha^*(u_{k+1,b})$ (possibly both) we let $a_{k+1} < a_{k+1}$. If $\alpha^*(u_{i,a}) = \alpha^*(u_{k+1,b})$ or $\alpha^*(u_{i,b}) = \alpha^*(u_{k+1,a})$ (possibly both) we let $b_{k+1} < a_{k+1}$.

Suppose next that $b_i < a_i$. If $\alpha^*(u_{i,a}) = \alpha^*(u_{k+1,a})$ or $\alpha^*(u_{i,b}) = \alpha^*(u_{k+1,b})$ (possibly both) we let $b_{k+1} < a_{k+1}$. If $\alpha^*(u_{i,a}) = \alpha^*(u_{k+1,b})$ or $\alpha^*(u_{i,b}) = \alpha^*(u_{k+1,a})$ (possibly both) we let $a_{k+1} < a_{k+1}$. 

M. Grohe, *Definable Graph Structure Theory*
b. We inductively define a linear order \( \leq_b \) on \( \{a_k, b_k \mid k \in [p]\} \) completely analogously as the order \( \leq_a \), except that we start the induction by letting \( b_1 <_b a_1 \).

Now we define the linear order \( \leq_{t,j,a} \) on \( \beta(t_{j,a}) \) by

\[
v \leq_{t,j,a} w \iff \begin{cases} v, w \in \{a_i, b_i \mid i \in V(I_j)\} \text{ and } v \leq_a w, \\
v \in \{a_i, b_i \mid i \in V(I_j)\} \text{ and } w \in \beta(t_{j,a}) \setminus \{a_i, b_i \mid i \in V(I_j)\} \\
v, w \in \beta(t_{j,a}) \setminus \{a_i, b_i \mid i \in V(I_j)\} \text{ and } v \leq_t w.\end{cases}
\]

We define the linear order \( \leq_{t,j,b} \) on \( \beta(t_{j,b}) = \beta(t_{j,a}) \) analogously using \( \leq_b \) instead of \( \leq_a \). This completes the definition of \( \leq \) and hence of the decomposition \( \Delta = (D, \sigma, \alpha, \leq) \). It is immediate from the definition that \( \leq_x \) is a linear order of \( \beta(x) \) for each \( x \in V(D) \). Hence \( \Delta \) is an ordered treelike decomposition of \( J \).

It remains to prove that the ordered decomposition \( \Delta \) is IFP-definable. The main difficulty is that it is not obvious how to represent the nodes of \( D \) by tuples of elements of \( G \). Suppose that the d-scheme \( \Lambda'(\overline{x}) \) that defines \( \Delta' \) is \( d \)-dimensional. Then we represent the nodes of \( D \) by \((d + 2)\)-tuples in the following way. Let \( t = (v_1, \ldots, v_d) \in V(\Lambda'(G, \overline{\pi}_J)) = V(D') \).

- The node \( t \in V(D) \) is represented by the tuple \((v_1, \ldots, v_d, v_d, v_d)\).
- Let \( j \in [\ell] \) and \( i := i_{\min}(j) \). Then the node \( t_{j,a} \) is represented by the tuple \((v_1, \ldots, v_d, a_i, b_i)\) and the node \( t_{j,b} \) is represented by the tuple \((v_1, \ldots, v_d, b_i, a_i)\).

With this representation of the nodes, it is straightforward to define a \((d + 2)\)-dimensional od-scheme \( \Lambda(\overline{x}) \) that defines \( \Delta \) within \((G, \overline{\pi}_J)\).

**Proof of the Q4C Lifting Lemma 10.3.1** Let \( \Lambda^1 := \Lambda_{q4c} \) be the d-scheme of the Q4C Decomposition Lemma 10.2.4. Let \( \Lambda^* \) be an od-scheme with \( C^* \subseteq OT_{\Lambda^*} \), and let \( \Lambda^2(\overline{x}) \) be the od-scheme obtained by applying Lemma 10.3.2 to \( \Lambda^* \). Then for every graph \( G \in C \) and every node \( \overline{\pi} \in V(\Lambda^1[G]) \) the scheme \( \Lambda^2(\overline{x}) \) defines an ordered treelike decomposition of \( \tau^{\Lambda^1[G]}(\overline{\pi}) \). An application of the Parametrised Ordered Decomposition Lifting Lemma 7.2.5 yields an o-scheme \( \Lambda \) with \( C^* \subseteq OT_{\Lambda} \). \(\square\)
Chapter 11

K₅-Minor Free Graphs

In this short chapter, we apply the decompositions into quasi-4-connected components to K₅-minor free graphs. The main result is a definable decomposition theorem stating that K₅-minor free graphs admit IFP-definable treelike decompositions into 3-connected planar graphs and the graph L displayed in Figure [11.1]; this theorem was already mentioned in the first paragraph of the introduction to this book. A consequence of the decomposition theorem and the Definability Lifting Lemma 5.4.3 is that the class of all K₅-minor free graphs is IFP-definable.

11.1 Decompositions

Let L be the graph displayed three times in Figure 11.1 and once, with labelled vertices, in Figure [11.2]. The second drawing of L in Figure 11.1 shows that K₃,₃ is a minor of L. Hence L is not planar. However, L does not contain K₅ as a minor. To see this, note that to obtain K₅ from L, we have to contract at least 5 edges to generate 5 vertices of degree 4. But then only 3 vertices remain.

An edge-maximal K₅-minor free graph is a K₅-minor free graph G such that for all G' with V(G') = V(G) and E(G') ⊃ E(G) it holds that K₅ ⪯ G'.

Fact 11.1.1 (Wagner [127]). Let G be an edge-maximal K₅-minor free graph that has no separating clique. Then either G is planar or G ≅ L.

Let L be the class of all graphs isomorphic to L, and recall that P denotes the class of planar graphs. It follows from Fact 11.1.1 that every K₅-minor free graph has a tree decomposition over P ∪ L. We shall prove later that the class of K₅-minor free graphs admits IFP-definable treelike decomposition over P ∪ L. The following consequence of Fact 11.1.1 is our starting point.

Lemma 11.1.2. Let G be a quasi-4-connected K₅-minor free graph. Then either G is planar or G ≅ L.

Proof. Suppose that the lemma is false and that G is a counterexample of minimum order. Then G is not planar and hence |G| ≥ 5. Let G' ⊇ G with V(G') = V(G) be edge-maximal K₅-minor free. If G' has no separating clique, then by Wagner’s Theorem (Fact 11.1.1), either G' ≅ L or G' is planar. If G' is planar, then G is planar as well. Suppose that G' is isomorphic...
to \(L\). We shall prove that in this case either \(G\) is isomorphic to \(G'\) or \(G\) is planar. Suppose that \(G\) is not isomorphic to \(G'\). Then there is at least one edge in \(E(G') \setminus E(G)\). Consider the third drawing of \(G' = L\) in Figure 11.2. If one of the horizontal edges (including the curved ones) is in \(E(G') \setminus E(G)\), then \(G\) is planar, as can easily be seen from the drawing. (By symmetry we may assume that one of the crossing edges is deleted.) So suppose that all horizontal edges are in \(E(G)\). If at least two vertical edges are in \(E(G') \setminus E(G)\), then again \(G\) is planar. To see this, suppose for example that the edges 26 and 48 are in \(E(G') \setminus E(G)\). Then we can redraw the graph by placing vertex 8 in the interior of the cycle 1237651. So suppose that \(E(G') \setminus E(G)\) consists of precisely one vertical edge. Then \(G\) is not quasi-4-connected, actually not even 3-connected. To see this, suppose for example that \(E(G') \setminus E(G) = \{26\}\). Then \(\{1, 3\}\) is a separator of \(G\) of order 2. This is a contradiction. Hence \(G \cong G' \cong L\).

In the following, suppose that \(G'\) has a separating clique. Let \(S\) be a minimal separating clique of \(G'\). Then \(|S| \geq 3\) because \(G' \supseteq G\) is 3-connected. Furthermore, \(|S| \leq 3\). To see this, suppose for contradiction that \(|S| \geq 4\). Let \(A\) be a connected component of \(G' \setminus S\). By the minimality of \(S\) it holds that \(N^{G'}(A) = S\). But then the minor \(G'/A\) of \(G'\) contains \(K_5\), which is a contradiction. Hence \(|S| = 3\).
Since \( V(G) = V(G') \) and \( E(G) \subseteq E(G') \), the set \( S \) is also a 3-separator of \( G \). As \( G \) is quasi-4-connected, \( S \) must be irrelevant. Thus \( G \setminus S \) has precisely two connected components, one of which consists of a single vertex \( v \). Let \( H := (G \setminus \{v\}) \cup K[S] \). Then \( H \) is \( K_5 \)-minor free, because \( H \subseteq G' \). Furthermore, \( H \) is quasi-4-connected, because \( G \) is. By the minimality of \( G \), it holds that \( H \) is either planar or isomorphic to \( L \). As \( S \) is a 3-clique in \( H \) and \( L \) contains no 3-clique, we have \( H \not\cong L \). Hence \( H \) is planar. We fix some planar embedding of \( H \). Then the 3-clique \( S \) becomes a simple closed curve in the plane. If one of the two regions of the plane bounded by this simple closed curve has an empty intersection with \( H \), then we can extend the embedding to \( G \) by embedding the vertex \( v \) into this region. This contradicts our assumption that \( G \) is not planar. Hence both regions of the plane bounded by the simple closed curve have a nonempty intersection with \( H \). But then \( S \) separates \( H \), and therefore, \( G \setminus S \) has at least three connected components. This is a contradiction.

**Corollary 11.1.3.** The class of all quasi-4-connected \( K_5 \)-minor free graphs admits \( \text{IFP}\)-definable orders.

**Proof.** This follows from Lemma 11.1.2, Theorem 9.3.1 (stating that the class of 3-connected planar graphs admits \( \text{IFP}\)-definable orders) and the Union Lemma for Definable Orders 3.2.10.

**Corollary 11.1.4.** The class of \( K_5 \)-minor free admits an \( \text{IFP}\)-definable ordered treelike decompositions.

**Proof.** This follows from Corollary 11.1.3 by the Q4C Lifting Lemma 10.3.1.

**Corollary 11.1.5.** The logic \( \text{IFP} + C \) captures PTIME on the class of \( K_5 \)-minor free graphs.

While Corollary 11.1.4 follows directly from Lemma 11.1.2 by an application the Q4C Lifting Lemma, the following more informative decomposition theorem follows from the lemma by an application of the Q4C Decomposition Lemma 10.2.4. However, it requires a few additional arguments, which may serve as a first illustration of how the detailed technical assertions of the Q4C Decomposition Lemma come into play.

**Theorem 11.1.6.** The class of \( K_5 \)-minor free graphs admits \( \text{IFP}\)-definable treelike decompositions over \( \mathbb{Z}_3^* \cap (P \cup L) \).

**Proof.** Let \( G \) be \( K_5 \)-minor free and \( \Delta_{q4c} := (D_{q4c}, \sigma_{q4c}, \alpha_{q4c}) := \Lambda_{q4c}[G] \) the decomposition of \( G \) obtained from the Q4C Decomposition Lemma 10.2.4. For all \( t \in V(D_{q4c}) \), let \( J_t := r_{q4c}(t) \) be the torso at \( t \) and \( M_t \subseteq E(J_t) \) be the matching associated with \( t \). Let \( J^*_t := J_t/M_t \). Then \( J^*_t \) is a minor of \( G \) by (iv) (of the Q4C Decomposition Lemma) and quasi-4-connected by (vi). Thus by Lemma 11.1.2 we have \( J_t^* \in P \cup L \). We shall prove that \( J_t^* \in P \cup L \). The theorem will follow, because \( J_t \in \mathbb{Z}_3^* \) by (v).

**Case 1:** \( J_t^* \in L \).

In this case, we have \( M_t = \emptyset \) and thus \( J_t = J_t^* \in L \). To see this, suppose for contradiction that \( M_t \neq \emptyset \). Let \( e \in M_t \), and let \( v_e \in V(J_t^*) \) be the vertex of \( J_t^* \) corresponding to \( e \). It follows from (ix) and (x) that \( v_e \) has degree 4. This is impossible, because \( J_t^* \cong L \) is 3-regular.
Case 2: \( J_t^* \in \mathcal{P} \).

We assume that \( J_t^* \) is embedded in the sphere and show how to extend the embedding to \( J_t \). Consider an edge \( e = vw \in M_t \), and let \( v_e \) be the vertex of \( J_t^* \) corresponding to \( e \). By (ix) we have \( |N_{J_t}(v) \setminus \{w\}| = |N_{J_t}(w) \setminus \{v\}| = 2 \). Say, \( N_{J_t}(v) \setminus \{w\} = \{v', v''\} \) and \( N_{J_t}(w) \setminus \{v\} = \{v', v''\} \). By (ix) and (x), \( v', v'', w', w'' \) correspond to four distinct vertices of \( J_t^* \), which for simplicity we denote by \( v', v'', w', w'' \) as well (even though they may actually be obtained by contracting other edges of \( M_t \)). Then \( N_{J_t}(v) = \{v', v'', w', w''\} \).

Since \( v'v'', w'w'' \in E(J_t^*) \) by (ix) both \( v', v'' \) and \( w', w'' \) must be adjacent in the cyclic ordering of the four vertices around \( v_e \) induced by the embedding of \( J_t^* \). But this means that we can “uncontract” the edge \( e \) and still embed the graph in the sphere by only changing the embedding locally around \( v_e \). We can do this independently for all edges in \( M_t \) and hence obtain an embedding of \( J_t \) in the sphere.

11.2 Definability

We now turn to the second main result of this chapter, the \( \text{IFP} \)-definability of the class of \( K_5 \)-minor free graphs. It follows from Corollary 2.2.4 that the class of all \( K_5 \)-minor free graphs is decidable in polynomial time. Combined with Corollary 11.1.5, this implies that the class is \( \text{IFP+C} \)-definable. To prove that is \( \text{IFP} \)-definable, we need the following lemma.

Lemma 11.2.1. The class \( \mathcal{P} \cup \mathcal{L} \) is \( \text{IFP} \)-definable.

Proof. This follows from Theorem 9.3.5.

Lemma 11.2.2. A graph is \( K_5 \)-minor free if and only if it has a tree decomposition over \( \mathcal{P} \cup \mathcal{L} \).

Proof. The forward direction follows from Theorems 11.1.6 and 4.6.8. To prove the backward direction, let \( \Delta = (T, \sigma, \alpha) \) be a tree decomposition of a graph \( G \) over \( \mathcal{P} \cup \mathcal{L} \). Suppose for contradiction that \( K_5 \subseteq G \), and let \( (Y_i)_{i \in [5]} \) be an image of \( K_5 \) in \( G \). Then by Fact 4.1.3(3) there is a \( t \in V(T) \) such that \( Y_i \cap \beta(t) \neq \emptyset \) for all \( i \in [5] \). This implies that \( K_5 \subseteq \tau(t) \in \mathcal{P} \cup \mathcal{L} \), which is a contradiction.

Theorem 11.2.3. The class of \( K_5 \)-minor free graphs is \( \text{IFP} \)-definable.

Proof. Follows from Theorem 11.1.6 and the previous two lemmas by Corollary 5.4.4 (to the Definability Lifting Lemma).

Remark 11.2.4. The results of this chapter can be proved rather effortlessly from the theory developed so far in this book. To appreciate what we achieved so far, the reader may try to find a direct \( \text{IFP} \)-definition of the class of \( K_5 \)-minor free graphs. This seems to be very difficult.
Chapter 12

Completions of Pre-Decompositions

We introduce pre-decompositions, which may be viewed as decompositions where the nodes have not yet been arranged in a digraph. Pre-decompositions will be used to construct treelike decompositions inductively; the inductive step of such a construction will be a completion of a pre-decomposition.

Pre-decompositions and their completions will play a crucial role in the inductive proof of our main theorem in Chapter 17. Until then, they will be a nuisance, because for several nice theorems stating the existence of certain definable decompositions, we will have to prove cumbersome versions that state the existence of definable completions of pre-decompositions.

12.1 Pre-Decompositions and Completions

Definition 12.1.1. (1) A pre-decomposition of a graph \( G \) is a triple \( \Phi = (V(\Phi), \sigma^\Phi, \alpha^\Phi) \), where \( V(\Phi) \) is a set and \( \sigma^\Phi, \alpha^\Phi : V(\Phi) \to 2^{V(G)} \).

(2) A pre-decomposition \( \Phi \) is proper if for all \( t \in V(\Phi) \) it holds that \( \sigma^\Phi(t) \cap \alpha^\Phi(t) = \emptyset \) and \( N^G(\alpha^\Phi(t)) \subseteq \sigma^\Phi(t) \).

(3) A pre-decomposition \( \Phi \) is tight if for all \( t \in V(\Phi) \) it holds that \( \alpha^\Phi(t) \) is connected in \( G \) and \( \sigma^\Phi(t) = N^G(\alpha^\Phi(t)) = \partial^G(\gamma^\Phi(t)) \), where \( \gamma^\Phi(t) := \sigma^\Phi(t) \cup \alpha^\Phi(G) \).

(4) The adhesion of a pre-decomposition \( \Phi \) is \( \max \{ |\sigma^\Phi(t)| \mid t \in V(\Phi) \} \).

Observe that every tight pre-decomposition is proper. Also observe that every decomposition \( (D, \sigma, \alpha) \) yields a pre-decomposition \( (V(D), \sigma, \alpha) \) obtained by dropping the digraph structure. Pre-decompositions are not particularly rich or interesting structures in themselves; the interesting question is whether they can be completed to treelike decompositions with certain desirable properties.

Definition 12.1.2. Let \( G \) be a graph and \( \Phi \) be a pre-decomposition of \( G \). A completion of \( \Phi \) over a class \( \mathcal{C} \) of graphs is a treelike decomposition \( \Delta \) of \( G \) such that for all \( t \in V(D^\Delta) \),

- either \( \tau^\Delta(t) \in \mathcal{C} \)
- or \( t \) is a leaf of \( D^\Delta \) and there is a \( t' \in V(\Phi) \) such that \( \sigma^\Delta(t) = \sigma^\Phi(t') \) and \( \alpha^\Delta(t) = \alpha^\Phi(t') \).

A completion of \( \Phi \) is a completion of \( \Phi \) over the class of all graphs.
Sometimes $\Phi$ may be viewed as a pre-decomposition of several graphs, in particular, of a graph $G$ and a subgraph $H$ of $G$. Then if $\Delta$ is a completion of $\Phi$, we speak of a completion of $\Phi$ in $G$ or in $H$ to make it clear which graph we are referring to. If $\Delta$ is a completion of $\Phi$, then we call nodes $t \in V(D^\Delta)$ and $t' \in V(\Phi)$ parallel (we write $t \parallel_{\Delta,\Phi} t'$ or just $t \parallel t'$) if $\sigma^\Delta(t) = \sigma^\Phi(t')$ and $\alpha^\Delta(t) = \alpha^\Phi(t')$. We call leaves $t \in V(D^\Delta)$ that are parallel to a node $t' \in V(\Phi)$ ground leaves of the completion $\Delta$, and we call all nodes of $\Delta$ that are not ground leaves completion nodes. Of course $\Delta$ may be a completion of more than one pre-decomposition, and it may have different ground leaves with respect to these pre-decompositions. Also note that $\Delta$ may have no ground leaves at all. In fact, for every pre-decomposition $\Phi$ of $G$ every treelike decomposition of $G$ over $\mathcal{C}$ is a completion of $\Phi$ over $\mathcal{C}$, possibly without any ground leaves.

**Lemma 12.1.3.** Let $\Phi$ be a tight pre-decomposition of a graph $G$, and let $\Delta = (D, \sigma, \alpha)$ be a completion of $\Phi$ in $G$. Then there exists a tight treelike decomposition $\Delta' = (D', \sigma', \alpha')$ of $G$ and a strong homomorphism $h$ from $D'$ to $D$ such that for every node $t' \in V(D')$ with $t := h(t')$ the following two conditions are satisfied.

(i) $\beta'(t') \subseteq \beta(t)$ and $\gamma'(t') \subseteq \gamma(t)$ and $\sigma'(t') \subseteq \sigma(t)$ and $\tau'(t') \subseteq \tau(t)$.

(ii) If $t$ is a ground leaf of $\Delta$ then $t'$ is a leaf of $D'$ with $t \parallel_{\Delta,\Delta'} t'$.

In particular, $t'$ is a ground leaf of $\Delta'$ (viewed as a completion of $\Phi$).

**Proof.** The construction of a tight decomposition in the proof of Lemma 4.4.2 yields this stronger result. \qed

We also need a version of the lemma for tree decompositions.

**Corollary 12.1.4.** Let $\Phi$ be a tight pre-decomposition of a connected graph $G$, and let $\Delta = (D, \sigma, \alpha)$ be a completion of $\Phi$ in $G$. Then there exists a tight tree decomposition $\Delta' = (T', \sigma', \alpha')$ of $G$ and a strong homomorphism $h$ from $T'$ to $D$ such that for every node $t' \in V(T')$ with $t := h(t')$ the following two conditions are satisfied:

(i) $\beta'(t') \subseteq \beta(t)$ and $\gamma'(t') \subseteq \gamma(t)$ and $\sigma'(t') \subseteq \sigma(t)$ and $\tau'(t') \subseteq \tau(t)$.

(ii) If $t$ is a ground leaf of $\Delta$ then $t'$ is a leaf of $T'$ with $t \parallel_{\Delta,\Delta'} t'$.

In particular, $t'$ is a ground leaf of $\Delta'$.

**Proof.** This follows from the previous lemma and Corollary 4.6.7. \qed

For our definable structure theory, we need a notion of definable pre-decomposition.

**Definition 12.1.5.** (1) A pd-scheme is a triple $\Psi := (\psi_V(\overline{x}), \psi_\sigma(\overline{x}, y), \psi_\alpha(\overline{x}, y))$ of $\forall \exists \exists$-formulae. The dimension of $\Psi$ is the length of the tuple $\overline{x}$.

(2) Let $G$ be a graph and $\Psi$ a pd-scheme. We let $V(\Psi[G]) := \psi_V(G, \overline{x})$, and for every $\overline{v} \in V(\Psi[G])$ we let $\sigma^{\Psi[G]}(\overline{v}) := \psi_\sigma(G, \overline{v}, \overline{x})$ and $\alpha^{\Psi[G]}(\overline{v}) := \psi_\alpha(G, \overline{v}, \overline{x})$. We let

$$\Psi[G] := \left( V(\Psi[G]), \sigma^{\Psi[G]}, \alpha^{\Psi[G]} \right)$$

and call it the pre-decomposition defined by $\Psi$ in $G$. \qed
Where necessary, we also use parametrised pd-schemes $\Psi(X)$ with the obvious meaning and a similar terminology as for parametrised d-schemes. For example, we may say that $\Psi(X)$ defines a pre-decomposition $\Phi$ within $(G, \overline{P})$.

**Example 12.1.6.** Let $G$ be a graph and $k \in \mathbb{N}$. The pre-decomposition $\Phi_{G,k}$ is defined as follows. For every tuple $(v_1, \ldots, v_{k+1}) \in V(G)^{k+1}$ we let $S_{\overline{v}} := \overline{v} \setminus \{v_1\}$, and we let $A_{\overline{v}}$ be the connected component of $G \setminus S_{\overline{v}}$ that contains $v_1$. We let

$$V(\Phi_{G,k}) := \{(v_1, \ldots, v_{k+1}) \in V(G)^{k+1} \mid S_{\overline{v}} = N^G(A_{\overline{v}}) = \partial^G(V(A_{\overline{v}} \cup S_{\overline{v}}))\}$$

and $\sigma_{\Phi_{G,k}}(\overline{v}) := S_{\overline{v}}$ and $\alpha_{\Phi_{G,k}}(\overline{v}) := V(A_{\overline{v}})$ for all $\overline{v} \in V(\Phi_{G,k})$. We call $\Phi_{G,k}$ the generic pre-decomposition of $G$ of adhesion $k$.

It is easy to see that there is a pd-scheme $\Psi_{G,k}$ such that for all graphs $G$ we have $\Psi_{k}[G] = \Phi_{G,k}$.

If we are only interested in completions of pre-decompositions, the names of the nodes are irrelevant. This leads to the following notions of containment and equivalence between pre-decompositions.

**Definition 12.1.7.** Let $\Phi, \Phi'$ be pre-decompositions of a graph $G$.

1. $\Phi$ is contained in $\Phi'$ (we write $\Phi \preceq \Phi'$) if for every $t \in V(\Phi)$ there is a $t' \in V(\Phi')$ such that $\sigma^\Phi(t) = \sigma^{\Phi'}(t')$ and $\alpha^\Phi(t) = \alpha^{\Phi'}(t')$.

2. $\Phi$ is equivalent to $\Phi'$ if $\Phi \preceq \Phi'$ and $\Phi' \preceq \Phi$.

Observe that if $\Phi$ and $\Phi'$ are equivalent pre-decompositions of a graph $G$ and $\Delta$ is a treelike decomposition of $G$, then $\Delta$ is a completion of $\Phi$ if and only if it is a completion of $\Phi'$.

**Definition 12.1.8.** Let $\Phi$ be a pre-decomposition of a graph $G$.

1. For every $U \subseteq V(\Phi)$, the subdecomposition $\Phi[U]$ of $\Phi$ with node set $U$ is the pre-decomposition with $V(\Phi[U]) := U$ and $\sigma^{\Phi[U]}(t) := \sigma^\Phi(t)$, $\alpha^{\Phi[U]}(t) := \alpha^\Phi(t)$ for all $t \in U$.

2. For every $W \subseteq V(G)$, the restriction $\Phi|_W$ of $\Phi$ to $W$ is the subdecomposition $\Phi[U]$ with node set $U := \{t \in V(\Phi) \mid \gamma^\Phi(t) \subseteq W\}$.

Let $\Phi$ be a pre-decomposition of a graph $G$. Note that for every $U \subseteq V(\Phi)$ the subdecomposition $\Phi[U]$ is contained in $\Phi$. Hence for all $W \subseteq V(G)$ the restriction $\Phi|_W$, viewed as a pre-decomposition of $G$, is contained in $\Phi$. We may also view $\Phi|_W$ as pre-decomposition of the induced subgraph $G[W]$.

**Lemma 12.1.9.** Let $\Psi$ be an $\ell$-dimensional pd-scheme.

1. There is a parametrised pd-scheme $\Psi'(X)$, where $X$ is $\ell$-ary, such that for every graph $G$ and every $U \subseteq V(\Psi[G])$, the scheme $\Psi'(X)$ defines the subdecomposition $\Psi[G][U]$ within $(G,U)$.

2. There is a parametrised pd-scheme $\Psi''(Y)$, where $Y$ is unary, such that for every graph $G$ and every $W \subseteq V(G)$, the scheme $\Psi''(Y)$ defines the restriction $\Psi[G]|_W$ within $(G,W)$.

**Proof.** Straightforward. □
12.2 Ordered Completions

**Definition 12.2.1.** Let \( G \) be a graph and \( \Phi \) a pre-decomposition of \( G \). An ordered completion of \( \Phi \) is an \( o \)-decomposition \( \Delta = (D, \sigma, \alpha, \leq) \) such that \( (D, \sigma, \alpha) \) is a completion of \( \Phi \) and for all completion nodes \( t \in V(\Delta) \) the binary relation \( \leq_t \) is a linear order of \( \beta(t) \).

Note that we do not require \( \leq_t \) to be a linear order of \( \beta(t) \) for ground nodes \( t \) of \( \Delta \). Hence an ordered completion is not necessarily an ordered decomposition. The following lemma is a generalisation of Lemma 7.1.8.

**Lemma 12.2.2.** Let \( \Psi \) be a pd-scheme, \( \Lambda \) a d-scheme, and \( \varphi(\bar{x}, y_1, y_2) \) an \( \text{IFP} \)-formula. Then there is an od-scheme \( \Lambda' \) such that the following holds. Let \( G \) be a graph such that

(i) \( \Psi[G] \) is proper pre-decomposition of \( G \);

(ii) \( \Lambda[G] \) is a completion of \( \Psi[G] \);

(iii) for every node \( t \in V(\Lambda[G]) \), either there is a tuple \( \bar{v} \in G^\varphi \) such that \( \varphi[G, \bar{v}, y_1, y_2] \) is a linear order of \( \beta^\Lambda(t) \), or there is a node \( t' \in V(\Psi[G]) \) such that \( t \parallel t' \).

Then \( \Lambda'[G] \) is an ordered completion of \( \Psi[G] \).

**Proof.** Similar to the proof of Lemma 7.1.8.

The following lemma is a generalisation of Lemma 7.1.14.

**Lemma 12.2.3.** Let \( \Lambda \) be an od-scheme and \( \Psi \) a pd-scheme. Then there is an od-scheme \( \Lambda' \) such that for every connected graph \( G \), if \( \Psi[G] \) is a tight pre-decomposition and \( \Lambda[G] \) is an ordered completion of \( \Psi[G] \), then \( \Lambda'[G] \) is a tight ordered completion of \( G \).

**Proof.** As the construction of Lemma 5.3.1 leaves tight nodes unaffected and just produces several copies of them, we can prove the lemma in the same way as Lemma 7.1.14.

12.3 Bounded Width Completions

**Definition 12.3.1.** Let \( \Phi \) be a pre-decomposition of a graph \( G \) and \( k \in \mathbb{N} \). A width-\( k \) completion of \( \Phi \) is a completion of \( \Phi \) over the class \( G_{k+1} \) of all graphs of order at most \( k + 1 \).

In other words, a width-\( k \) completion is a completion where the bags of all completion nodes have cardinality at most \( k + 1 \).

**Lemma 12.3.2 (Bounded Width Completion Lemma).** Let \( \Psi \) be a pd-scheme and \( k \in \mathbb{N} \). Then there exists a d-scheme \( \Lambda \) such that for every connected graph \( G \) the following holds. If \( \Psi[G] \) is a tight pre-decomposition of \( G \) that has a width-\( k \) completion, then \( \Lambda[G] \) is a width-\( k \) completion of \( \Psi[G] \).

**Proof.** The proof of this lemma is a more or less straightforward extension of the proof of the Definable Structure Theorem for Graphs of Bounded Tree Width 6.1.1. As in that proof, it is more convenient to work with decompositions of width \( k - 1 \) instead of \( k \), because for such decomposition be can represent the bags by \( k \)-tuples. So let \( k \in \mathbb{N}^+ \), and let \( G \) be a graph such that \( \Phi := \Psi[G] \) is a tight pre-decomposition of \( G \) that has a width-(\( k - 1 \)) completion. Without loss of generality we assume that \( |G| \geq 2 \).
In the first step of the proof, we shall define a completion $\Delta' = (D', \sigma', \alpha')$ of $\Phi$, and in the second step we will delete nodes from $\Delta'$ to turn it into a width-$(k - 1)$ completion.

**Step 1. Definition of $\Delta'$.
Suppose that the pd-scheme $\Psi$ is $\ell$-dimensional. Let
\[
k' := 1 + k + \max\{k, \ell\}.
\]
To simplify the notation, for $k'$-tuples $\bar{v} = (v_1, \ldots, v_{k'})$ we let
\[
\begin{align*}
\bar{v}_I & := (v_2, \ldots, v_{k+1}) \\
\bar{v}_{II} & := (v_{k+2}, \ldots, v_{2k+1}), \\
\bar{v}_{III} & := (v_{k+2}, \ldots, v_{k+\ell+1}).
\end{align*}
\]
The decomposition $\Delta'$ will have three kinds of nodes: r-nodes (root nodes), c-nodes (completion child nodes), and g-nodes (ground nodes). Let $\bar{v} \in V(G)^{k'}$.

(A) $\bar{v}$ is an r-node if $v_1 = v_2$ and $\bar{v}_{II} = \bar{v}_{III}$ and there is a connected component $A_{\bar{v}}$ of $G$ such that $\bar{v}_{II} \subseteq V(A_{\bar{v}})$.

(B) $\bar{v}$ is a c-node if $v_1 = v_2$ and $\bar{v}_{II} \setminus \bar{v}_I \neq \emptyset$ and there is a connected component $A_{\bar{v}}$ of $G \setminus \bar{v}_I$ such that $\bar{v}_{II} \setminus \bar{v}_I \subseteq V(A_{\bar{v}})$ and $N^G(A_{\bar{v}}) = \bar{v}_I \cap \bar{v}_{II}$.

(C) $\bar{v}$ is a g-node if $v_1 \neq v_2$ and $\bar{v}_{II} \subseteq V(\Phi)$ and $\sigma^\Phi(\bar{v}_{III}) \subseteq \bar{v}_I$ and $\alpha^\Phi(\bar{v}_{III}) \cap \bar{v}_I = \emptyset$.

We let $V_r, V_c$, and $V_g$ be the sets of r-nodes, c-nodes, and g-nodes, respectively, and $V(D') := V_r \cup V_c \cup V_g$. For all $\bar{v} \in V_r \cup V_c$, we let $S_{\bar{v}} := N(A_{\bar{v}})$. Note that $S_{\bar{v}} = \emptyset$ if $\bar{v} \in V_r$ and $S_{\bar{v}} = \bar{v}_I \cap \bar{v}_{II}$ if $\bar{v} \in V_c$. For all $\bar{v} \in V_g$ we let $A_{\bar{v}} := G[\alpha^\Phi(\bar{v}_{III})]$ and $S_{\bar{v}} := \sigma^\Phi(\bar{v}_{III})$. As $\Phi$ is tight, $A_{\bar{v}}$ is a connected component of $G \setminus S_{\bar{v}}$, and $S_{\bar{v}} = N(A_{\bar{v}})$.

To define the edge relation $E(D')$, let $\bar{v}, \bar{w} \in V(D')$. Then $\bar{v}\bar{w} \in E(D')$ if

(D) $\bar{v} \not\in V_g$ and $\bar{w} \not\in V_r$ and $\bar{v}_{II} = \bar{w}_{II}$ and $V(A_{\bar{v}}) \subseteq V(A_{\bar{w}})$.

We define the mappings $\sigma', \alpha' : V(D') \to 2^{V(G)}$ by

(E) $\sigma'(\bar{v}) := S_{\bar{v}}$ and $\alpha'(\bar{v}) := V(A_{\bar{v}})$

for all $\bar{v} \in V(D')$. This completes the definition of the decomposition $\Delta'$.

Claim 1. $\Delta'$ is a treelike decomposition.

Proof. We shall verify the axioms in the order $[\text{TL.5}] [\text{TL.2}] [\text{TL.3}] [\text{TL.1}] [\text{TL.4}]$

To prove $[\text{TL.5}]$, let $A$ be a connected component of $G$. Then for each $v \in V(A)$, the $k'$-tuple $\bar{v} := (v, \ldots, v)$ is an r-node with $\sigma'(\bar{v}) = V(A_{\bar{v}}) = V(A)$ and $\sigma'(\bar{v}) = \emptyset$.

$[\text{TL.2}]$ follows immediately from the definitions and the assumption that $\Phi$ is a tight (and hence proper) pre-decomposition.

To prove $[\text{TL.3}]$, let $\bar{v} \in V(D')$ and $\bar{w} \in N^D_+(\bar{v})$. Then $\alpha'(\bar{w}) \subseteq \alpha'(\bar{v})$ follows immediately from $[D]$. As we have $\sigma'(\bar{w}) = N(\alpha'(\bar{w}))$ and $\sigma'(\bar{v}) = N(\alpha'(\bar{v}))$, this implies $\gamma'(\bar{w}) \subseteq \gamma'(\bar{v})$.

To prove $[\text{TL.1}]$, suppose for contradiction that there is a cycle $C \subseteq D'$. By $[D]$ all r-nodes are roots and all g-nodes are leaves of $D'$, and thus all nodes of $C$ are c-nodes. We
prove that for all \( \overline{v}, \overline{w} \in V_e \) with \( \overline{vw} \in E(D') \) we have \( \alpha'(''v''') \supset \alpha'(''w''') \); this will lead to a contradiction. So let \( \overline{v}, \overline{w} \in V_e \) with \( \overline{vw} \in E(D') \). Then

\[
\alpha'(''w''') \subseteq V(A_{\overline{v}}) \setminus \overline{w}_I = V(A_{\overline{v}}) \setminus \overline{v}_{II} \subset V(A_{\overline{v}}) = \alpha'(''v''').
\]

The inclusion \( V(A_{\overline{v}}) \setminus \overline{v}_{II} \subset V(A_{\overline{v}}) \) is strict, because \( \overline{v}_{II} \setminus \overline{v}_I \neq \emptyset \) and \( \overline{v}_{II} \setminus \overline{v}_I \subseteq V(A_{\overline{v}}) \).

It remains to verify (TL.4). Let \( \overline{v} \in V(D') \) and \( \overline{w}_1, \overline{w}_2 \in N^D_+(\overline{v}) \). Then \( \overline{v} \in V_e \cup V_c \) and \( \overline{w}_1, \overline{w}_2 \in V_e \cup V_g \). It follows from the definitions and the tightness of \( \Phi \) that for \( i = 1, 2 \), the graph \( A_{\overline{w}_i} \) is a connected component of \( G \setminus \overline{v}_{II} \) and that \( \sigma'(''w''') = S_{\overline{w}_i} = N^G(A_{\overline{w}_i}) \). Thus (TL.4) follows from Lemma 4.2.10.

Claim 2. There is a d-scheme \( \Lambda' \) (depending on \( \Psi \), but not on \( G \)) such that \( \Lambda'[G] = \Delta' \).

Proof. Straightforward.

Step 2. Pruning the Decomposition.

For every \( \overline{v} \in V(D') \), we let \( G_{\overline{v}} := G[\gamma'(''v''')] \). Observe that \( G_{\overline{v}} \) is connected. We inductively define a sequence of sets \( U_h \subseteq V(D') \), for all \( h \in \mathbb{N}^+ \).

- \( U_1 \) consists of all \( \overline{v} \in V_e \cup V_c \) with \( \gamma(''v'') = \overline{v}_{II} \) and all \( \overline{v} \in V_g \).
- \( U_{h+1} \) is the union of \( U_h \) with the set of all \( \overline{v} \in V_e \cup V_c \) such that for every connected component \( A \) of \( G_{\overline{v}} \setminus \overline{v}_{II} \) there is a node \( \overline{w} \in N^D_+(\overline{v}) \cap U_h \) with \( A_{\overline{w}} = A \).

Recall that the height of a tree decomposition \((T, \beta)\) of a graph \( H \) is the height of the tree \( T \).

Claim 3. Let \( \overline{v} \in V(D') \setminus V_g \) such that \( G_{\overline{v}} \) has a tree decomposition \((T_{\overline{v}}, \beta_{\overline{v}})\) of height at most \( h \) such that

1. \((T_{\overline{v}}, \beta_{\overline{v}})\) is a width-k completion of the induced pre-decomposition \( \Phi_{\mid \gamma'(''v''')} \);
2. \( \overline{v}_{II} = \beta_{\overline{v}}(r_{\overline{v}}) \) for the root \( r_{\overline{v}} \) of \( T_{\overline{v}} \).

Then \( \overline{v} \in U_{h+1} \).

Proof. We prove the claim by induction on \( h \). For the base step \( h = 0 \), suppose that \((T_{\overline{v}}, \beta_{\overline{v}})\) is a tree decomposition of \( G_{\overline{v}} \) of height 0 satisfying (i) and (ii). Then, as \( \overline{v} \not\in V_g \),

\[
\gamma(''v''') = V(G_{\overline{v}}) = \beta_{\overline{v}}(r_{\overline{v}}) = \overline{v}_{II}
\]

and thus \( \overline{v} \in U_1 \).

For the inductive step, let \( h > 0 \) and suppose that \((T_{\overline{v}}, \beta_{\overline{v}})\) is a tree decomposition of \( G_{\overline{v}} \) of height \( h \) satisfying (i) and (ii). Let \( A \) be a connected component of \( G_{\overline{v}} \setminus \overline{v}_{II} = G_{\overline{v}} \setminus \beta_{\overline{v}}(r_{\overline{v}}) \). We shall prove that there is a node \( \overline{w} \in N^D_+(\overline{v}) \cap U_h \) with \( A_{\overline{w}} = A \). This will imply \( \overline{v} \in U_{h+1} \).

If there is an \( x \in V(\Phi) \) with \( \alpha(\overline{v}) = V(A) \), then there is a \( \overline{w} \in V_g \subseteq U_1 \subseteq U_h \) such that \( A_{\overline{w}} = A \) and \( \overline{w} \in N^D_+(\overline{v}) \). In the following, we assume that there is no such \( x \).

Let \( S := N^G(A) \) and \( H := G[V(A) \cup S] \). Let \( t \in V(T_{\overline{v}}) \) be \( <T_{\overline{v}}\)-maximal with \( V(A) \subseteq \alpha_{T_{\overline{v}}}(t) \), that is, \( V(A) \subseteq \alpha_{T_{\overline{v}}}(t) \) and \( V(A) \nsubseteq \alpha_{T_{\overline{v}}}(u) \) for any \( u \in N^T_{T_{\overline{v}}}(t) \). Clearly, such a node \( t \) exists, because \( V(A) \subseteq V(G_{\overline{v}}) = \alpha_{T_{\overline{v}}}(r_{\overline{v}}) \). We have

\[
V(A) \cap \beta_{\overline{v}}(t) \neq \emptyset.
\]  

M. Grohe, *Definable Graph Structure Theory*
To see this, suppose for contradiction that $V(A) \cap \beta_\tau(t) = \emptyset$. As $V(A) \subseteq \gamma_\tau(t)$, there is at least one child $u \in N^\tau_+(t)$ such that $V(A) \cap \alpha_\tau(u) \neq \emptyset$. Then it follows from Fact 4.1.3(2) that $V(A) \subseteq \alpha_\tau(u)$, which is a contradiction. This proves (12.3.1). Note that (12.3.1) implies $t \neq \tau$. Furthermore,

$$S \subseteq \sigma_\tau(t) \subseteq \beta_\tau(t).$$

The second inclusion is trivial. To prove the first, note $S = N^G(A) \subseteq \gamma_\tau(t)$, because $V(A) \subseteq \alpha_\tau(t)$. Moreover, $S \subseteq \beta_\tau(r_\tau)$, because $A$ is a connected component of $G_\tau \setminus \beta_\tau(r_\tau)$. Hence $S \subseteq \gamma_\tau(t) \cap \beta_\tau(r_\tau) \subseteq \sigma_\tau(t)$ by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9.

(12.3.2) implies that $t$ is not a ground node of $(T_\tau, \beta_\tau)$. To see this, suppose for contradiction that there is an $x \in V(\Phi)$ with $t \parallel x$. As $V(A) \subseteq \alpha_\tau(t) = \alpha_\Phi(x)$ and $N^G(A) \subseteq \sigma_\tau(t) = \sigma_\Phi(x) = N^G(\alpha_\Phi(x))$ and both $V(A)$ and $\alpha_\Phi(x)$ are connected, it follows that $V(A) = \alpha_\Phi(x)$. But this contradicts our assumption that there is no $x \in V(\Phi)$ with $\alpha_\Phi(x) = V(A)$.

Let $T_A := T_\tau[\{u \in V(T_\tau) \mid t \not\leq u\}]$ be the full subtree of $T_\tau$ rooted at $t$, and define $\beta_A : V(T_A) \to 2^{V(H)}$ by $\beta_A(u) := \beta_\tau(u) \cap V(H)$. Then $(T_A, \beta_A)$ is a tree decomposition of $H$ of height at most $(h - 1)$. If $x$ is a ground node of $(T_\tau, \beta_\tau)$, then $\alpha_\tau(x)$ is connected and $\sigma_\tau(x) = N^G(\alpha_\tau(x))$, because $\Phi$ is tight. Moreover, $\alpha_\tau(x) \subseteq G_\tau \setminus \beta_\tau(r_\tau)$, and as $A$ is a connected component of $G_\tau \setminus \beta_\tau(r_\tau)$, it follows that $\alpha_\tau(x) \subseteq V(A)$ and $\gamma_\tau(x) \subseteq V(H)$. Hence $x$ is a node of the restriction $\Phi|_{V(H)}$. But this implies that $(T_A, \beta_A)$ is a width-$(k - 1)$ completion of $\Phi|_{V(H)}$.

Observe that $\beta_\tau(t) \cap \beta_\tau(r_\tau) = S$ by (12.3.2) and $\beta_\tau(t) \setminus \beta_\tau(r_\tau) \neq \emptyset$ by (12.3.1). As $t$ is a not a ground node of $(T_\tau, \beta_\tau)$, we have $|\beta_\tau(t)| \leq |\beta_\tau(r_\tau)| \leq k$. Let $\overline{w} = (w_1, \ldots, w_k)$ be a tuple with $w_1 = w_2$ and $\overline{w}_I = \overline{w}_{II}$ and $\overline{w}_{II} = \beta_\tau(t)$. Then $\overline{w} \in V_c$ with $A_{\overline{w}} = A$ and $S_{\overline{w}} = S$. By [D], we have $\overline{w} \in N^G_d(\overline{w})$, and by the induction hypothesis we have $\overline{w} \in U_h$.

We let $D$ be the induced subgraph of $D'$ with universe $V(D) := \bigcup_{h \in \mathbb{N}^+} U_h$, and we let $\sigma, \alpha : V(D) \to 2^{V(G)}$ be the restrictions of $\sigma', \alpha'$ to $V(D)$. We let $\Delta := (D, \sigma, \alpha)$.

**Claim 4.** $\Delta$ is a tree decomposition of $G$.

**Proof.** $\Delta$ inherits (TL.1) (TL.4) from $\Delta$. To see that it satisfies (TL.5) let $A$ be a connected component of $G$. Let $(T_A, \beta_A)$ be a tree decomposition of $A$ that is a width-$(k - 1)$ completion of $\Phi|_{V(A)}$. To see that such a decomposition exists, observe first that there is a width-$(k - 1)$ completion of $\Phi|_{V(A)}$ in $A$, because there is a width-$(k)$ completion of $\Phi$ in $G$. Then it follows from Corollary 4.6.7 that there is a tree decomposition $(T_A, \beta_A)$ of $A$ that is a width-$(k - 1)$ completion of $\Phi|_{V(A)}$. Without loss of generality we may assume that $\beta(r_A) \neq \emptyset$ for the root $r_A$ of $T_A$.

Let $h$ be the height of $T_A$. If $r_A$ is a ground node, then there is a node $\overline{v} \in V_g \subseteq U_1 \subseteq V(D)$ such that $\alpha(\overline{v}) = \alpha_A(r_A) = V(A)$ and $\sigma(\overline{v}) = N(\alpha(\overline{v})) = \emptyset$. In the following, assume that $r_A$ is not a ground node. Then $|\beta_A(r_A)| \leq k$, and we can find a node $\overline{v} \in V_h$ with $\overline{v}_I = \overline{v}_{II} = \beta_A(r_A)$ and $\sigma'(\overline{v}) = \emptyset$ and $\alpha'(\overline{v}) = A_{\overline{v}} = A$. By Claim 3, we have $\overline{v} \in U_{h+1} \subseteq V(D)$. Hence $\Delta$ is treelike.

**Claim 5.** For all $\overline{v} \in V(D)$, either $\beta(\overline{v}) \subseteq \overline{v}_{II}$ or $\overline{v} \in V_g$.

**Proof.** Let $\overline{v} \in V(D)$. Let $h \in \mathbb{N}^+$ such that $\overline{v} \in U_h$. The claim is obvious if $h = 1$. If $h > 1$, let $A_1, \ldots, A_m$ be the connected components of $G_\tau \setminus \overline{v}_{II}$. Then for every $i \in [m]$ there is a
\[ \overline{w}_i \in N^\Gamma_+ (\overline{v}) \cap U_{h-1} \subseteq N^\Gamma_+(\overline{v}) \text{ such that } A_{\overline{w}_i} = A_i. \]

Hence

\[ \beta(\overline{v}) = \gamma(\overline{v}) \setminus \bigcup_{\overline{w} \in N^\Gamma_+(\overline{v})} \alpha(\overline{w}) \subseteq \gamma(\overline{v}) \setminus \bigcup_{i=1}^{m} \alpha(\overline{w}_i) = V(D) \setminus \bigcup_{i=1}^{m} V(A_i) = \overline{v}_II. \]

Claim 5 implies that \( \Delta \) is a width-(\( k - 1 \)) completion of \( \Phi \). It follows from Claim 2 and the inductive definition of \( V(D) = \bigcup_{h \in \mathbb{N}^+} U_h \), which can easily be formalised in IFP, that there is a d-scheme \( \Lambda \) (not depending on \( G \)) such that \( \Lambda[G] = \Delta. \)

Remark 12.3.3. The assumption that the pre-decomposition \( \Phi = \Psi[G] \) be tight is crucial for our proof of the Bounded Width Completion Lemma 12.3.2. Actually, we do not need all tightness conditions, but only that \( \alpha(t) \) is connected and that \( \sigma(t) = R(\alpha(t)) \) for all \( t \in V(\Phi) \). It is an interesting open problem whether the lemma can be strengthened to hold without these assumptions. This would considerably simplify some of the proofs later in this book.

Corollary 12.3.4. Let \( \Psi \) be a pd-scheme and \( k \in \mathbb{N} \). Then there exists an od-scheme \( \Lambda \) such that for every connected graph \( G \) the following holds. If \( \Psi[G] \) is a tight pre-decomposition of \( G \) that has a width-(\( k - 1 \)) completion, then \( \Lambda[G] \) is an ordered completion of \( \Psi[G] \).

12.4 Derivations of Pre-Decompositions

Definition 12.4.1. Let \( d, d' \in \mathbb{N} \). Let \( G \) be a graph and \( \Phi \) a pre-decomposition of \( G \). The \((d, d')\)-derivation of \( \Phi \) is the pre-decomposition \( \Phi^{(d, d')} \) defined as follows:

- The node set \( V(\Phi^{(d, d')}) \) consists of all triples \((t, S, A)\), where \( t \in V(\Phi) \) and \( S \subseteq V(G) \) with \(|S| \leq d\) and \( A \) is a connected component of \( G \setminus S \) such that \(|(V(A) \cup S) \setminus \gamma(\Phi)(t)| \leq d'\) and \( S = N^G(A) = \partial^G(V(A) \cup S) \).

- For every node \((t, S, A) \in V(\Phi^{(d, d')})\) we let \( \alpha^{\Phi^{(d, d')}}(t, S, A) := V(A) \) and \( \sigma^{\Phi^{(d, d')}}(t, S, A) := S \).

Obviously, \( \Phi^{(d, d')} \) is a tight pre-decomposition of \( G \). If \( \Phi \) is a tight pre-decomposition of adhesion \( p \leq d \), then \( \Phi \preceq \Phi^{(d, d')} \), that is, for every node \( t \in V(\Phi) \) there is a node \( t' \in V(\Phi^{(d, d')}) \) such that \( \sigma\Phi(t) = \sigma\Phi^{(d, d')}(t') \) and \( \alpha\Phi(t) = \alpha\Phi^{(d, d')}(t') \); just let \( t' := (t, \sigma\Phi(t), G[\alpha\Phi(t)])[i] \). The next lemma shows that if \( \Phi \) is definable then its \((d, d')\)-derivation is definable as well.

Lemma 12.4.2. Let \( \Psi \) be a pd-scheme and \( d, d' \in \mathbb{N} \). Then there is a pd-scheme \( \Psi^{(d, d')} \) such that for all graphs \( G \), the pre-decomposition \( \Psi^{(d, d')}[G] \) is equivalent (cf. Definition 12.1.7) to the \((d, d')\)-derivation of \( \Psi[G] \).

Proof. For simplicity, we only define \( \Psi^{(d, d')} \) in such a way that it works for all graphs of order at least 2. It can easily be extended to all graphs. Let \( G \) be a graph of order at least 2 and \( \Phi := \Psi[G] \). Suppose that \( \Psi \) is \( \ell \)-dimensional. Then \( V(\Phi) \subseteq V(G)^\ell \). Let \( k := \ell + d + 3 \). For every tuple \( \overline{v} = (v_1, \ldots, v_k) \) we define a triple \((\overline{v}, S_{\overline{v}}, A_{\overline{v}})\) as follows:

- \( l_{\overline{v}} := (v_1, \ldots, v_\ell) \).

- If \( v_{\ell+1} = v_{\ell+2} \) then \( S_{\overline{v}} = \emptyset \). Otherwise, \( S_{\overline{v}} := \{v_{\ell+3}, \ldots, v_{\ell+d+2}\} \).

- \( A_{\overline{v}} \) is the connected component of \( G \setminus S_{\overline{v}} \) that contains \( v_{\ell+d+3} \).
We define the \( k \)-dimensional pd-scheme \( \Psi^{(d,d')} = (\psi^{(d,d')}_{V}(\overline{x}), \psi^{(d,d')}_{\sigma}(\overline{x}, y), \psi^{(d,d')}_{\alpha}(\overline{x}, y)) \) in such a way that \( V(\Psi^{(d,d')}[G]) = \psi^{(d,d')}_{V}(G, \overline{x}) \) is the set of all \( v \in V(G)^k \) such that \((t_{\overline{x}}, S_{\overline{x}}, A_{\overline{x}}) \in V(\Phi^{(d,d')})\), that is, \( t_{\overline{x}} \in V(\Phi) \) and \(|(V(A_{\overline{x}}) \cup S_{\overline{x}}) \setminus \gamma^{\Phi}(t)| \leq d'\) and \( S_{\overline{x}} = \partial^{G}(V(A_{\overline{x}}) \cup S_{\overline{x}}) \). Note that \( |S_{\overline{x}}| \leq d \), because \( S_{\overline{x}} \) is at most the set of entries of a \( d \)-tuple. We define \( \psi^{(d,d')}_{\sigma}(\overline{x}, y) \) and \( \psi^{(d,d')}_{\alpha}(\overline{x}, y) \) such that for every \( \overline{x} \in V(\Psi^{(d,d')}[G]) \) we have \( \alpha^{\Psi^{(d,d')}[G]}(\overline{v}) = V(A_{\overline{x}}) \) and \( \sigma^{\Psi^{(d,d')}[G]}(\overline{x}, y) = S_{\overline{x}} \). Then obviously the pre-decompositions \( \Psi^{(d,d')}[G] \) and \( \Phi^{(d,d')} \) are equivalent.

**Lemma 12.4.3.** Let \( \Phi \) be a pre-decomposition of a graph \( G \) and \( d_1, d_1', d_2, d_2' \in \mathbb{N} \). Then

\[
(\Phi^{(d_1,d_1')}(d_2,d_2')) \preceq \Phi^{(d_2,d_2')}(d_1,d_1').
\]

**Proof.** Straightforward. \( \square \)

### 12.5 The Finite Extension Lemma for Ordered Completions

When completing pre-decompositions, we will often be in a situation where we start with a pre-decomposition \( \Phi \) of a graph \( G \), go to a subgraph or minor \( G' \) of \( G \) that is “simpler” than \( G \) and for which we know how to complete a pre-decomposition \( \Phi' \) “induced” by \( \Phi \). Then we complete \( \Phi' \) in \( G' \), and finally we extend the resulting decomposition to \( G \). Usually, it will not work that smoothly, and we can only complete a derivation of \( \Phi' \) in \( G' \) and then extend the decomposition to a completion of a derivation of \( \Phi \) in \( G \). This will be good enough. The difficulty with this approach is that it is not always obvious how to define the induced pre-decomposition \( \Phi' \), or that the obvious definition leads to unexpected results such as \( \Phi' \) not being a proper pre-decomposition (even if \( \Phi \) was). Fortunately, this is not really a problem if we only work with derivations of \( \Phi \), because derivations are tight by definition.

In this section, we only consider pre-decompositions induced on subgraphs. (We will deal with specific minors, the quasi-4-connected components of a graph, in Section 12.6.) Let \( \Phi := (V(\Phi), \sigma, \alpha) \) be a pre-decomposition of a graph \( G \) and \( G' \subseteq G \). We define the pre-decomposition \( \Phi' = (V(\Phi'), \sigma', \alpha') \) induced by \( \Phi \) on \( G' \) by \( V(\Phi') := V(\Phi) \) and \( \sigma'(t) := \sigma(t) \cap V(G') \) and \( \alpha'(t) := \alpha(t) \cap V(G') \) for all \( t \in V(\Phi') \). Observe that if \( \Phi \) is a proper pre-decomposition of \( G \), then \( \Phi' \) is a proper pre-decomposition of \( G' \). However, if \( \Phi \) is tight, then \( \Phi' \) is not necessarily tight as well.

**Lemma 12.5.1.** Let \( d, d', k \in \mathbb{N} \). Let \( G \) be a graph, \( X \subseteq V(G) \) with \( |X| \leq k \), and \( G' := G \setminus X \). Let \( \Phi \) be a pre-decomposition of \( G \), and let \( \Phi' \) be the pre-decomposition induced by \( \Phi \) on \( G' \). Let \( \Delta' = (D', \sigma', \alpha') \) be a completion of \( (\Phi')^{(d,d')} \) in \( G' \).

Then there is a treelike decomposition \( \Delta = (D, \sigma, \alpha) \) of \( G \) with the following properties:

(i) \( D = D' \).

(ii) For all \( t \in V(D) \),

\[
\sigma'(t) \subseteq \sigma(t) \subseteq \sigma'(t) \cup X,
\]
\[
\alpha'(t) \subseteq \alpha(t) \subseteq \alpha'(t) \cup X
\]

(iii) \( \Delta \) is a completion of \( \Phi^{(d+k,d'+k)} \) in \( G \). Furthermore, every ground node of \( \Delta' \) (with respect to \( (\Phi')^{(d,d')} \)) is a ground node of \( \Delta \) (with respect to \( \Phi^{(d+k,d'+k)} \)).
Proof. Without loss of generality we may assume that $G$ is connected. If not, we treat each connected component separately.

We let $D := D'$. Let $t \in V(D)$. If $\alpha'(t) = V(G')$ we let $\alpha(t) := V(G)$ and $\sigma(t) := \emptyset$. Otherwise, we let $X_t := N^G(\alpha'(t)) \cap X$ and $Y_t := \partial^G(\gamma'(t) \cup X_t) \cap X_t$ and

$$\sigma(t) := \sigma'(t) \cup Y_t,$$

$$\alpha(t) := \alpha'(t) \cup (X_t \setminus Y_t).$$

Note that $\gamma(t) = \gamma'(t) \cup X_t$.

**Claim 1.** $\Delta$ is a treelike decomposition of $G$.

**Proof.** (TL.1) is trivially satisfied. To prove (TL.2) let $t \in V(D)$ with $\alpha(t) \neq V(G)$ and $v \in \alpha(t)$ and $w \in N^G(v)$. We shall prove that $w \in \gamma(t)$. Suppose first that $v \in \alpha'(t)$. If $w \in V(G')$ then $w \in N^G(v) \subseteq \gamma'(t) \subseteq \gamma(t)$. If $w \in X$ then $w \in X_t \subseteq \gamma(t)$. Now suppose that $v \in X_t \setminus Y_t$. Then $v \notin \partial^G(\gamma'(t) \cup X_t)$ and thus $w \in \gamma'(t) \cup X_t = \gamma(t)$.

To prove (TL.3) let $t \in V(D)$ and $u \in N^+(D)$. If $\alpha(t) = V(G)$, the assertion is trivial, so we may assume that $\alpha(t) \neq V(G)$. Observe that $X_u \subseteq X_t$, because $\alpha'(u) \subseteq \alpha'(t)$ and $\alpha'(t) \cap X = \emptyset$. Thus $\gamma(u) = \gamma'(u) \cup X_u \subseteq \gamma'(t) \cup X_t = \gamma(t)$. To prove that $\alpha(u) \subseteq \alpha(t)$, let $v \in \alpha(u)$. If $v \in \alpha'(t)$ then $v \in \alpha'(t) \subseteq \alpha(t)$. So suppose that $v \in X_u \setminus Y_u$. Then $v \in X_t$. Suppose for contradiction that $v \notin Y_t$. Then $v \notin \partial^G(\gamma'(t) \cup X_t)$. Thus there is a $w \in N^G(v) \setminus (\gamma'(t) \cup X_t) \subseteq N^G(v) \setminus (\gamma'(u) \cup X_u)$, and therefore $v \notin \partial^G(\gamma'(u) \cup X_u)$. As $v \in X_u$, this implies $v \in Y_u$, which is a contradiction.

To prove (TL.4) let $t \in V(D)$ and $u_1, u_2 \in N^+(D)$. If $u_1 \parallel^{\Delta'} u_2$ then $u_1 \parallel^{\Delta} u_2$. Suppose that $u_1 \not\perp^{\Delta'} u_2$. Then

$$\gamma(u_1) \cap \gamma(u_2) = (\gamma'(u_1) \cap \gamma'(u_2)) \cup (X_{u_1} \cap X_{u_2}) = (\sigma'(u_1) \cap \sigma'(u_2)) \cup (X_{u_1} \cap X_{u_2}).$$

To prove that the last term is equal to $\sigma(u_1) \cap \sigma(u_2)$, it suffices to prove that $X_{u_1} \cap X_{u_2} = Y_{u_1} \cap Y_{u_2}$. However, this follows directly from the definitions and the fact that $\alpha'(u_1)$ and $\alpha'(u_2)$ are disjoint.

To prove (TL.5) let $t \in V(D)$ such that $\alpha'(t) = V(G')$. Then $\alpha(t) = V(G)$ and $\sigma(t) = \emptyset$.

The decomposition $\Delta$ obviously satisfies (i) and (ii). To prove (iii), let $t \in V(D')$ be a ground node of $\Delta'$. Let $(t', S', A') \in V((\Phi')^{d,d'})$ such that $t \parallel^{\Delta', (\Phi')^{d,d'}} (t', S', A')$. Then $|\gamma'(t) \setminus \gamma(t)| \leq d'$. Let $S := \sigma(t)$ and $A := G[\alpha(t)]$.

**Claim 2.** $(t', S, A) \in \Phi(d+k,d'+k)$.

**Proof.** Note first that $\gamma(t) \setminus \gamma(t) \subseteq (\gamma(t) \setminus \gamma(t)) \cup X$ and thus $|\gamma(t) \setminus \gamma(t)| \leq d' + k$. Moreover, we have $S = \sigma(t) \subseteq \sigma(t) \cup X = S' \cup X$ and thus $|S| \leq d + k$.

Furthermore, $A$ is connected because $A' = G[\alpha'(t)]$ is connected and $A' \subseteq A \subseteq G[V(A') \cup N^G(A')]$.

It remains to prove that $\sigma(t) = N^G(\alpha(t)) = \partial^G(\gamma(t))$. Let $v \in \sigma(t)$. If $v \in \sigma'(t) = N^G(\alpha'(t)) = \partial^G(\gamma'(t))$, let $w \in N^G(v) \cap \alpha(t)$ and $w' \in N^G(v) \setminus \gamma(t)$. Then $w \in N^G(v) \cap \alpha(t)$ and $w' \in N^G(v) \setminus \gamma(t)$. This implies that $v \in N^G(\alpha(t)) \cap \partial^G(\gamma(t))$. If $v \in Y_t$ then $v \in \partial^G(\gamma'(t) \cup X_t) = \partial^G(\gamma(t))$. Furthermore, $v \in X_t \subseteq N^G(\alpha'(t))$, and as $v \notin \alpha(t)$, it follows that $v \in N^G(\alpha(t))$. This shows that $\sigma(t) \subseteq N^G(\alpha(t))$ and $\sigma(t) \subseteq \partial^G(\gamma(t))$. For the converse inclusions, first consider a $v \in N^G(\alpha(t))$. Let $w \in N^G(v) \cap \alpha(t)$. If $w \in \alpha'(t)$
then \( v \in N^G(\alpha'(t)) = \sigma'(t) \subseteq \sigma(t) \). If \( w \in X_t \setminus Y_t \), then \( w \not\in \partial^G(\gamma'(t) \cup X_t) \). Thus \( N^G(w) \subseteq \gamma'(t) \cup X_t = \gamma(t) \). It follows that \( v \in \gamma(t) \setminus \alpha(t) = \sigma(t) \). Now consider a \( v \in \partial^G(\gamma(t)) = \partial^G(\gamma'(t) \cup X_t) \). If \( v \in X_t \) then \( v \in Y_t \) and thus \( v \in \sigma(t) \). Otherwise, \( v \in \gamma'(t) \).

Suppose for contradiction that \( v \in \alpha'(t) \). Let \( w \in N^G(v) \setminus \gamma(t) \). Then \( w \in N^G(\alpha'(t)) \). If \( w \not\in X \) then \( w \in \sigma'(t) \subseteq \gamma(t) \), which is a contradiction. If \( w \in X \) then \( w \in X_t \subseteq \gamma(t) \), again a contradiction. \( \blacksquare \)

As \( t \models \Deltaartz (t', S, A) \), assertion (iii) follows. \( \square \)

Let us now turn to the definability of induced pre-decompositions:

**Lemma 12.5.2.** Let \( \Psi(\mathcal{X}) \) be a parametrised pd-scheme and \( \Theta(\mathcal{Y}) \) a subgraph transduction. Then there is a parametrised pd-scheme \( \Psi'(\mathcal{X}, \mathcal{Y}) \) such that for every graph \( G \) and all \( \mathcal{PQ} \in G^{\mathcal{XY}} \) the scheme \( \Psi'(\mathcal{X}, \mathcal{Y}) \) defines the pre-decomposition induced by \( \Psi[G, \mathcal{P}] \) on \( \Theta[G, \mathcal{Q}] \) within \( (G, \mathcal{P}, \mathcal{Q}) \).

**Proof.** Straightforward. \( \square \)

**Corollary 12.5.3.** For every pd-scheme \( \Psi \) there is a parametrised pd-scheme \( \Psi'(\mathcal{X}) \), where \( \mathcal{X} \) is a unary relation variable, such that for every graph \( G \) and every set \( P \subseteq V(G) \) the scheme \( \Psi'(\mathcal{X}) \) defines the pre-decomposition induced by \( \Psi[G] \) on \( G \setminus P \) within \( (G, P) \).

The next lemma is "definable" version of Lemma [12.5.1].

**Lemma 12.5.4 (Finite Extension Lemma for Ordered Completions).** Let \( d, d', k \in \mathbb{N} \). Let \( \Psi \) a pd-scheme and \( \Lambda(\mathcal{X}) \) an od-scheme, where \( \mathcal{X} \) is a unary relation variable. Then there is an od-scheme \( \Lambda' \) such that the following holds. Let \( G \) be a graph and \( P \subseteq V(G) \) such that \( |P| \leq k \). Let \( \Phi := \Psi[G] \) and \( G' := G \setminus P \), and let \( \Phi' \) be the pre-decomposition induced by \( \Phi \) on \( G' \). Suppose that \( \Lambda(\mathcal{X}) \) defines an ordered completion of \( (\Phi'(d, d')) \) in \( G' \) within \( (G, P) \).

Then \( \Lambda'[G] \) is an ordered completion of \( \Phi'(d+k, d'+k) \).

**Proof.** It is easy to see that the completion constructed in Lemma [12.5.1] is lfp-definable. To turn it into an ordered completion, we use the fact that the bags of the decomposition defined by \( \Lambda' \) and the new decomposition differ by at most \( k \) vertices. \( \square \)

### 12.6 The Q4C Completion Lemma

Our goal in this section is to prove a generalisation of the Q4C Lifting Lemma [10.3.1] that lets us define ordered completions of pre-decompositions for graphs whose quasi-4-connected components admit such completions. Essentially, we will prove the following. Let \( G \) be a graph and \( \Psi \) a pd-scheme that defines a tight pre-decomposition \( \Phi := \Psi[G] \) on \( G \). Further assume that there is an od-scheme that for all proper quasi-4-connected components \( J^* \) of \( G \) defines an ordered completion of the pre-decomposition derived from the pre-decomposition "induced" by \( \Phi \) on \( J^* \). Then we can define an ordered completion of a pre-decomposition derived from \( \Phi \) on \( G \).

Let \( G \) be a graph and \( \Phi = (V(\Phi), \sigma, \alpha) \) a pre-decomposition on \( G \). Let \( J^* \) a quasi-4-connected component of \( G \), and let \( J \) and \( M \) be the torso and matching associated with \( J^* \). We use the notation introduced in Proviso [10.3.3]. We define the pre-decompositions \( \Phi^+ = (V(\Phi^+), \sigma^+, \alpha^+) \) and \( \Phi^* = (V(\Phi^*), \sigma^*, \alpha^*) \) induced by \( \Phi \) on \( J \) and \( J^* \), respectively, by \( V(\Phi^+) := V(\Phi^*) := V(\Phi) \) and \( \sigma^+(t) := \sigma(t) \cap V(J), \sigma^*(t) := \sigma^*(t) \) and \( \alpha^+(t) := \alpha(t) \cap V(J), \)
α*(t) := α+(t)*. This implies γ*(t) = γ+(t)*. Note that even if Φ is a tight pre-decomposition, Φ+ and Φ* are not necessarily proper, let alone tight pre-decompositions. But this will not matter, because we will always work with derivations of Φ+ and Φ*, and they are tight by definition.

**Lemma 12.6.1.** Suppose that Φ is a tight pre-decomposition of G. Then for every t ∈ V(Φ) we have |∂J(γ+(t))| ≤ 2|σ(t)|.

**Proof.** For every v ∈ σ(t) we shall define a set Z_v ⊆ V(J) with |Z_v| ≤ 2, and then we shall prove that ∂J(γ+(t)) ⊆ ∪v∈σ(t) Z_v. This will obviously imply the assertion of the lemma.

Let v ∈ σ(t). If v ∈ V(J), we let Z_v := {v}. Otherwise, let A be the connected component of G \ J that contains v. By the Q4C Decomposition Lemma [10.2.4(ii)] we have |N(A)| ≤ 3. If N(A) ⊆ γ(t) we let Z_v := ∅. Otherwise, we let Z_v := N(A) \ γ(t).

To see that ∂J(γ+(t)) ⊆ ∪v∈σ(t) Z_v, let w ∈ ∂J(γ+(t)) and x ∈ V(J \ γ+(t)) such that wx ∈ E(J). If w ∈ σ(t) then Z_w = {w}, because w ∈ V(J), and hence w ∈ Z_w. In particular, this is the case if wx ∈ E(G), because then then w ∈ ∂G(γ(t)) = σ(t). Suppose that w ∈ α(t) and hence wx ∈ E(J \ E(G). Then there is a connected component A of G \ J such that w, x ∈ N(A). As w ∈ α(t) and x ∈ V(J \ γ(t), the set σ(t) separates w from x in G, and thus V(A) ∩ σ(t) = ∅. Let v ∈ V(A) ∩ σ(t). Then w ∈ Z_v.

**Lemma 12.6.2 (Q4C Completion Lemma).** Let p, d, d′ ∈ N such that d ≥ 2d + 2d′ + 6p + 3 and d′ ≥ 2d + 2p. Let Ψ be a pd-scheme and Λ*(π) a parametrised od-scheme, where |π| matches the dimension of Λq4c. Then there is an od-scheme Λ such that for all graphs G the following holds. Suppose that

(i) Φ := Ψ[G] is a tight pre-decomposition of G of adhesion at most p;

(ii) for all v ∈ V(Λq4c[G]), if Jv is the quasi-4-connected component of G with index v and Φv is the pre-decomposition induced by Φ on Jv, then the scheme Λ*(π) defines an ordered completion of (Φv)(d,d′) in Jv within (G, π).

Then Λ[G] is an ordered completion of Φ(d,d′).

To prove the Q4C Completion Lemma, we need a lemma that corresponds to Lemma [10.3.2] in the proof of the Q4C Lifting Lemma [10.3.1].

**Lemma 12.6.3.** Let p, d, d′ ∈ N. Let Λ*(π) be a parametrised od-scheme, where the length of π matches the dimension of the d-scheme Λq4c, and let Ψ be a pd-scheme. Then there is a parametrised od-scheme Λ(π) such that the following holds. Let G be a graph, Jπ a quasi-4-connected component of G with index π, and let Jv be the torso associated with π. Suppose that the pre-decomposition Φ := Ψ[G] is tight and of adhesion at most p. Let Φ+, Φ be the pre-decompositions induced by Φ on Jπ, Jv, respectively. Furthermore, suppose that Λ*(π) defines an ordered completion of (Φv)(d,d′) in Jv within (G, π).

Then Λ(π) defines an ordered completion of (Φv)(2d,2d′+2p) in Jπ within (G, π).

**Proof.** Let G be a graph such that the pre-decomposition Φ := Ψ[G] is tight and of adhesion at most p. Let J be a quasi-4-connected component of G, and let J, M be the torso and matching associated with J.

Let Φ = (V, σ, α) and Φ = (V, σ, α+) be the pre-decompositions induced by Φ on J* and J, respectively. Here we use V to denote the set V(Φ*) = V(Φ+) = V(Φ). Let
\( \Phi^* := (V^*, \alpha^*, \sigma^*) := (\Phi^*)(d_*, \nu_s^*) \) in \( J^* \) and \( \Phi^{++} := (V^{++}, \alpha^{++}, \sigma^{++}) := (\Phi^+)(2d_*, 2\nu_s^* + 2\nu_t) \) in \( J \).

**Claim 1.** For every node \( t^* \in V^* \) there is a node \( t^{++} \in V^{++} \) such that

\[
\begin{align*}
\gamma^{++}(t^{++}) &= (\gamma^*(t^*))^0, \\
\sigma^{++}(t^{++}) &\subseteq (\sigma^*(t^*))^0, \\
\alpha^{++}(t^{++}) &\supseteq (\alpha^*(t^*))^0. 
\end{align*}
\]

**Proof.** Let \( t^* \in V^* \). Let \( X := (\gamma^*(t^*))^0 \) and \( S := \partial^J(X) \) and \( A := J[X \setminus S] \).

**Subclaim 1a.**

\[
\gamma^*(t^*) = X^* \tag{12.6.1}
\]

and

\[
\sigma^*(t^*) = S^* \tag{12.6.2}
\]

and

\[
\alpha^*(t^*) = (V_0 \cap V(A)) \cup \{v_e | e \in M \text{ such that } e \subseteq V(A)\}. \tag{12.6.3}
\]

**Proof.** \hspace{1em} [12.6.1] follows immediately from the definition of \( X \).

To prove the inclusion \( \sigma^*(t^*) \supseteq S^* \) of \hspace{1em} [12.6.2], let \( v \in S = \partial^J(X) \). Let \( w \in V(J) \setminus X \) such that \( vw \in E(J) \). Then \( v^* \in X^* \) and \( w^* \in V(J^*) \setminus X^* \), the latter because \( X = (X^*)^0 \). Thus \( v^* \in \partial^J(X^*) = \sigma^*(t^*) \). To prove the converse inclusion \( \alpha^*(t^*) \subseteq S^* \), let \( v^* \in \sigma^*(t^*) = \partial^J(\gamma^*(t^*)) \), and let \( w^* \in V(J^*) \setminus \gamma^*(t^*) \) such that \( v^*w^* \in E(J^*) \). Then there are \( v, w \in V(J) \) such that \( v^* = v^* \) and \( w^* = w^* \) and \( vw \in E(J) \). Then \( v \in X \) and \( w \in V(J) \setminus X \) and thus \( v \in S = \partial^J(X) \), which implies \( v^* = v^* \in S^* \). \hspace{1em} [12.6.3] follows from \hspace{1em} [12.6.1] and \hspace{1em} [12.6.2]. This completes the proof of Subclaim 1a.

**Subclaim 1b.** \( A \) is connected.

**Proof.** Let \( v_1, v_2 \in V(A) \). We shall prove that there is a path \( P \subseteq A \) from \( v_1 \) to \( v_2 \). As \( v_1^*, v_2^* \in X^* = \gamma^*(t^*) \) and \( \alpha^*(t^*) \) is connected and \( \sigma^*(t^*) = N^J(\alpha^*(t^*)) \), there is a path \( P^* \subseteq J^* \) from \( v_1^* \) to \( v_2^* \) with all internal vertices in \( \alpha^*(t^*) \). If \( v_1^*, v_2^* \in \alpha^*(t^*) \), then it follows \hspace{1em} [12.6.3] that there is a path \( P \subseteq A \) from \( v_1 \) to \( v_2 \).

Suppose that \( v_1^* \not\in \alpha^*(t^*) \). We shall prove that there is \( w_i \in N^J(v_i) \cap V(A) \) such that \( w_i^* \in \alpha^*(t^*) \). Then we can find a path \( Q \subseteq A \) from \( w_i \) to \( v_{3-i} \) (if \( v_{3-i} \in V(A^*) \)) or \( v_{3-i} \) (otherwise), and from \( Q \) we obtain the desired path \( P \).

As \( v_i^* \not\in \alpha^*(t^*) \) we have \( v_i^* \in S^* \). Thus by \hspace{1em} [12.6.2], \( v_i^* = v_e \) for some edge \( e = v_iv_i' \in M \) with \( v_i' \in S = \partial^J(X) \). Let \( w \in N^J(v_i') \setminus X \). By the Q4C Decomposition Lemma \[10.2.4\] \( v_i \not\in N^J(v_i') \) we have \( N^J(v_i') = \{v_i, w, w'\} \) for some \( w' \in N^J(w) \). As \( w \in V(J) \setminus X \), we have \( w' \in V(J) \setminus V(A) \) and thus \( N^J(v_i') \cap V(A) = \{v_i\} \). As \( v_i^* \in S^* = \gamma^*(t^*) = N^J(\alpha^*(t^*)) \), there is a \( w_i^* \in \alpha^*(t^*) \) with \( v_i^*w_i^* \in E(J^*) \). If \( w_i^* \in V_0 \) with , we let \( w_i := w_i^* \). Then \( w_i \) is \( N^J(v_i) \) and \( w_i \), \hspace{1em} [12.6.3], both endvertices of \( e \) are in \( V(A) \), and at least one endvertex \( w_i \) of \( e \) is adjacent to \( v_i \). This completes the proof of Subclaim 1b.

**Subclaim 1c.** \( N^J(A) = S \).

**Proof.** As we trivially have \( N^J(A) \subseteq S \), we only need to prove the converse inclusion. Let \( v \in S \). Then \( v^* = \gamma^*(t^*) = N^J(\alpha^*(t^*)) \), and thus there is a \( w^* \in \alpha^*(t^*) \) such that
$v^*w^* \in E(J^*)$. By \cite{12.6.2} and \cite{12.6.3} there is a $w \in V(A)$ such that either $vw \in E(J)$ or $v'w \in E(J)$ for some $v'$ such that $vv' \in M$. If $vw \in E(J)$, then $v \in N^d(A)$ as desired.

So assume that $v \in M$ and $v'w \in E(J)$. If $v' \in V(A)$, then again we have $v \in N^d(A)$. So assume that $v' \in \partial^d(X)$. Then there is a $w' \in N^d(v') \setminus X$. By the Q4C Decomposition Lemma \cite{10.2.4} we have $N^d(v') = \{v, w, w'\}$ and $ww' \in E(J)$. Hence $w \in \partial^d(X) = S$, a contradiction. This completes the proof of Subclaim 1c.

Thus $A$ is connected and $S = N^d(A) = \partial^d(X)$, where $X = V(A) \cup S$. Moreover, we have $S \subseteq (\sigma^*(t^*))^\circ$ and thus $|S| \leq 2|\sigma^*(t^*)| \leq 2d_*$. Subclaim 1d. There is a node $t^{++} \in V^+$ with $\sigma^{++}(t^{++}) = S$ and $\alpha^{++}(t^{++}) = V(A)$.

**Proof.** Let $t \in V$ such that $|\gamma^*(t^*) \setminus \gamma^*(t)| \leq d'_*$. By the definition of $\Phi^\circ$ we have $\gamma^*(t) = \gamma^+(t)^*$. We shall prove that $|X \setminus \gamma^+(t)| \leq 2d'_* + 2p$. This will imply the claim.

Let $v \in X \setminus \gamma^+(t)$. Then $v^* \in \gamma^*(t)$, and either $v^* \not\in \gamma^*(t)$, or $v$ is an endvertex of an edge $e = vw \in M$ such that $w \in \gamma^+(t)$ and hence $v^* = v_e \in \gamma^+(t)^* = \gamma^*(t)$. We let

$$Y_1 := V_0 \cap (\gamma^*(t^*) \setminus \gamma^*(t)),$$

$$Y_2 := \{v, w \mid e = vw \in M \text{ with } v_e \in \gamma^*(t^*) \setminus \gamma^*(t)\},$$

$$Y_3 := \{v \not\in \gamma^+(t) \mid \exists w : vw \in M \text{ and } w \in \gamma^+(t)\}.$$

Then $X \setminus \gamma^+(t^*) \subseteq Y_1 \cup Y_2 \cup Y_3$. We have $|Y_1 \cup Y_2| \leq 2|\gamma^*(t^*) \setminus \gamma^*(t)| \leq 2d'_*$. Furthermore, for each edge $vw$ that contributes to $Y_3$ we have $w \in \partial^d(\gamma^+(t^*))$, because $v \not\in \gamma^+(t)$. Thus $|Y_3| \leq |\partial^d(\gamma^+(t))| \leq 2p$ by Lemma \cite{12.6.1}. This proves Subclaim 1d.

To complete the proof of Claim 1, just note that $\gamma^+(t^{++}) = X = (\gamma^*(t^*))^\circ$ and that $\sigma^{++}(t^{++}) \subseteq (\sigma^*(t^*))^\circ$ and $\alpha^{++}(t^{++}) \supseteq (\alpha^*(t^*))^\circ$ follow from \cite{12.6.2} and \cite{12.6.3}.

We let $\pi_J \in V(\Lambda_{q4c}[G])$ be an index of $J^*$. We assume that $\Delta^* := (D^*, \sigma^*, \alpha^*, \leq^*)$ is the ordered completion of $\Phi^\circ$ in $J^*$ defined by $\Lambda^*(\pi)$ within $(G, \pi_J)$. By Lemma \cite{12.2.3} we may assume without loss of generality that $\Delta^*$ is tight.

We define a decomposition $\Delta' = (D', \sigma', \alpha')$ of $J$ as follows. We let $D' := D^*$. For all completion nodes $t \in V(D)$ we let $\sigma'(t) := (\sigma^*(t))^\circ$ and $\alpha'(t) := (\alpha^*(t))^\circ$. For all ground nodes $t \in V(D)$ we let $\gamma'(t) := (\gamma^*(t))^\circ$ and $\sigma'(t) := \partial^J(\gamma'(t))$

**Claim 2.** $\Delta'$ is a treeleaf decomposition of $J$.

**Proof.** Straightforward.

It follows from Claim 1 that for every ground node $t \in V(D') = V(D^*)$ there is a $t^{++} \in \Phi^+$ such that $t \parallel \Delta', \Phi^+, t^{++}$. Thus $\Delta'$ is a completion of $\Phi^+$ in $J$ that has the same ground nodes as the completion $\Delta^*$ of $\Phi^\circ$.

Observe, furthermore, that for all nodes $t \in V(D') = V(D^*)$ we have $\beta'(t) \subseteq (\beta^*(t))^\circ$.

From here on, it is straightforward to adapt the proof of Lemma \cite{10.3.2} to complete the proof of the lemma.

**Proof of the Q4C Completion Lemma** \cite{12.6.2}. To explain the definition of $\Lambda$, we let $G$ be a graph that satisfies (i) and (ii). Let $\Delta_{q4c} = (D_{q4c}, \sigma_{q4c}, \alpha_{q4c}) := \Lambda_{q4c}[G]$. By the Normalisation Lemma for Definable Treelike Decompositions \cite{5.2.3} we may assume without loss of generality that $\Delta_{q4c}$ is strict and normal. For every $t \in V(D_{q4c})$, let $J_t := \tau_{q4c}(t)$. Let $M_t$ be the matching defined by the formula $\mu$ of the Q4C Decomposition Lemma \cite{10.2.4} and let $J_t^* = J_t / M_t$ be the quasi-4-connected component with torso $J_t$ and matching $M_t$. For every $t \in$
Observe that $J$ be confused with $\leq$ the set of all $D_x$ which are leafs, into internal nodes. Instead, we have to define the decomposition of $\Phi$. Lemmas 12.2.3 and 4.3.7, we may assume that $\Delta$ lift the decompositions $\Delta$. Note that $\Delta'$ is precisely the set of $\Delta$ of $\Phi^{(d, d')}$.

Unfortunately, we cannot apply the Decomposition Lifting Lemma 5.6.2, mainly because it rearranges the decomposition of the torsos and may turn ground nodes, which are leafs, into internal nodes. Instead, we have to define the decomposition of $G$ “from scratch”. Nevertheless, our construction is similar to the construction of the Decomposition Lifting Lemma.

To simplify the notation, we write $N_{q4c}, \leq_{q4c}$ instead of $N_{q4c}^+, \leq_{q4c}$ and $N_t, \leq_t$ (not to be confused with $\leq_t$) instead of $N_{q4c}^+, \leq_{D_t}$ for all $t \in V(D_{q4c})$ and $N_t, \leq_t'$ instead of $N_{q4c}^+, \leq_t'$ for the directed graph $D'$ defined next. We define a decomposition $\Delta' = (D', \sigma', \alpha')$ of $G$ as follows.

(A) We let $V(D') := \{(t, x) \mid t \in V(D_{q4c}), x \in V(D_t)\}$.

(B) For all $(t, x), (u, y) \in V(D')$ there is an edge from $(t, x)$ to $(u, y)$ if either $t = u$ and $xy \in E(D_t)$ or $tu \in E(D_{q4c})$ and $\sigma_{q4c}(u) \cap \alpha_t(x) \neq \emptyset$ and $\sigma_{q4c}(u) \cap \alpha_t(x') = \emptyset$ for all $x' \in N_t(x)$.

Observe that $\sigma_{q4c}(u) \cap \alpha_t(x') \neq \emptyset$ implies $\sigma_{q4c}(u) \subseteq \gamma_t(x)$, because $\sigma_{q4c}(u)$ is a clique in $J_t = \tau_{q4c}(t)$. Thus the conditions $\sigma_{q4c}(u) \cap \alpha_t(x) \neq \emptyset$ and $\sigma_{q4c}(u) \cap \alpha_t(x') = \emptyset$ for all $x' \in N_t(x)$ imply that $\sigma_{q4c}(u) \subseteq \beta_t(x)$ and $\sigma_{q4c}(u) \subseteq \beta_t(x')$ for all $x' \leq_t x$. We let $U(t, x)$ be the set of all $u \in N_{q4c}(t)$ such that $\sigma_{q4c}(u) \cap \alpha_t(x') \neq \emptyset$. Then $U(t, x)$ is precisely the set of all $u \in N_{q4c}(t)$ such that $(t, x) \leq' (u, y)$ for some $y \in V(D_u)$.

(C) For all $(t, x) \in V(D')$, we let $\sigma'(t, x) := (\sigma_{q4c}(t) \cap \gamma_t(x)) \cup \sigma_t(x)$.

(D) For all $(t, x) \in V(D')$, we let $\gamma'(t, x) = \gamma_t(x) \cup \bigcup_{u \in U(t, x)} \sigma_{q4c}(u)$ and $\alpha'(t, x) := \gamma'(t, x) \setminus \sigma'(t, x)$.

Note that $\alpha'(t, x) = (\alpha_t(x) \setminus \sigma_{q4c}(t)) \cup \bigcup_{u \in U(t, x)} \sigma_{q4c}(u)$.

Claim 1. Let $(t, x) \in V(D')$ such that $x$ is $\leq_t$-minimal. Then $\sigma'(t, x) = \sigma_{q4c}(t)$ and $\alpha'(t, x) = \alpha_{q4c}(t)$.

Proof. As $J$ is connected and $\Delta_t$ is normal, we have $\sigma_t(x) = \emptyset$ and $\alpha_t(x) = V(J) = \beta_{q4c}(t)$. Thus $\sigma'(t, x) = \sigma_{q4c}(t)$ follows immediately from the definition of $\sigma'$. Moreover, we have $U(t, x) = N_{q4c}(t)$, because by Lemma 4.3.8 we have $\sigma_{q4c}(u) \cap \alpha_t(x) = \sigma_{q4c}(u) \neq \emptyset$ for all $u \in N_{q4c}(t)$. Thus $\gamma'(t, x) = \beta_{q4c}(t) \cup \bigcup_{u \in N_{q4c}(t)} \sigma_{q4c}(u) = \gamma_{q4c}(t)$ and hence $\alpha'(t, x) = \alpha_{q4c}(t)$.

Claim 2. $\Delta'$ is a treelike decomposition of $G$.

Proof. Straightforward.

Claim 3. For all $(t, x) \in V(D')$ it holds that $\beta'(t, x) = \beta_t(x) \cup (\sigma_{q4c}(t) \cap \gamma_t(x))$.

In particular, if $x$ is a leaf of $D_t$ then $\beta'(t, x) = \gamma_t(x)$.
Claim 4.

Let $(α, Case 1: Since $\emptyset \subseteq \Delta_3$, we have $x \in \bigcup_{(u,y) \in N'_3(t,x)} \alpha'(u,y)$.

If $x$ is a leaf of $D_t$, then $\beta_t(x) = \gamma_t(x)$ and thus $\beta'(t,x) = \gamma_t(x)$.

Let $(t,x) \in V(D')$. Then $(t,x)$ is an potential ground node (for short: pg-node) if $x$ is a ground node of $\Delta_t$, and $(t,x)$ is an potential completion node (for short: pc-node) if $x$ is a completion node of $\Delta_t$. Note that there is an IFP-formula that defines a linear order on all bags $\beta'(t,x)$ of pc-nodes $(t,x) \in V(D')$, because $\leq_t$ is a linear order on $\beta_t(x)$ and $|\beta'(t,x) \setminus \beta_t(x)| \leq |\sigma_{q4c}(t)| \leq 3$. However, we cannot directly use this formula to turn $\Delta'$ into an ordered completion of $\Phi^{(d,d')}$ whose ground nodes are the pg-nodes, for the following two reasons. First, the pg-nodes $(t,x)$ are parallel to nodes of $\Phi_t$ and not to nodes of $\Phi^{(d,d')}$. And second, the pg-nodes are not necessarily leaves of $D'$.

Consider a pg-node $(t,x) \in V(D')$. Let $z^{++} \in V(\Phi_t)$ such that $z^{++} \parallel \Phi_t, \Delta_t, x$. Recall that $\Phi_t = (\Phi_t^+)^{(2d',2d'+2p)}$, and let $z = (t,x) \in V(\Phi_t^+)$ such that $|\gamma_{\Phi_t}(z^{++}) \setminus \gamma_{\Phi_t^+}(z)| \leq 2d'_s + 2p$.

As $\gamma_{\Phi_t}(z) \subseteq \gamma_{\Phi}(z)$, we thus have

$$|\gamma_t(x) \setminus \gamma_{\Phi}(z)| = |\gamma_{\Phi_t}(z^{++}) \setminus \gamma_{\Phi_t^+}(z)| \leq 2d'_s + 2p. \quad (12.6.4)$$

As $x$ is a leaf of $D_t$, for all nodes $(u,y) \in N'_3(t,x)$ we have $u \in N_{q4c}(t)$, and $y$ is $\leq_u$-minimal. By Claim 1 and the tightness of the decomposition $\Delta_{q4c}$, this implies that $\alpha'(u,y) = \sigma_{q4c}(u)$ is connected in $G$ and that $\sigma'(u,y) = \sigma_{q4c}(u) = N^{G'}(\alpha'(u,y)) = \partial^G(\gamma'(u,y))$. Moreover, $|\sigma'(u,y)| = |\sigma_{q4c}(u)| \leq 3$.

We partition $N'_3(t,x)$ into three parts:

- $N_1 = N_1(t,x)$ is the set of all $(u,y) \in N'_3(t,x)$ with $\sigma'(u,y) \subseteq \sigma'(t,x)$.
- $N_2 = N_2(t,x)$ is the set of all $(u,y) \in N'_3(t,x)$ with $\sigma'(u,y) \not\subseteq \sigma'(t,x)$ and $\alpha'(u,y) \subseteq \gamma_{\Phi}(z)$.
- $N_3 = N_3(t,x) := N'_3(t,x) \setminus (N_1 \cup N_2)$

Claim 4. Let $(u,y) \in N_3$. Then either $\sigma'(u,y) \subseteq \sigma_{\Phi}(z) \cup (\gamma_t(x) \setminus \gamma_{\Phi}(z))$ or $\alpha'(u,y) \cap \sigma_{\Phi}(z) \neq \emptyset$.

Proof. Since $(u,y) \in N_3$, we have $\alpha'(u,y) \setminus \gamma_{\Phi}(z) \neq \emptyset$.

Case 1: $\alpha'(u,y) \cap \gamma_{\Phi}(z) \neq \emptyset$.

Remember that $\alpha'(u,y) = \sigma_{q4c}(u)$ is connected because $\Delta_{q4c}$ is tight. Thus $\alpha'(u,y) \cap \sigma_{\Phi}(z) \neq \emptyset$, because $\sigma_{\Phi}(z)$ separates $\gamma_{\Phi}(z)$ from $\alpha'(u,y) \setminus \gamma_{\Phi}(z)$.

M. Grohe, Definable Graph Structure Theory
Case 2: $\alpha'(u, y) \cap \gamma^\Phi(z) = \emptyset$.

We have $\sigma'(u, y) = \sigma_{q4c}(u) \subseteq \gamma_t(x)$, because $\sigma_{q4c}(u) \cap \alpha_t(x) \neq \emptyset$ and $\sigma_{q4c}(u)$ is a clique in $J_t$. As $\alpha'(u, y) \cap \gamma^\Phi(z) = \emptyset$ and $\sigma'(u, y) = N^G(\alpha'(t, y)) = N^G(\alpha'(u, y))$, it follows that $\sigma'(u, y) \cap \gamma^\Phi(z) \subseteq \partial^G(\gamma^\Phi(z)) = \sigma^\Phi(z)$, because $\Phi$ is tight. Thus $\sigma'(u, y) \subseteq \sigma^\Phi(z) \cup \gamma^\Phi(z)$.

Let

$$S_3(t, x) := \bigcup_{(u, y) \in N_3} \sigma'(u, y).$$

Claim 5. $|S_3(t, x)| \leq 2d'_e + 6p$.

Proof. Let $(u_1, y_1), \ldots, (u_m, y_m) \in N_3$ be a system of representatives of the $\|\Delta'$-equivalence classes in $N_3$. Without loss of generality we may assume that for some $\ell \in [m]$ we have $\sigma'(u_i, y_i) \subseteq \sigma^\Phi(z) \cup \gamma_t(x) \setminus \gamma^\Phi(z)$ for all $i \in [\ell]$ and $\sigma'(u_i, y_i) \subseteq \sigma^\Phi(z) \cup \gamma_t(x) \setminus \gamma^\Phi(z)$ for all $i \in [\ell + 1, m]$. Then

$$S_3(t, x) \subseteq \sigma^\Phi(z) \cup \gamma_t(x) \setminus \gamma^\Phi(z) \cup \bigcup_{i=1}^{\ell} \sigma'(u, y).$$

By Claim 4 we have $\alpha'(u_i, y_i) \cap \sigma^\Phi(z) \neq \emptyset$ for all $i \in [\ell]$. As the sets $\alpha'(u_i, y_i)$ are mutually disjoint for all $i \in [m]$ and as $|\sigma^\Phi(z)| \leq p$, it follows that $\ell \leq p$. Thus

$$|S_3(t, x)| \leq |\sigma^\Phi(z)| + |\gamma_t(x) \setminus \gamma^\Phi(z)| + \sum_{i=1}^{\ell} |\sigma'(u, y)| \leq p + 2d'_e + 2p + 3p = 2d'_e + 6p$$

by (12.6.4).

Let

$$H(t, x) := G \left[ \beta'(t, x) \cup \bigcup_{(u, y) \in N_2} \alpha'(u, y) \right],$$

$$S(t, x) := \sigma'(t, x) \cup S_3(t, x).$$

Note that the union in the definition of $H(t, x)$ is over $N_2 = N_2(t, x)$. This means that $H(t, x)$ is “mostly” contained in $\gamma^\Phi(z)$. For all $a \in V(H(t, x)) \setminus S(t, x)$, let $A(t, x, a)$ be the connected component of $H(t, x) \setminus S(t, x)$ that contains $a$. We define a new decomposition $(D, \sigma, \alpha)$ by modifying $\Delta'$ locally at all pg-nodes as follows.

- For all pg-nodes $(t, x)$ of $\Delta'$ and all $a \in V(H(t, x)) \setminus S(t, x)$, we add a new node $(t, x, a)$ and an edge from $(t, x)$ to $(t, x, a)$. Moreover, we delete all edges from $(t, x)$ to nodes $(u, y) \in N_2(t, x)$. That is, we let $D$ be the digraph with

$$V(D) := V(D') \cup \{(t, x, a) \mid (t, x) \in V(D') \text{ pg-node}, a \in V(H(t, x)) \setminus S(t, x)\},$$

$$E(D) := E(D') \setminus \{(t, x)(u, y) \mid (t, x) \in V(D') \text{ pg-node}, (u, y) \in N_2(t, x)\}$$

$$\cup \{(t, x)(t, x, a) \mid (t, x) \in V(D') \text{ pg-node}, a \in V(H(t, x)) \setminus S(t, x)\}.$$
• For all pg-nodes \((t, x)\) of \(\Delta'\) and all \(a \in V(H(t, x)) \setminus S(t, x)\), we let
\[
\begin{align*}
\gamma(t, x, a) &:= V(A(t, x, a)) \cup N^G(A(t, x, a)), \\
\sigma(t, x, a) &:= \partial^G(\gamma(t, x, a)), \\
\alpha(t, x, a) &:= \gamma(t, x, a) \setminus \sigma(t, x, a).
\end{align*}
\]

• For all \((t, x)\) \(\in V(D')\) we let
\[
\begin{align*}
\sigma(t, x) &:= \sigma'(t, x), \\
\alpha(t, x) &:= \alpha'(t, x).
\end{align*}
\]

**Claim 6.** Let \((t, x)\) be a pg-node of \(\Delta'\) and \(a \in V(H(t, x)) \setminus S(t, x)\). Then \(\sigma(t, x, a) \subseteq S(t, x)\).

**Proof.** Let \(v \in \sigma(t, x, a) = \partial^G(\gamma(t, x, a)) \subseteq N^G(A(t, x, a))\) and \(w \in N^G(v) \setminus \gamma(t, x, a)\). If \(w \in V(H(t, x))\), then \(v \in N^H(t, x)(A(t, x, a)) \subseteq S(t, x)\). So let us assume that \(w \notin V(H(t, x))\).

Then \(v \in \partial^G(H(t, x))\). By the definition of \(H(t, x)\) and \(N_1, N_2, N_3\), this implies that either \(v \in \sigma'(t, x)\) or \(v \in \sigma'(u, y)\) for some \((u, y) \in N_3\). Thus \(v \in S(t, x)\).

Since \(S(t, x) \subseteq \beta'(t, x) \subseteq V(H(t, x))\), it follows from Claim 6 that \(\gamma(t, x, a) \subseteq V(H(t, x))\).

**Claim 7.** \(\Delta\) is a treelike decomposition of \(G\).

**Proof.** The graph \(D\) is obviously acyclic, because \(D'\) is. Hence \(\Delta\) satisfies [TL1].

Axiom [TL2] for \(\Delta\) is inherited from [TL2] for \(\Delta'\) except maybe at the new nodes \((t, x, a)\), where it follows directly from the definitions.

Axiom [TL3] for \(\Delta\) is inherited from [TL3] for \(\Delta'\) except at the pg-nodes and their new children. So let \((t, x) \in V(D')\) be a pg-node and let \(a \in V(H(t, x)) \setminus S(t, x)\). We have \(\gamma(t, x, a) \subseteq V(H(t, x)) \subseteq \gamma'(t, x) = \gamma(t, x)\). Suppose for contradiction that \(\alpha(t, x, a) \nsubseteq \alpha'(t, x) = \alpha(t, x)\). Then \(\alpha(t, x, a) \cap \sigma'(t, x) \neq \emptyset\). Let \(v \in \alpha(t, x, a) \cap \sigma'(t, x)\). Recall that \(\sigma'(t, x) \subseteq \sigma_{q_{4\text{c}}}(t) \cup \sigma_t(x)\).

**Case 1:** \(v \in \sigma_{q_{4\text{c}}}(t)\).

Then \(v \in \partial^G(\gamma_{q_{4\text{c}}}(t))\), because \(\Delta_{q_{4\text{c}}}\) is tight. As \(\gamma(t, x, a) \subseteq \gamma'(t, x) \subseteq \gamma_{q_{4\text{c}}}(t)\), it follows that \(v \in \partial^G(\gamma(t, x, a)) = \sigma(t, x, a)\). This is a contradiction.

**Case 2:** \(v \in \sigma_t(x) \setminus \sigma_{q_{4\text{c}}}(t)\).

Then \(v \in \partial^I(\gamma_t(x))\), because \(\Delta_t\) is tight. Let \(w \in N^H(v) \setminus \gamma_t(x)\). Then \(w \in \beta_{q_{4\text{c}}}(t) \setminus \gamma_t(x) = \beta_{q_{4\text{c}}}(t) \setminus \gamma'(t, x)\). Thus if \(vw \in E(G)\), then as in Case 1 we have \(v \in \partial^G(\gamma(t, x, a))\), which leads to a contradiction. Hence \(vw \notin E(J_h) \setminus E(G)\), which means that either \(v, w \in \sigma_{q_{4\text{c}}}(t)\) or \(v, w \in \sigma_{q_{4\text{c}}}(u)\) for some \(u \in N_{q_{4\text{c}}}(t)\). Since \(v \notin \sigma_{q_{4\text{c}}}(t)\) by the assumption of Case 2, only the latter is possible. Let \(u \in N_{q_{4\text{c}}}(t)\) such that \(v, w \in \sigma_{q_{4\text{c}}}(u)\). As \(w \notin \gamma_t(x)\), we have \(\sigma_{q_{4\text{c}}}(u) \nsubseteq \gamma_t(x)\) and thus \(\alpha_{q_{4\text{c}}}(u) \cap \gamma'(t, x) = \emptyset\). As the decomposition \(\Delta_{q_{4\text{c}}}\) is tight, there is a \(w' \in N^G(v) \cap \alpha_{q_{4\text{c}}}(u) \subseteq N^G(v) \setminus \gamma'(t, x)\). Again, this implies \(v \in \partial^G(\gamma(t, x, a))\), which leads to a contradiction.

Axiom [TL4] for \(\Delta\) is inherited from [TL4] for \(\Delta'\) except at the pg-nodes. So let \((t, x) \in V(D')\) be a pg-node and \(z_1, z_2 \in N^D_+(t, x)\).

M. Grohe, *Definable Graph Structure Theory*
Case 1: There are $a_1, a_2 \in V(H(t, x)) \setminus S(t, x)$ such that $z_i = (t, x, a_i)$ for $i = 1, 2$.
If $A(t, x, a_1) = A(t, x, a_2)$ then $z_1 \parallel z_2$. Otherwise, $A(t, x, a_1) \cap A(t, x, a_2) = \emptyset$. Furthermore, $N^G(A(t, x, a_1)) \cap V(A(t, x, a_2)) = \emptyset$, because $N^{H(t, x)}(A(t, x, a_1)) \cap V(A(t, x, a_2)) = \emptyset$ and $A(t, x, a_2) \subseteq H(t, x)$. Similarly, $N^G(A(t, x, a_2)) \cap V(A(t, x, a_1)) = \emptyset$. Thus

$$\gamma(t, x, a_1) \cap \gamma(t, x, a_2) \subseteq N^G(A(t, x, a_1)) \cap N^G(A(t, x, a_2)).$$

Let $v \in N^G(A(t, x, a_1)) \cap N^G(A(t, x, a_2))$. Then $v$ has a neighbour in $A(t, x, a_2)$ and thus $v \in \partial^G(\gamma(t, x, a_1)) = \sigma(t, x, a_1)$, and similarly $v \in \sigma(t, x, a_1)$. Thus $\gamma(t, x, a_1) \cap \gamma(t, x, a_2) \subseteq \sigma(t, x, a_1) \cap \sigma(t, x, a_2)$.

Case 2: $z_1 = (u, y) \in N_1(t, x) \cup N_3(t, x)$ and there is an $a \in V(H(t, x)) \setminus S(t, x)$ such that $z_2 = (t, x, a)$.
Then $\alpha(z_1) = \alpha_{q_4c}(u)$ and $\sigma(z_1) = \sigma_{q_4c}(u)$. Thus

$$\alpha(z_1) \cap \gamma(z_2) \subseteq \alpha_{q_4c}(u) \cap V(H(t, x)) \subseteq \alpha_{q_4c}(u) \cap \left(\beta_{q_4c}(t, x, a) \cup \bigcup_{(u', y') \in N_2} \alpha_{q_4c}(u')\right) = \emptyset,$$

because if $(u', y) \in N_2$ then $u \parallel^{q_4c} u'$. Suppose for contradiction that $\gamma(z_1) \cap \alpha(z_2) = \gamma_{q_4c}(u) \cap \alpha(t, x, a) \neq \emptyset$. Let $v \in \gamma_{q_4c}(u) \cap \alpha(t, x, a)$. Then $v \in \sigma_{q_4c}(u)$. As $\Delta_{q_4c}$ is tight, $v$ has a neighbour in $\alpha_{q_4c}(u)$, which implies that $v \in \partial^G(H(t, x))$. As above, it follows that thus $v \in \partial^G(\gamma(t, x, a)) = \sigma(t, x, a)$, which is a contradiction.

Case 3: $z_2 = (u, y) \in N_1(t, x) \cup N_3(t, x)$ and there is an $a \in V(H(t, x)) \setminus S(t, x)$ such that $z_1 = (t, x, a)$.
Symmetric to Case 2.

Case 4: $z_i = (u_i, y_i) \in N_1(t, x) \cup N_3(t, x)$ for $i = 1, 2$.
The claim follows from (TL.4) for $\Delta'$.

Finally, Axiom (TL.5) for $\Delta$ is inherited from (TL.5) for $\Delta'$.

Claim 8. Let $(t, x) \in V(D')$.

(1) If $(t, x)$ is a pg-node of $\Delta'$ then $\beta(t, x) \subseteq S(t, x)$.

(2) If $(t, x)$ is a pc-node then $\beta(t, x) = \beta'(t, x)$.

Proof. Assertion (2) is obvious, because $(t, x)$ has the same children in $D$ and $D'$. To prove (1), suppose that $(t, x)$ is a pg-node. For $i = 1, 2, 3$, let $N_i := N_i(t, x)$. Let $N_4$ be the set of all nodes $(t, x, a)$ for $a \in H(t, x) \setminus S(t, x)$. Then $N^D(t, x) = N_1 \cup N_3 \cup N_4$ and thus

$$\beta(t, x) = \gamma(t, x) \setminus \left(\bigcup_{(u, y) \in N_1 \cup N_3} \alpha(u, y) \cup \bigcup_{(t, x, a) \in N_4} \alpha(t, x, a)\right)$$

$$= \left(\gamma_1(t, x) \cup \bigcup_{(u, y) \in N_1 \cup N_2 \cup N_3} \alpha_{q_4c}(u)\right) \setminus \left(\bigcup_{(u, y) \in N_1 \cup N_3} \alpha_{q_4c}(u) \cup \bigcup_{(t, x, a) \in N_4} \alpha(t, x, a)\right)$$

$$= \left(\gamma_1(t, x) \cup \bigcup_{(u, y) \in N_2} \alpha_{q_4c}(u)\right) \setminus \bigcup_{(t, x, a) \in N_4} \alpha(t, x, a).$$
As \(\sigma(t, x, a) \subseteq N^G(A(t, x, a))\) we have \(\alpha(t, x, a) \supseteq V(A, t, x, a))\) for all \((t, x, a) \in N_4\), we have

\[
\bigcup_{(t, x, a) \in N_4} \alpha(t, x, a) \supseteq \bigcup_{a \in H(t, x) \setminus S(t, x)} V(A(t, x, a))
\]

\[
= V(H(t, x)) \setminus S(t, x)
\]

\[
= \left(\beta'(t, x) \cup \bigcup_{(u, y) \in N_2} \alpha'(u, y)\right) \setminus S(t, x)
\]

\[
= \left(\gamma_t(x) \cup \bigcup_{(u, y) \in N_2} \alpha_{q_4c}(u)\right) \setminus S(t, x)
\]

(by Claim 3).

Thus

\[
\beta(t, x) \subseteq \left(\gamma_t(x) \cup \bigcup_{(u, y) \in N_2} \alpha_{q_4c}(u)\right) \setminus \left(\left(\gamma_t(x) \cup \bigcup_{(u, y) \in N_2} \alpha_{q_4c}(u)\right) \setminus S(t, x)\right) = S(t, x).
\]

Claim 9. Let \((t, x)\) be a pg-node of \(\Delta'\). Then

\[
|S(t, x)| \leq 2d_s + 2d_s' + 6p + 3.
\]

Proof. Recall that \(S(t, x) = \sigma'(t, x) \cup S_3(t, x) \subseteq \sigma_{q_4c}(t) \cup \sigma_t(x) \cup S_3(t, x)\). We have \(|\sigma_{q_4c}(t)| \leq 3\) and \(|\sigma_t(x)| = |\sigma^{\Phi_t}(z^{++})| \leq 2d_s\) and \(|S_3(t, x)| \leq 2d'_s + 6p\) by Claim 5. The claim follows.

Claim 10. Let \((t, x)\) be a pg-node of \(\Delta'\) and \(a \in V(H(t, x)) \setminus S(t, x)\). Then \((t, x, a)\) is a leaf of \(D\), and there is a node \(t' \in V(\Phi^{(d, d')})\) such that \((t, x, a) \parallel_{\Delta', \Phi^{(d, d')}} t'\).

Proof. It is immediate from the definitions that \((t, x, a)\) is a leaf of \(D\) and that \(\sigma(t, x, a) = \partial^G(\gamma(t, x, a)) = N^G(\alpha(t, x, a))\) and that \(\alpha(t, x, a)\) is connected.

We have \(\sigma(t, x, a) \subseteq S(t, x)\) and thus

\[
|\sigma(t, x, a)| \leq 2d_s + 2d'_s + 6p + 3 \leq d.
\]

Furthermore,

\[
\gamma(t, x, a) \subseteq V(H(t, x))
\]

\[
= \beta'(t, x) \cup \bigcup_{(u, y) \in N_2(t, x)} \alpha'(u, y)
\]

\[
= \gamma_t(x) \cup \bigcup_{(u, y) \in N_2(t, x)} \alpha'(u, y)
\]

(by Claim 3)

\[
\subseteq \gamma_t(x) \cup \gamma^\Phi(z(t, x))
\]

(by the definition of \(N_2(t, x)\)).

Thus

\[
|\gamma(t, x, a) \setminus \gamma^\Phi(z(t, x))| \leq |\gamma_t(x) \setminus \gamma^\Phi(z(t, x))| \leq 2d'_s + 2p \leq d'
\]

by (12.6.4). This proves the claim.

Thus \(\Delta\) is a completion of \(\Phi^{(d, d')}\). It is clear from the construction that \(\Delta\) is definable, that is, there is a \(d\)-scheme \(\Lambda\) such that \(\Lambda[G] = \Delta\).
It remains to turn $\Delta$ into an ordered completion of $\Phi^{(d,d')}$, The bags of the pc-nodes of $\Delta'$ remain unchanged in $\Delta$, and thus they admit a definable linear order. By Claim 8 and 9 the pg-nodes $(t,x)$ of $\Delta'$ have bags of cardinality bounded by $2d_* + 2d'_* + 6p + 3$ in $\Delta$. Thus these bags also admit a definable linear order (with $2d_* + 2d'_* + 6p + 3$ parameters). The new nodes $(t,x,a)$ are all ground nodes and thus need not be ordered. Thus we can apply Lemma 12.2.2 to obtain the desired ordered completion of $\Phi^{(d,d')}$. \qed
Robertson and Seymour [111] proved a structure theorem for graphs with excluded minors, which essentially says that all graphs excluding some fixed graph as a minor have a tree decomposition whose torsos are “almost embeddable” in some fixed surface. We will discuss this structure theorem in Chapter 17. In this chapter, we will start dealing with “almost embeddable” graphs by considering graphs that are almost embeddable in the plane.

In the previous chapters we have assembled our machinery for dealing with treelike decompositions. Now we are at the point where we seriously start to prove our Definable Structure Theorem 17.2.1. This means, in particular, that the graph theory gets much heavier. Indeed, for me the results of this chapter were the most difficult in the whole book.

Our goal is to prove that “almost planar graphs” admit IFP-definable ordered treelike decompositions. Almost planar graphs can be drawn onto a disk except for a subgraph, called “vortex” or “ring”, which is attached to the boundary of the disk. Recall that we proved that 3-connected planar graphs admit IFP-definable linear orders by defining the faces of the embedding in IFP. We could do this because 3-connected planar graphs have a unique embedding in the sphere. It turns out that almost planar graphs, even if 3-connected, no longer have a unique embedding. A key technical result of this chapter, Lemma 13.3.3, states that all “almost embeddings” of an almost planar graphs have the same facial cycles for faces that are sufficiently “central”, that is, far from the vortex, and that we can define these facial cycles in IFP. Then we can use the faces to define a linear order on the central vertices of the graph. The subgraph consisting of the vertices that are not central, which we call the “skeleton”, has bounded tree width, and we can define a bounded width treelike decomposition of this skeleton. We can extend it to an ordered treelike decomposition of the whole graph by the Ordered Extension Lemma 7.3.2. Intuitively, we just add the linearly ordered central vertices to each bag of the bounded width decomposition of the skeleton.

13.1 Relaxations of Planarity

13.1.1 Rings and Vortices

**Definition 13.1.1.** (1) A path decomposition of a graph $G$ is a tree decomposition $(P, \beta)$ of $G$ where $P$ is a (directed) path.

(2) A tuple $(v_1, \ldots, v_n) \in V(G)$ is increasing in a path decomposition $(P, \beta)$ of a graph $G$.
if there are \( t_1, \ldots, t_n \in V(P) \) such that \( t_1 \triangleleft P t_2 \triangleleft P \cdots \triangleleft P t_n \) and \( v_i \in \beta^P(t_i) \) for all \( i \in [n] \).

(3) Let \( p \in \mathbb{N} \). A \( p \)-ring is a pair \( (R, \pi) \), where \( R \) is a graph and \( \pi = (r_1, \ldots, r_n) \in V(G)^n \) such that there exists a path decomposition \( (P, \beta) \) of \( R \) of width at most \( (p - 1) \) in which \( \pi \) is increasing.

Note that for a tuple \( (v_1, \ldots, v_n) \in V(G) \) to be \emph{increasing} in a path decomposition \( (P, \beta) \), the vertices \( v_1, \ldots, v_n \) are not required to be distinct, but the nodes \( t_1, \ldots, t_n \in V(P) \) are. If \( v_i = v_j \) for some \( i < j \), then \( v_i = v_j \) is contained in all bags \( \beta(t) \) for \( t \in P(P) \) with \( t_1 \triangleleft P t \triangleleft P t_j \), because the path decomposition \( (P, \beta) \) satisfies axiom \([T.1]\) for tree decompositions.

The name “ring” for the objects we just defined may seem odd, because our rings are not closed. It seems more natural to use decompositions over cycles rather than paths in the definition of a ring. Indeed, this would be possible, but it would not make a big difference for the theory we shall develop here. The advantage of using path decompositions is that we can use the well-developed theory of tree decompositions to reason about path decompositions. One particularly important fact about tree decompositions that we shall use frequently is that every clique of the graph is contained in a bag of the decomposition. This is not the case for cycle decompositions, at least not if they are defined in the obvious way.

The name “ring” is motivated by the role rings play in arrangements (to be defined below), where they will be attached to boundary cycles of surfaces.

**Lemma 13.1.2.** Let \( p \in \mathbb{N} \). Let \( (R, \pi) \) be a \( p \)-ring with \( \pi = (r_1, \ldots, r_n) \). Then for all \( i \in [n - 1] \) there is a set \( S \subseteq V(R) \) with \( |S| \leq p \) such that \( S \) separates \( \{r_1, \ldots, r_i\} \) from \( \{r_{i+1}, \ldots, r_n\} \) in \( R \).

**Proof.** Let \( (P, \beta) \) be a path decomposition of \( R \) of width at most \( p - 1 \) such that \( \pi \) is increasing in \( (P, \beta) \). For \( i \in [n] \), let \( t_i \in V(P) \) such that \( r_i \in \beta(t_i) \). Then it follows from Fact \([4.1.3]\) that \( \beta(t_i) \) separates \( \{r_1, \ldots, r_i\} \) from \( \{r_{i+1}, \ldots, r_n\} \).

**Definition 13.1.3.** Let \( p \in \mathbb{N} \). A \( p \)-vortex is a pair \( (R, \pi) \), where \( R \) is a graph and \( \pi = (r_1, \ldots, r_n) \in V(G)^n \) such that for all \( i, j \in [n + 1] \) with \( i \leq j \) there is a set \( S \subseteq V(R) \) with \( |S| \leq p \) such that \( S \) separates \( \{r_1, \ldots, r_i\} \cup \{r_{j-1}, \ldots, r_n\} \) from \( \{r_j, \ldots, r_{j-1}\} \), where we take \( \{r_1, r_0\} \) and \( \{r_{n+1}, r_n\} \) to denote the empty set.

**Corollary 13.1.4.** Every \( p \)-ring is a \( 2p \)-vortex.

**Remark 13.1.5.** The converse of the previous corollary does not hold. However, Robertson and Seymour proved that all vortices are rings with other components attached to the bags (see Corollary \([A.0.4]\) for the precise statement). Robertson and Seymour’s definition of “vortex” slightly differs from ours, but the difference is not essential (see page \(469\) in Appendix A).

### 13.1.2 Arrangements in a Disk

Recall that for a graph \( G \) and a mapping \( \pi \) from \( V(G) \) to some set \( S \), by \( \pi(G) \) we denote the graph with vertex set \( \pi(V(G)) \) and edge set \( \{\pi(v)\pi(w) \mid vw \in E(G)\} \). Also recall that a graph is \emph{normally} embedded in a disk if it is embedded in such a way that the boundary curve of the disk intersects the graph only in vertices and not in the interior of any edges.

M. Grohe, \textit{Definable Graph Structure Theory}
Definition 13.1.6. An arrangement of a graph $G$ in a closed disk $D$ is a tuple

$$(G_0, \pi, R, \bar{r}),$$

where $G_0$ is a graph normally embedded in $D$ and $\pi: V(G_0) \to V(G)$ and $R \subseteq G$ and $\bar{r} = (r_1, \ldots, r_n) \in V(G_0)^n$ such that the following conditions are satisfied.

(AD.1) For all $v, w \in V(G_0)$ with $v \neq w$, if $\pi(v) = \pi(w)$ then $v, w \in \bar{r}$.

(AD.2) $G = \pi(G_0) \cup R$.

(AD.3) $E(\pi(G_0)) \cap E(R) = \emptyset$.

(AD.4) $V(\pi(G_0)) \cap V(R) = \pi(\bar{r})$.

(AD.5) $G_0 \cap bd(D) = \bar{r}$, and the vertices $r_1, \ldots, r_n$ appear on $bd(D)$ in cyclic order.

If the mapping $\pi$ is injective, then we call $(G_0, \pi, R, \bar{r})$ an injective arrangement of $G$ in $D$.

Let me introduce some terminology that we shall use throughout this chapter. Let $(G_0, \pi, R, \bar{r})$ be an arrangement of a graph $G$ in a disk $D$. As usual, for every $v \in V(G)$ we let $\pi^{-1}(v)$ be the set of all vertices $v_0 \in V(G_0)$ with $\pi(v_0) = v$. By [AD.1] and [AD.2] for every vertex $v \in V(G \setminus R)$ we have $|\pi^{-1}| = 1$. We define an embedding $\Pi$ of $G \setminus R$ into $D$ by letting $\Pi(v)$ be the unique vertex in $\pi^{-1}(v)$ for every $v \in V(G \setminus R)$ and letting $\Pi(e)$ be the edge of the embedded graph $G_0$ from $\Pi(v)$ to $\Pi(w)$ for every $e \in E(G \setminus R)$. If the arrangement is injective, then we can extend this embedding to $\pi(G_0)$.

By choosing the disk $D$ appropriately we may usually assume without loss of generality that the mapping $\pi$ in an arrangement $(G_0, \pi, R, \bar{r})$ is the identity on $V(G_0) \setminus \bar{r}$, and if the arrangement is injective we can even assume that $\pi$ is the identity and thus that $G_0$ is a subgraph of $G$.

Example 13.1.7. For some $n \geq 3$, we let $G$ be the graph obtained from a wheel with $n$ spokes by connecting all vertices of distance 2 on the outer cycle (see Figure 13.1). Formally, the wheel $W_n$ is the graph with $V(W_n) := [0, n]$ and $E(W_n) := \{0i \mid i \in [n]\} \cup \{i(i+1) \mid i \in [n-1]\} \cup \{nn\}$, and we let

$$G := W_n + \{\{i(i+2) \mid i \in [n-2]\} \cup \{(n-1)1, n2\}\}.$$

The graph $G$ has a natural injective arrangement $(G_0, \pi, R, \bar{r})$ in a disk $D$: we let $G_0$ be $W_n$ embedded into $D$ in such a way that the vertices $1, \ldots, n$ appear on the boundary of $D$ in cyclic order and 0 is in the interior. We let $\pi$ be the identity on $V(G_0) = V(W_n)$ and

$$R := ([n], \{i(i+2) \mid i \in [n-2]\} \cup \{(n-1)1, n2\})$$

Figure 13.1. The graph $G$ of Example 13.1.7 for $n = 8$
and $\tau := (1, \ldots, n)$.

**Example 13.1.8.** Every graph $G$ has a trivial arrangement $(G_0, \pi, R, \tau)$ in any disk $D$, where

- $R := G$.
- $\tau$ is an arbitrary tuple of vertices of $G$.
- $G_0 := (\tau, \emptyset)$ with the vertices of $\tau$ embedded on the boundary of $D$ in cyclic order.
- $\pi$ is the identity on $\tau$.

Obviously, the arrangement of Example 13.1.7 is much closer to our intention than the trivial arrangements defined in Example 13.1.8. The reason is that the graph $R$ in Example 13.1.7 follows the cyclic structure on the vertices in $\tau$ rather closely. The following definition tries to measure how well $R$ follows the cyclic structure on the vertices in $\tau$ in an arrangement.

**Definition 13.1.9.** Let $p \in \mathbb{N}$, and let $(G_0, \pi, R, \tau)$ be an arrangement of a graph $G$ in a disk $D$.

1. $(G_0, \pi, R, \tau)$ is a $p$-arrangement of $G$ in $D$ if $(R, \pi(\tau))$ is a $p$-ring.
2. $(G_0, \pi, R, \tau)$ is a local $p$-arrangement of $G$ in $D$ if $(R, \pi(\tau))$ is a $2p$-vortex.

A graph $G$ is $p$-almost planar if it has a $p$-arrangement in some closed disk. The class of $p$-almost planar graphs is denoted by $\text{AP}_p$.

**Example 13.1.10.** The arrangement defined in Example 13.1.7 is a 5-arrangement. To prove this, we define a path decomposition $(P, \beta)$ of $R$ of width 4: we let $P$ be the path with vertex set $[n]$ and edge set $\{i(i+1) \mid i \in [n-1]\}$. For each $i \in [n]$, we let $\beta(i) := \{1, 2, i, i+1, i+2\}$.

Note that the path width of $R$, being a cycle or a union of two cycles, is just 2, but there is no path decomposition of $R$ of width 2 in which the tuple $\tau$ is increasing.

By Corollary 13.1.4, every $p$-arrangement is a local $p$-arrangement. The following example shows that the converse fails drastically. Local arrangements are mainly interesting if substantial parts of the graph are embedded into the interior of the disk.

**Example 13.1.11.** Every graph $G$ has a trivial local 0-arrangement $(G_0, \pi, R, \tau)$. Just let $R := G$, $G_0 := \emptyset$, $\pi$ the empty mapping, and $\tau$ the empty tuple.

Injective arrangements seem more natural than arbitrary arrangements, but it is often more convenient to work with arrangements that are not necessarily injective. The following lemma shows that, in some sense, the difference between injective and general arrangements is not so big; every graph that has a $p$-arrangement in a disk is a minor of a graph that has an injective $(p+1)$-arrangement in a disk.

**Lemma 13.1.12.** Let $(G_0, \pi, R, \tau)$ be a $p$-arrangement of a graph $G$ in a disk $D$. Then there is a graph $G'$, a subgraph $R' \subseteq G'$, and an injective mapping $\pi' : V(G_0) \to V(G')$ such that the following two conditions are satisfied:

(i) $(G_0, \pi', R', \tau)$ is an injective $(p+1)$-arrangement of $G'$ in $D$.

(ii) There is a set $E' \subseteq E(R')$ such that $G = G'/E'$.
Proof. Without loss of generality we assume that \( \pi \) is the identity on \( V(G_0) \setminus \bar{r} \). Suppose that \( \bar{r} = (r_1, \ldots, r_n) \). Let \((P, \beta)\) be a path decomposition of \( R \) of width at most \((p - 1)\) such that \( \pi(\bar{r}) \) is increasing in \( P \). Without loss of generality we assume that \( P \) is the natural path on \([m]\). Let \( i_1 < i_2 < \ldots < i_n \in [m] \) such that \( \pi(r_j) \in \beta(i_j) \) for all \( j \in [n] \). We define a graph \( G' \) as follows:

- We let \( V(G') := (V(G) \setminus \pi(\bar{r})) \cup \bar{r} \), where we assume without loss of generality that \( V(G) \cap \bar{r} = \emptyset \).

- To define \( E(G') \), for each edge \( e \in E(G) \) we define a set \( E_e \subseteq (V(G'))^2 \), and for each \( r \in \pi(\bar{r}) \) we define a set \( E_r' \subseteq (V(G'))^2 \), and then we let

\[
E(G') := \bigcup_{e \in E(G)} E_e \cup \bigcup_{r \in \pi(\bar{r})} E_r'.
\]

Let \( e \in E(G) \).

Case 1: Both endvertices of \( e \) are in \( V(G) \setminus \pi(\bar{r}) \).
We let \( E_e := \{e\} \).

Case 2: At least one endvertex of \( e \) is in \( \pi(\bar{r}) \) and \( e = \pi(e_0) \) for some \( e_0 \in E(G_0) \).
We let \( E_e := \pi^{-1}(e) \).

Case 3: \( e = rv \) for some \( r \in \pi(\bar{r}) \) and \( v \in V(R) \setminus \pi(\bar{r}) \).
Suppose that \( \pi^{-1}(r) = \{r_{j_1}, \ldots, r_{j_\ell}\} \), where \( j_1 < j_2 < \ldots < j_\ell \). Choose an arbitrary \( i \in [m] \) such that \( e \subseteq \beta(i) \). We let

\[
j := \begin{cases} j_1 & \text{if } i < i_{j_2}, \\
j_k & \text{if } i_{j_k} \leq i < i_{j_{k+1}} \text{ for some } k \in [2, \ell - 1], \\
j_\ell & \text{if } \geq i_{j_\ell}
\end{cases}
\]

and then let \( E_e := \{r_{j}v\} \).

Case 4: \( e = r'r' \in E(R) \) for some \( r, r' \in \pi(\bar{r}) \).
We define \( E_e \) similarly to the previous case. Suppose that \( \pi^{-1}(r) = \{r_{j_1}, \ldots, r_{j_\ell}\} \), where \( j_1 < j_2 < \ldots < j_\ell \), and \( \pi^{-1}(r') = \{r'_{j'_1}, \ldots, r'_{j'_{\ell'}}\} \), where \( j'_{j'_1} < j'_{j'_2} < \ldots < j'_{j'_{\ell'}} \).
Note that the sets \( \{j_1, \ldots, j_\ell\} \) and \( \{j'_{j'_1}, \ldots, j'_{j'_{\ell'}}\} \) are disjoint. Choose an arbitrary \( i \in [m] \) such that \( e \subseteq \beta(i) \). We let

\[
j := \begin{cases} j_1 & \text{if } i < i_{j_2}, \\
j_k & \text{if } i_{j_k} \leq i < i_{j_{k+1}} \text{ for some } k \in [2, \ell - 1], \\
j_\ell & \text{if } \geq i_{j_\ell}
\end{cases}
\]

\[
j' := \begin{cases} j'_{j'_1} & \text{if } i < i_{j'_{j'_2}}, \\
j'_k & \text{if } i_{j'_{j_k}} \leq i < i_{j'_{j_{k+1}}} \text{ for some } k \in [2, \ell' - 1], \\
j'_{\ell'} & \text{if } \geq i_{j'_{j'_{\ell'}}}
\end{cases}
\]

and then let \( E_e := \{r_{j}r'_{j'}\} \).

Preliminary Version
Now let \( r \in \pi(\overline{r}) \). Suppose that \( \pi^{-1}(r) = \{ r_{j_1}, \ldots, r_{j_k} \} \), where \( j_1 < j_2 < \ldots < j_k \). We let \( E'_r := \{ r_{j_k} r_{j_{k+1}} \mid k \in \{ \ell - 1 \} \} \).

This completes the definition of \( E(G') \) and thus of \( G' \).

Let \( E' := \bigcup_{r \in \pi(\overline{r})} E'_r \) and observe that \( G = G'/E' \). Also observe that \( G_0 \subseteq G' \). We let \( \pi' \) be the identity on \( V(G_0) \). We define a subgraph \( R' \subseteq G' \) by \( V(R') := (V(R) \setminus \pi(\overline{r})) \cup \overline{r} \) and \( E(R') := \bigcup_{e \in E(R)} E_e \cup \bigcup_{r \in \pi(\overline{r})} E'_r \). Then \( (G_0, \pi', R', \overline{r}) \) is an injective arrangement of \( G' \) in \( D \). It remains to prove that it is a \((p+1)\)-arrangement.

Let \( r \in \pi(\overline{r}) \) with \( \pi^{-1}(r) = \{ r_{j_1}, \ldots, r_{j_k} \} \), where \( j_1 < j_2 < \ldots < j_k \). For every \( i \in [m] \), we let

\[
b(r, i) := \begin{cases}
\emptyset & \text{if } r \notin \beta(i), \\
\{ r_{j_i} \} & \text{if } r \in \beta(i) \text{ and } i < j_k, \\
\{ r_{j_{k-1}}, r_{j_k} \} & \text{if } i = j_k \text{ for some } k \in [2, \ell], \\
\{ r_{j_k} \} & \text{if } i < j < j_{k+1} \text{ for some } k \in [2, \ell - 1], \\
\{ r_{j_k} \} & \text{if } r \in \beta(i) \text{ and } i > j_k.
\end{cases}
\]

Note that for all \( i \in [j_1, j_k] \) we have \( r \in \beta(i) \), because \((P, \beta)\) is a path decomposition of \( R \) and \( r = \pi(j_1) = \pi(j_k) \in \beta(i_{j_1}) \cap \beta(i_{j_k}) \). Thus if \( b(r, i) \neq \emptyset \) then \( r \in \beta(i) \). Note furthermore that for every \( i \in [m] \) there is at most one \( r \in \pi(\overline{r}) \) such that \( |b(r, i)| = 2 \), because there is at most one \( j \in [n] \) such that \( i = j \). We let

\[
\beta'(i) := (\beta(i) \setminus \pi(\overline{r})) \cup \bigcup_{r \in \pi(\overline{r})} b(r, i).
\]

Then \((P, \beta')\) is a path decomposition of \( R' \) of width at most \( p \), and \( \overline{r} \) is increasing in \((P, \beta')\). Thus \((R', \overline{r})\) is a \((p+1)\)-ring.

The following lemma collects a few useful observations about arrangements of 3-connected graphs. Recall that \( \angle(G) \) denotes the set of angles of an embedded graph and that \( \overline{v} \searrow \overline{w} \) denotes that the angles \( \overline{v} \searrow \overline{w} \) are aligned (cf. Section 9.2).

Lemma 13.1.13. Let \( G \) be a 3-connected graph and \((G_0, \pi, R, \overline{r})\) an arrangement of \( G \) in a closed disk \( D \).

(1) For all faces \( f \in F(G_0) \) with \( bd(f) \cap bd(D) = \emptyset \), the facial subgraph \( Bd(f) \subseteq G_0 \) is a cycle.

(2) For all \( \overline{v} \in \angle(G_0) \) with \( \overline{v} \cap \overline{r} = \emptyset \) there is exactly one face \( f \in F(G_0) \) such that \( \overline{v} \in \angle(f) \).

(3) For all \( \overline{v} = (v_1, v_2, v_3) \in \angle(G_0) \) with \( (\overline{v} \cup N^{G_0}(\overline{v})) \cap \overline{r} = \emptyset \) there is exactly one \( v_4 \in V(G_0) \) such that \( (v_2, v_3, v_4) \in \angle(G_0) \) and \( (v_1, v_2, v_3) \searrow (v_2, v_3, v_4) \).

Proof. Let us first assume that \(|\pi(\overline{r})| \geq 3 \). Suppose that \( \overline{r} = (r_1, \ldots, r_n) \) for some \( n \in \mathbb{N} \). To simplify the notation, we let \( r_{n+1} := r_1 \). Let \( G'_0 \) be the graph obtained from \( G_0 \) by adding a new vertex \( v^* \), edges from \( v^* \) to all vertices in \( \overline{r} \), and adding an edge from \( r_i \) to \( r_{i+1} \) for all \( i \in [n] \) such that there is no edge from \( r_i \) to \( r_{i+1} \) in \( G_0 \).

Claim 1. \( G'_0 \) is 3-connected.

Proof. Let \( v_1, v_2 \in V(G'_0) \). Suppose for contradictions that \( S \subseteq V(G'_0) \setminus \{ v_1, v_2 \} \) separates \( v_1 \) from \( v_2 \) in \( G'_0 \). Then at most one of the vertices \( v_1, v_2 \) is in \( \overline{r} \cup \{ v^* \} \), because the graph

M. Grohe, Definable Graph Structure Theory
Let $G \in \tilde{G}$ are at most 2 vertices in $\tilde{G}$. Let $w_1 := v_1$. If $v_2 \notin \tilde{r} \cup \{v^*\}$, let $w_2 := v_2$, and otherwise let $w_2 \in \tilde{r}$ such that $\pi(w_2) \notin \pi(S)$ (such a vertex exists by our assumption $|\pi(\tilde{r})| \geq 3$). Observe that for $i = 1, 2$ there is a path, possibly of length 0, from $v_i$ to $w_i$ in $G_0' \setminus S$. For $v_2$ and $w_2$ this is the case because the graph $G_0'\tilde{r} \cup \{v^*\}$ is 3-connected. Hence $\tilde{S}$ also separates $w_1$ from $w_2$ in $G_0'$.

As $G$ is 3-connected, there is a path $Q$ from $\pi(w_1)$ to $\pi(w_2)$ in $G \setminus \pi(S)$. Let $x_1$ be the first vertex of $Q$ in $R$, and let $x_2$ be the last vertex of $Q$ in $R$ (possibly $x_2 = \pi(w_2)$). Then for $i = 1, 2$, there is a vertex $x_i' \in \tilde{r}$ with $\pi(x_i') = x_i$ and a path $Q_i \in G_0 \setminus S$ from $w_i$ to $x_i'$ such that $\pi(Q_i) = w_i \Pi x_i$. Furthermore, there is a path $Q' \subseteq G_0'\tilde{r} \cup \{v^*\} \setminus S$ from $x_1'$ to $x_2'$, because the graph $G_0'\tilde{r} \cup \{v^*\}$ is 3-connected. Then $Q_1 \cup Q' \cup Q_2$ is path from $w_1$ to $w_2$ in $G_0' \setminus S$, which is a contradiction.

Let $S \supseteq D$ be a sphere. We can extend the embedding of $G_0$ in $D$ to an embedding of $G_0'$ in $S$ by placing $v^*$ and all new edges into the disk $S \setminus int(D)$ in such a way that $v^*$ and the interior of all new edges are contained in $S \setminus D$. In the following, we view $G_0'$ as a graph embedded in the sphere $S$. Note that the embeddings of $G_0$ and $G_0'$ strongly coincide on $G_0$ (cf. page 188). Thus assertions (1) and (2) of the lemma follow from the corresponding assertions for the 3-connected planar graph $G_0'$ (Fact 9.1.21 and Lemma 9.2.1). Statement (3) follows from the fact that all facial subgraphs of $G_0'$ are cycles.

It remains to deal with the case that $|\pi(\tilde{r})| \leq 2$. If $V(G_0) \setminus \tilde{r} = \emptyset$, then the assertions of the lemma are void, thus we may assume without loss of generality that $V(G_0) \setminus \tilde{r} \neq \emptyset$. Then $V(R) \setminus \pi(\tilde{r}) = \emptyset$, because $G$ is 3-connected. If $|\tilde{r}| \leq 2$, then $G$ is planar, and it has an embedding that strongly coincides with the embedding of $G_0$ on $G_0 \setminus \tilde{r} = G \setminus \pi(\tilde{r})$. As $G$ is 3-connected, we can argue as above. So suppose that $|\tilde{r}| \geq 3$. Then we define a graph $G_0''$ as above by turning $\tilde{r}$ into a cycle and adding a vertex $v^*$ connected to all vertices in $\tilde{r}$. However, now it may happen that $G_0''$ is not 3-connected. But this can only happen if there are at most 2 vertices in $\tilde{r}$ that have neighbours in $V(G_0) \setminus \tilde{r}$. If this is the case, we define a 3-connected graph $G_0'''$ by adding edges from at most two vertices in $\tilde{r}$ that have no neighbour in $V(G_0) \setminus \tilde{r}$ to vertices in $V(G_0'') \setminus \tilde{r}$. We can easily add the edges in such a way that they can be drawn in faces of $V(G_0')$. Then $G_0'''$ is still planar and has a plane embedding that strongly coincides with the embedding of $G_0$ on $G_0 \setminus \tilde{r}$. Now we can argue as above.

\subsection{Central Vertices}

Definition 13.2.1. Let $k \in \mathbb{N}^+$.

1. Let $G$ be a graph embedded in a surface $S$. Then a set $W \subseteq V(G)$ is $k$-central in $G$ if there are closed disks $D_1, D_2, \ldots, D_k \subseteq S$ such that $W \subseteq int(D_1)$ and $D_i \subseteq int(D_{i+1})$ for all $i \in [k - 1]$, and for all $i \in [k]$ there is a cycle $C_i \subseteq G$ such that $bd(D_i) = C_i$.

If this is the case, we also say that $W$ is $k$-central within the disks $D_1, \ldots, D_k$ or within the cycles $C_1, \ldots, C_k$.

2. Let $G$ be a 3-connected planar graph. Then a set $W \subseteq V(G)$ is $k$-central in $G$ if for the unique (up to homeomorphism) embedding $\Pi$ of $G$ into $S_0$, if the set $\Pi(W)$ is $k$-central in the embedded graph $\Pi(G)$, we say that $W$ is $k$-central within the cycles $C_1, \ldots, C_k \subseteq G$ if $\Pi(W)$ is $k$-central within $\Pi(C_1), \ldots, \Pi(C_k)$ in $\Pi(G)$.

Let $G$ be either an embedded graph in a surface or a 3-connected planar graph. Then a vertex $w \in V(G)$, tuple $\overline{w} \in V(G)^\ell$, or subgraph $H \subseteq G$ is $k$-central in $G$ if the the set $\{w\}$, $\overline{w}$,
In this chapter we are interested in graphs embedded in a closed disk. We start with a few preliminary facts about such graphs. Let $D$ be a closed disk. For every simple closed curve $g \subseteq D$, let $\text{ins}(g)$ denote the (unique) arcwise connected component of $D \setminus g$ homeomorphic to an open disk. Let $\text{clins}(g) := d(\text{ins}(g))$. Now let $G$ be a graph embedded in $D$. For a cycle $C \subseteq G$, let $\text{Clins}(C) := G \cap \text{clins}(C)$ and $\text{Ins}(C) := G \cap \text{ins}(C) = \text{Clins}(C) \setminus C$. To characterise the central vertices in graphs embedded in a disk, we need a simple lemma. Though the lemma is probably well-known, in lack of a reference I sketch a proof (cf. the related Proposition 5.5.1 of [92]).

**Lemma 13.2.2.** Let $G$ be a graph normally embedded in a closed disk $D$, and let $f \in F(G)$. Then there is a closed disk $d \subseteq D$ such that $f \subseteq d$ and $\text{bd}(f) \supseteq \text{bd}(d)$.

**Proof.** We prove the lemma by induction on $|E(G)|$. It is trivial if $|E(G)| = 0$. So suppose that $|E(G)| > 0$ and suppose that the lemma is proved for smaller graphs. Let $H := \text{Bd}(f)$.

**Case 1:** There is a cycle $C \subseteq H$ such that $f \subseteq \text{ins}(C)$.

In this case, we apply the induction hypothesis to the disk $\text{clins}(C)$ and the graph $\text{Clins}(C) - E(C)$.

**Case 2:** There is a path $P \subseteq H$ with both endvertices in $\text{bd}(D)$.

We choose such a path such that all its internal vertices are in $\text{int}(D)$. Then there are two closed disks $D_1, D_2$ such that $D_1 \cup D_2 = D$ and $D_1 \cap D_2 = P$ and $\text{bd}(D_i) \subseteq P \cup \text{bd}(D)$. Let $i \in \{1, 2\}$ such that $f \subseteq D_i$. We apply the induction hypothesis to the disk $D_i$ and the graph $(G \cap D_i) - E(P)$.

**Case 3:** Neither Case 1 nor Case 2.

It is easy to see that in this case we have $\text{bd}(D) \subseteq \text{bd}(f)$, and we let $d := D$.

**Definition 13.2.3.** Let $G$ be a graph embedded in a closed disk $D$ and $v \in V(G)$. The centrality of $v$ is the minimum number $c(v)$ of vertices of $G$ on a $G$-normal simple curve from $v$ to $\text{bd}(D)$.

**Lemma 13.2.4.** Let $k \in \mathbb{N}^+$, and let $G$ be a graph normally embedded in a closed disk $D$. Then for every $v \in V(G)$ the following statements are equivalent:

(i) $v$ is $k$-central in $G$.

(ii) $v$ is $k$-central in $G$ within cycles $C_1, \ldots, C_k$ such that for all $i \in [k]$ and all $w \in V(C_i)$ it holds that $c(w) = k - i + 1$.

(iii) $c(v) \geq k + 1$.

**Proof.** Note that the implication (ii) $\Rightarrow$ (i) is trivial, and the implication (i) $\Rightarrow$ (iii) is also obvious, because every curve $g$ from a $k$-central vertex $v$ to $\text{bd}(D)$ contains $v$ and at least one vertex from each of the cycles witnessing that it is $k$-central. Thus we only need to prove (iii) $\Rightarrow$ (ii). The proof is by induction on $k$.

$k = 1$: Let $F_1$ be the set of all faces $f$ of $G$ with $\text{bd}(f) \cap \text{bd}(D) \not\subseteq V(G)$, and let

$$G_1 := \bigcup_{f \in F_1} \text{Bd}(f).$$

M. Grohe, *Definable Graph Structure Theory*
We view $G_1$ as a graph embedded in $D$. Note that for all $w \in V(G)$ it holds that $c(w) = 1 \iff w \in V(G_1)$.

Let $v \in V(G)$ such that $c(v) \geq 2$. Then $v \not\in V(G_1)$. Hence there is a face $f \in F(G_1)$ such that $v \in f$. Since $c(v) > 1$, we have $bd(f) \cap bd(D) \subseteq V(G)$. Thus there is a subgraph $H \subseteq G_1$ such that $bd(f) = H$. By Lemma 13.2.2 we can find a closed disk $d \subseteq D$ such that $f \subseteq d$ and $bd(d) \subseteq bd(f) = H$. Let $C_1 \subseteq H$ be a cycle such that $bd(d) = C_1$. We have $v \in ins(C_1)$. Thus $v$ is 1-central within $C_1$. Moreover, $C_1 \subseteq H \subseteq G_1$ and hence $c(w) = 1$ for all $w \in V(C_1)$.

$k \geq 2$: Let $v \in V(G)$ with $c(v) \geq k + 1$. By the induction hypothesis, there are cycles $C_2, \ldots, C_k$ such that $v$ is $(k-1)$-central within these cycles, and for all $i \in \{2, k\}$ and all $w \in V(C_i)$ it holds that $c(w) = (k - 1) - (i - 1) + 1 = k - i + 1$. Consider the graph $G_2 := Ins(C_2)$, viewed as a graph embedded in the disk $D_2 := clins(C_2)$.

Then $v \in V(G_2)$, and for every $G_2$-normal curve $g \subseteq D_2$ from $v$ to $C_2$ we have $|g \cap V(G)| \geq 2$. To see this, suppose for contradiction that there is a $G_2$-normal curve $g \subseteq D_2$ from $v$ to $C_2$ with $|g \cap V(G)| < 2$, or equivalently, $g \cap V(G) = \{v\}$. Then we can find a $G$-normal curve $g'$ from $v$ to a vertex $w \in V(C_2)$ with $g' \cap V(G) = \{v, w\}$. Let $g''$ be a $G$-normal curve from $w$ to $bd(D)$ with $|g'' \cap V(G)| = k - 1$. Then $g' \cup g''$ is a $G$-normal curve from $v$ to $bd(D)$ with $|g'' \cap V(G)| = k$. This contradicts $c(v) \geq k + 1$.

By the induction hypothesis applied to $G_2$ and $v$, we obtain a cycle $C_1 \subseteq G_2$ such that $v$ is 1-central in $G_2$ within $C_1$, and for all $w \in V(C_1)$ there is a $G_2$-normal curve $g$ from $w$ to $C_2$ with $|g \cap V(G_2)| = 1$. It follows that $v$ is $k$-central in $G$ within $C_1, \ldots, C_k$, and for every $w \in V(C_1)$ it holds that $c(w) = k$. \hfill \Box

13.2.1 Central Vertices with Respect to an Arrangement

Throughout this subsection, we make the following assumption:

**Assumption 13.2.5.** $(G_0, \pi, R, \overline{R})$ is an arrangement of a graph $G$ in a closed disk $D$.

As usually, $\Pi$ denotes the embedding of $G \setminus R$ into $D$ induced by the inverse of $\pi$. Obviously, for all vertices $v \in V(G_0)$ that are at least 1-central we have $\pi(v) \in V(G \setminus R)$ and thus $\Pi(\pi(v)) = v$.

**Lemma 13.2.6.** Let $H \subseteq G_0$ be 2-central in $G_0$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus \pi(H)$. Then there is an embedding $\Pi^*$ of the graph $G/A_1/A_2/\ldots/A_m$ into the open disk $int(D)$ such that $\Pi$ and $\Pi^*$ strongly coincide on $\pi(H)$.

**Proof.** Let $H$ be 2-central within the cycles $C_1, C_2$. Without loss of generality we assume that $N^G(A_i) \neq \emptyset$ for every $i \in [m]$. If not, $A_i$ will be contracted to an isolated vertex in $G/A_1/A_2/\ldots/A_m$ and does not affect the embedding. This implies that for every vertex $v \in V(G) \setminus V(\pi(H))$ there is a path from $v$ to a vertex in $\pi(H)$. We further assume that $\pi(C_1) \subseteq A_1$. Let $G' := \pi(Clins(C_1))$ and $A'_1 := G' \cap A_1$. Note that $Clins(C_1)$ is 1-central in $G_0$. Hence $G' \subseteq G \setminus R$ and therefore $\Pi$ induces an embedding of $G'$ into the open disk $int(D)$.

**Claim 1.** $G \setminus A_1 = G' \setminus A'_1$.

**Proof.** The inclusion $G \setminus A_1 \supseteq G' \setminus A'_1$ is trivial. For the converse inclusion, let $v \in V(G) \setminus V(G')$. We shall prove that $v \in V(A_1)$. Let $P$ be a path from $v$ to $V(\pi(H))$. Then $P \cap \pi(C_1) \neq \emptyset$, Preliminary Version
because $\pi(C_1)$ separates $V(\pi(H)) \subseteq V(G')$ from $V(G) \setminus V(G')$. Let $w$ be the first vertex of $P$ in $V(\pi(C_1))$. Then $v$ and $w$ belong to the same connected component of $G \setminus \pi(H)$, and as $w \in V(\pi(C_1)) \subseteq V(A_1)$ it follows that $v \in V(A_1)$.

\[ \text{Claim 2. } G/A_1 \cong G'/A_1'. \]

\textit{Proof.} We first observe that $A_1'$ is connected. To see this, let $v, w \in V(A_1')$. We shall prove that there is a path from $v$ to $w$ in $A_1'$. Let $P$ be a path from $v$ to $w$ in $A_1$. If $P \subseteq A_1'$, then there is nothing to prove, hence we assume that $P \not\subseteq A_1'$. Then $P \cap \pi(C_1) \neq \emptyset$. Let $v'$ be the first vertex on $P$ in $V(\pi(C_1))$, and let $w'$ be the last vertex on $P$ in $V(\pi(C_1))$. Furthermore, let $Q$ be a path from $v'$ to $w'$ in $\pi(C_1)$. Then $vPv'Qw'Pw$ is a path from $v$ to $w$ in $A_1'$.

To prove the claim, it remains to prove that $N^G(A_1) = N^{G'}(A_1')$. This is obvious, because if $v \in N^G(w) \setminus V(A_1)$ for some $w \in V(A_1)$, then $v \in V(H) \subseteq V(\pi(Ins(C_1)))$ and hence $w \in V(\pi(Clins(C_1))) = V(G')$. But then $w \in V(A_1) \cap V(G') = V(A_1')$ and $v \in N^{G'}(w)$.

The claims imply that the graphs $G/A_1/A_2/\ldots/A_m$ and $G'/A_1'/A_2'/\ldots/A_m$ are isomorphic. Now the lemma follows from Corollary \[9.1.19\] \[ \square \]

\textbf{Lemma 13.2.7.} Let $H$ be a 3-connected graph, and let $J \subseteq H$ be a 2-connected subgraph. Let $A_1, \ldots, A_m$ be connected components of $H \setminus J$ (not necessarily all of them). Let $H^* := H/A_1/A_2\cdots/A_m$. Then $H^*$ is 3-connected.

\textit{Proof.} Without loss of generality we may assume that $m \geq 1$. Then $|H^*| \geq 4$, because $3 \leq |J| < |H^*|$. For $i \in [m]$, let $a_i$ be the vertex of $H^*$ that corresponds to the contracted component $A_i$, and let

$$H' := H \setminus \bigcup_{i=1}^{m} A_i = H^* \setminus \{a_1, \ldots, a_m\}.$$ 

Since $J$ is 2-connected and $H$ is 3-connected, $H'$ is 2-connected.

Suppose for contradiction that $H'$ is not 3-connected. Let $S \subseteq V(H^*)$ be a separator of $H^*$ of order at most 2, and let $v_1, v_2$ be in different connected components of $H^* \setminus S$. For $j = 1, 2$, if $v_j = a_i$ for some $i \in [m]$, let $w_j \in N^{H^*}(a_i) \setminus S = N^H(A_i) \setminus S$. Such a $w_j$ exists because $|N^H(A_i)| \geq 3$. If $v_j \in V(H')$, let $w_j := v_j$. We shall prove that there is a path from $w_1$ to $w_2$ in $H^* \setminus S$; this will also give us a path from $v_1$ to $v_2$. Assume first that $S \cap \{a_1, \ldots, a_m\} = \emptyset$. Then $S \subseteq V(H')$. Let $P$ be a path from $w_1$ to $w_2$ in $H \setminus S$. Such a path exists because $H$ is 3-connected. But by contracting all edges of $P$ in $\bigcup_{i=1}^{m} E(A_i)$, we obtain a walk from $w_1$ to $w_2$ in $H^* \setminus S$, which is a contradiction. This proves that $S \cap \{a_1, \ldots, a_m\} \neq \emptyset$. Hence $|S \cap V(H')| \leq 1$. As $H'$ is 2-connected, there is a path from $w_1$ to $w_2$ with all internal vertices in $H' \setminus S$, and thus there is a path from $w_1$ to $w_2$ in $H^* \setminus S$. \[ \square \]

\textbf{Corollary 13.2.8.} Assume that $G$ is 3-connected. Let $H \subseteq G_0$ be 2-connected and 2-central in $G_0$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus \pi(H)$. Then $G^* := G/A_1/\ldots/A_m$ is a 3-connected planar graph. Furthermore, for every triple $\overline{\pi} \in V(H)^3$ it holds that $\overline{\pi} \in \angle(G_0) \iff \pi(\overline{\pi}) \in \angle(G^*)$.

\textit{Proof.} By Lemma \[13.2.7\] the graph $G^*$ is 3-connected. By Lemma \[13.2.6\] it has an embedding $\Pi^*$ into $\text{int}(D)$ that strongly coincides with $\Pi$ on $\pi(H)$. Thus $G^*$ is planar. Let $\overline{\pi} \in V(H)^3$. As $\Pi^*$ and $\Pi$ strongly coincide on $\Pi(H)$, we have $\pi(\overline{\pi}) \in \angle(G^*) = \angle(\Pi^*) \iff \pi(\overline{\pi}) \in \angle(\Pi)$. By the definition of $\Pi$, we have $\pi(\overline{\pi}) \in \angle(\Pi) \iff \overline{\pi} \in \angle(G_0)$. \[ \square \]

M. Grohe, \textit{Definable Graph Structure Theory}
13.3 Defining the Central Faces

Our goal for this section is to generalise Lemma 9.3.6 to almost planar graphs, that is, to define the angles and the alignment relation for a 3-connected almost planar graph. The following two examples show why the two proofs of the lemma given in Chapter 9 cannot easily be generalised from planar to almost planar graphs.

Example 13.3.1. In Example 9.3.11 we computed the set \( W_\infty \) with respect to the angle \((5, 0, 10)\) for the planar graph shown in Figure 9.6. We found that \( V(G) \setminus W_\infty = \{0, 5, 6, 7, 8, 9, 10\} \) is the vertex set of the facial cycle determined by \((5, 0, 10)\). We also computed \( W_\infty \) with respect to the triple \((5, 0, 15)\), which is not an angle, and found that \( V(G) \setminus W_\infty \) is not the vertex set of a chordless and nonseparating cycle.

Now consider the 4-almost planar graph \( G' \) obtained from \( G \) by adding edges between 25 and 30 and between 27 and 40 (see Figure 13.2(a)). If we compute \( W_\infty \) with respect to the angle \((5, 0, 10)\) we find that \( V(G) \setminus W_\infty = \{0, 5, 25, 30, 10\} \), which is the vertex set of a chordless and nonseparating cycle, but not the facial cycle determined by the angle \((5, 0, 10)\).

Let \( G'' \) be the 4-almost planar graph obtained from \( G \) by adding an edges between 25 an 35 and between 27 and 40 (see Figure 13.2(b)). If we compute \( W_\infty \) with respect to the triple \((5, 0, 15)\), which is not an angle, we find that \( V(G) \setminus W_\infty = \{0, 5, 25, 35, 15\} \), which again is the vertex set of a chordless and nonseparating cycle, but of course not a facial cycle. □

Example 13.3.2. Figure 13.3 shows a 5-almost planar graph (the graph without the dashed edge). Note that the graph is 3-connected and “highly” nonplanar. Also note that the facial cycles of the plane part of the graph all have length 7. Recall that in our proof that classes of graphs embeddable in a fixed surface admit ordered treelike decompositions it was crucial that sufficiently large embedded graphs always have facial cycles of length 6.
By a similar construction, for every surface $S$ and every $k \geq 4$ we obtain a 3-connected almost planar graph that is not embeddable in $S$ and only has facial cycles of length $k$.

Furthermore, not all facial cycles of the graph shown in Figure 13.3 are chordless. If we add the dashed edge to the graph, it remains 5-almost planar, but now has a cycle of length 5 that is chordless and nonseparating, but not facial.

What we can do is define the angles and the alignment relation for all faces of an almost planar graph that are sufficiently central. This is the assertion of the following lemma, which may be viewed as the main technical result of this chapter.

**Lemma 13.3.3.** For all $p \in \mathbb{N}$ there are $\mathbf{IFP}$-formulae

\[
\text{angle}_p(x_1, x_2, x_3) \quad \text{and} \quad \text{aligned}_p(x_1, x_2, x_3, x_4)
\]

with the following properties. Let $G$ be 3-connected graph and $(G_0, \pi, R, \tau)$ a local $p$-arrangement of $G$ in a closed disk $D$, and let $\Pi$ be the embedding of $G \setminus R$ into $D$ induced by $\pi$. Then for all vertices $v_1, v_2, v_3, v_4 \in V(G \setminus R)$ such that $\Pi(v_2)$ is $(5p + 15)$-central in $G_0$ it holds that

\[
G \models \text{angle}_p[v_1, v_2, v_3] \iff (\Pi(v_1), \Pi(v_2), \Pi(v_3)) \text{ is an angle of } G_0,
\]

\[
G \models \text{aligned}_p[v_1, v_2, v_3, v_4] \iff (\Pi(v_1), \Pi(v_2), \Pi(v_3), (\Pi(v_2), \Pi(v_3), \Pi(v_4)) \text{ are aligned angles of } G_0.
\]

The rest of this section is devoted to a proof of Lemma 13.3.3. We make the following assumptions.

**Assumption 13.3.4.** $G$ is a 3-connected graph that is not planar. $(G_0, \pi, R, \tau)$ is a local $p$-arrangement of $G$ in a closed disk $D$. We assume that $\tau = (r_1, \ldots, r_n)$, and we fix the orientation of the disk $D$ in such a way that the vertices $r_1, \ldots, r_n$ appear in clockwise order on $\text{bd}(D)$. We assume that $\pi$ is the identity on $V(G_0) \setminus \tau$.

As indicated before, the assumption that $\pi$ be the identity on $V(G_0) \setminus \tau$ is harmless, but very convenient. We thus identify $G \setminus R$ with the embedded graph $G_0 \setminus \tau$ in $\text{int}(D)$ and view $G_0 \setminus \tau$ as an embedded subgraph of the abstract graph $G$. This way, we completely avoid any
13.3. Defining the Central Faces

reference to the embedding $\Pi$. Note, however, that this identification cannot be extended to
the whole graph $G_0$, because the mapping $\pi$ may not be injective on $\tilde{r}$.

We will illustrate the graph $G$ as in Figure 13.4. The embedded subgraph $G \setminus R = G_0 \setminus \tilde{r}$ is
drawn inside a disk, and $R$ is indicated by a spiral around the disk. This (and the following)
figures are overly simplistic in that they assume $\pi$ to be injective. For a more accurate picture,
we would have to identify some of the vertices on the boundary of the disk. Fortunately, it
turns out that this is only a minor issue that makes no real difference.

13.3.1 Generic Facial Cycles

In this section we will prove a lemma that will later be crucial in identifying facial cycles
of $G_0$, as long as they have at least one vertex that is 3-central in $G_0$. The criterion for
identifying these cycles is that they are facial cycles of a 3-connected planar graph that is
a minor of $G$ and satisfies some additional conditions. As 3-connected planar graphs have
unique embeddings, this criterion makes no reference to the specific arrangement of $G$, except
for the rather mild condition that the cycle contains one vertex that is at least 3-central. In
this sense, we may call the cycles identified by this criterion “generic”.

Lemma 13.3.5. Let $C, C' \subseteq G$ be disjoint cycles. Let $A$ be the connected component of
$G \setminus C'$ that contains $C$. Let $F \subseteq E(G \setminus (A \cup C'))$ and $H := G/F$. Suppose that the following
conditions are satisfied:

(i) $H$ is a 3-connected planar graph.

(ii) $C$ is a facial cycle of $H$.

(iii) There are mutually disjoint cycles $C_1, \ldots, C_{p+2}$ such that $C_{p+2} = C'$ and $C$ is $(p + 2)$-
        central in $H$ within $C_1, \ldots, C_{p+2}$.

(iv) There is a vertex in $s \in V(C \setminus R)$ that is 3-central in $G_0$.

Then $C$ is a facial cycle of $G_0$. 

Preliminary Version
Note that the condition (iii) states that $C$ is $(p + 2)$-central in $H$ and not in $G_0$. Except in condition (iv), the assumptions of the lemma say nothing about how the cycles $C, C^1, \ldots, C^{p + 2} = C'$ are embedded in $G_0$, or if they belong to $G_0$ at all. However, the lemma essentially claims that “inside” the cycles $C^1, \ldots, C^{p + 2}$, the unique plane embedding of $H$ coincides with the embedding of $G_0$.

**Proof of Lemma [13.3.5]** Recall that a $C'$-bridge in $G$ is either a graph consisting of a single edge that is a chord of $C'$ together with its endvertices, or a connected component of $G \setminus C'$ together with its neighbours in $C'$, called the vertices of attachment of the bridge, and the edges from the component to the vertices of attachment. We call a $C'$-bridge in $G$ external if it contains an edge in $F$. As no edge in $F$ has an endvertex in $C'$, all external bridges contain at least one vertex in $V(G) \setminus V(C')$. We define the exterior graph $X \subseteq G$ to be the union of $C'$ with all external $C'$-bridges.

**Claim 1.** $X \cap R \neq \emptyset$.

**Proof.** Suppose for contradiction that $X \subseteq G \setminus R$. Then $X$ is a plane graph embedded in $\text{int}(D_i)$. It will actually be more convenient for us to view $X$ as being embedded in some sphere $S \supseteq D_i$. Let $B_1, \ldots, B_n$ be the external $C'$-bridges, and for each $i \in [n]$, let $Y_i := B_i \setminus C'$. Then $Y_i$ is a connected component of $G \setminus C'$, and it holds that $F \subseteq \bigcup_{i=1}^{n} E(Y_i)$. Let $H' := G/Y_1/\ldots/Y_n$. Then $H'$ is a minor of $H$, and hence $H'$ is a planar graph.

We can combine plane embeddings of $X$ and $H'$ to a plane embedding of $G$ as follows. Let $\Pi'$ be an embedding of $H'$ into a sphere $S'$. For every $i \in [n]$, we proceed as follows. Let $y_i$ be the vertex of $H'$ that corresponds to $Y_i$, and let $e_1, \ldots, e_k$ be the edges of $H'$ incident with $y_i$. For $j \in [k]$, let $v_j$ be the other endvertex of $e_j$. As $N^{H'}(y_i) = N^G(Y_i) \subseteq V(C')$, it holds that $v_j \in V(C')$. Let $D' \subseteq S'$ be the closed disk bounded by $\Pi'(C')$ that contains $\Pi'(y_i)$. Then $\Pi'(e_i)$ is a simple curve in $D'$ from $\Pi'(y_i)$ to $\Pi'(v_j)$. Let $N \subseteq \text{int}(D_i)$ be a small open neighbourhood of $\{\Pi'(y_i)\} \cup \bigcup_{i=1}^{k} (\Pi'(e_i) \setminus \{\Pi'(v_j)\})$ such that $N \cap \Pi'(H') = \{\Pi'(y_i)\} \cup \bigcup_{i=1}^{k} (\Pi'(e_i) \setminus \{\Pi'(v_j)\})$.

Now consider the plane graph $C' \cup B_i \subseteq X$ in the sphere $S$. There is a closed disk $D_i \subseteq S$ such that $C' = \text{bd}(D_i)$ and $C' \cup B_i \subseteq D_i$. It is not hard to see that there is a homeomorphism $h$ from $D_i$ to $D'$ that maps $C'$ to $\Pi'(C')$ and $B_i$ into $N \cup \{v_1, \ldots, v_k\}$. This yields an extension of the embedding $\Pi'$ to $H' \setminus \{y_i\} \cup B_i$. We can do this simultaneously for all $i \in [n]$ and obtain an embedding of $G$ into a sphere. This contradicts our general assumption (see Assumption [13.3.4]) that $G$ is not planar.

**Claim 2.** $C \cap X = \emptyset$, and $C^i \cap X = \emptyset$ for $i = 1, \ldots, p + 1$.

**Proof.** Note first that $C \cap X \subseteq A \cap X = \emptyset$, because $E(A) \cap F = \emptyset$. Let $B$ be the $C'$-bridge of $G$ with $A \subseteq B$. Then there is a path $P$ from $C'$ to $C$ in $B$. By assumption (iii), for $i \in [p + 1]$ the cycle $C^i$ separates $C'$ from $C$. Thus $P \cap C^i \neq \emptyset$ and therefore $C^i \subseteq A$. It follows that $C^i \cap X = \emptyset$.

Let $s \in V(C \setminus R)$ be 3-central in $G_0$. Such an $s$ existst by assumption (iii). We construct two cycles $C_1, C_2 \subseteq G \setminus R$ such that $s$ is 2-central within $C_1, C_2$ in $G_0$ and such that $C_1, C_2$ are “as close to $s$ as possible”. The construction is similar to the proof of Lemma [13.2.4]. Let $C'_1, C'_2, C'_3 \subseteq G_0$ such that $s$ is 3-central within $C'_1, C'_2, C'_3$ in $G_0$, and let $D'_1, D'_2, D'_3 \subseteq D$ be the closed disks bounded by these cycles. Let $F_1 \subseteq F(G_0)$ be the set of all faces of $G_0$ incident with $s$, and let $L_1 := \bigcup_{f \in F_1} \text{bd}(f)$. As $f \subseteq \text{int}(D'_1)$ for all $f \in F$, we have
Let $L_1 \subseteq D_1' \subseteq \text{int}(D_2')$ and hence $L_1 \cap C_2' = \emptyset$. The graph $L_1$ is a 2-connected graph embedded in $\text{int}(D)$. Hence there is a face $f_1 \in F(L_1)$ such that $C_2' \subseteq f_1$. We let $C_1 := \text{Bd}(f_1)$. By Fact 9.1.21, $C_1$ is a cycle. We let $D_1 := \text{clins}(C_1) \subseteq D$ be the closed disk bounded by $C_1$. As $C_1 \subseteq L_1 \subseteq \text{int}(D_2') \subseteq \text{int}(D)$, we have $C_1 \subseteq G \setminus R$. We construct the cycle $C_2$ in a similar way. Let $F_2 \subseteq F(G_0)$ be the set of all faces of $G_0$ incident with a vertex of $L_1$, and let $L_2 := \bigcup_{f \in F_2} \text{Bd}(f)$. As $L_1 \subseteq D_1' \subseteq \text{int}(D_2')$, for all $f \in F_2$ we have $f \subseteq \text{int}(D_2')$, and hence $L_2 \subseteq D_2' \subseteq \text{int}(D_2')$. The graph $L_2$ is a 2-connected graph embedded in $\text{int}(D)$. Hence there is a face $f_2 \in F(L_2)$ such that $C_2' \subseteq f_2$. We let $C_2 := \text{Bd}(f_2)$. By Fact 9.1.21, $C_2$ is a cycle. We let $D_2 := \text{clins}(C_2) \subseteq D$ be the closed disk bounded by $C_2$. Then

$$s \in \text{int}(D_1) \subseteq D_1 \subseteq \text{int}(D_2) \subseteq D_2 \subseteq \text{int}(D),$$

and therefore $s$ is 2-central within $C_1, C_2$.

Claim 3.

1. For every point $x \in C_1$ there is a simple curve $h \subseteq D_1$ from $x$ to $s$ that is internally disjoint from $G_0$.

2. For every point $x \in C_2$ there is a simple curve $h \subseteq D_2$ from $x$ to a vertex $v \in V(C_1)$ that is internally disjoint from $G_0$.

3. For every path $P \subseteq G_0 \setminus \{s\}$ with endvertices in $V(G) \setminus \text{int}(D_1)$ it holds that $P \cap \text{int}(D_1) = \emptyset$.

Proof. To prove (1), recall that every $x \in C_1 \subseteq L_1$ is in the boundary of some face in $F_1$. Let $f \in F_1$ with $x \in \text{bd}(f)$. Since $s$ is also in $\text{bd}(f)$, we can let $h$ be a simple curve from $x$ to $s$ with interior in $f$.

(2) can be proved similarly.

To prove (3), suppose for contradiction that $P \subseteq G \setminus \{s\}$ is a path with endvertices $v, w \in V(G) \setminus \text{int}(D_1)$ and $P \cap \text{int}(D_1) \neq \emptyset$. Without loss of generality we assume that $v, w \in V(C_1)$ and $P \setminus \{v, w\} \subseteq \text{int}(D_1)$. Let $x, y \in C_1 \setminus V(G)$ be two points separating $v$ from $w$ in $C_1$, that is, $v$ and $w$ belong to different arcwise connected components of $C_1 \setminus \{x, y\}$. By (1), there is a simple curve $h \subseteq D_1$ from $x$ to $y$ with interior in $\text{int}(D_1)$ and $h \cap G_0 \subseteq \{s\}$. Then $v$ and $w$ belong to different arcwise connected components of $D_1 \setminus h$. Hence $P \cap h \neq \emptyset$. But this is impossible, because $P \subseteq G_0 \setminus \{s\}$.

Let $V(F) := \bigcup F$ be the set of all endvertices of the edges in $F$, and let $A_1, \ldots, A_m$ be the connected components of $G[V(F)]$. Then $H = G/A_1/\ldots/A_m$. Without loss of generality, we may assume that $F = \bigcup_{i=1}^m E(A_i)$. Let $a_1, \ldots, a_m$ be the vertices of $H$ corresponding to the components $A_1, \ldots, A_m$, respectively. For every $v \in V(G)$, let

$$v^* := \begin{cases} a_i & \text{if } v \in V(A_i), \text{ for some } i \in [m], \\ v & \text{if } v \in V(G) \setminus V(F). \end{cases}$$

For every subgraph $J \subseteq G$, let $J^*$ be the subgraph of $H$ with $V(J^*) = \{v^* \mid v \in V(J)\}$ and

$$E(J^*) = \{v^*w^* \mid \exists v_0w_0 \in E(J) : v_0^* = v^* \text{ and } w_0^* = w^*\}.$$

Let $H_0 := G \setminus V(F) = H \setminus \{a_1, \ldots, a_m\}$. Note that $A, C, C' \subseteq H_0$. By (ii) and Fact 9.1.22, $C$ is a chordless and nonseparating cycle of $H$. 

Preliminary Version
Claim 4. $C$ is a chordless and nonseparating cycle of $G$.

Proof. Note first that if an edge $e \in E(G)$ is a chord of $C$ in $G$, then $e \in E(H_0)$ and hence $e$ is a chord of $C$ in $H$, which contradicts $C$ being chordless in $H$.

Suppose next that $C$ separates $v \in V(G \setminus C)$ from $w \in V(G \setminus C)$. As $C$ is nonseparating in $H$, there is a path $P' \subseteq H$ from $v^*$ to $w^*$. But then there is a path $P \subseteq G$ from $v$ to $w$ with $P^* = P'$. This is a contradiction.

Hence if $C \subseteq G \setminus R \subseteq G_0$, then it follows from Lemma 9.1.15 that $C$ is a facial cycle of $G_0$. In the following, we assume that

$$C \cap R \neq \emptyset \quad (13.3.1)$$

and show that this leads to a contradiction.

$C \cap R \neq \emptyset$ implies that all connected components of $C \setminus R$ are paths. Let $Q' \in C \setminus R$ be the connected component that contains $s$. Let $q'_1, q'_2 \in V(C) \cap V(R)$ with $q'_i \in N_C(q'_i)$. Let $Q := C[V(Q') \cup \{q'_1, q'_2\}]$ be the extension of $Q'$ by these vertices. Then $q'_1, q'_2 \in \pi(\tau)$, and either $q'_1 = q'_2$ and $Q = C$ or $Q$ is a path. Furthermore, $Q \subseteq \pi(G_0)$, and hence there is a path or cycle $Q_0 \subseteq G_0$ such that $\pi(Q_0) = Q$. Let $q_1, q_2 \in V(Q_0)$ such that $\pi(q_1) = q'_1$ and $\pi(q_2) = q'_2$. Then $q_1, q_2 \in \tau$, and either $q_1 = q_2$ and $Q_0$ is a cycle through $q_1$ or $q_1 \neq q_2$ and $Q_0$ is a path with endvertices $q_1, q_2$. Thus either $Q_0$ is a simple closed curve in $D$ or $Q_0$ is a simple curve in $D$ with endpoints $q_1, q_2$, and $Q_0 \cap \partial D = \{q_1, q_2\}$. Furthermore, $Q_0 \setminus \{q_1, q_2\} = Q' \subseteq G \setminus R$. Let $i_1, i_2 \in [n]$ such that $q_1 = r_{i_1}$ and $q_2 = r_{i_2}$. Without loss of generality we may assume that $i_1 \leq i_2$. Let $f_1, f_2$ be the two arcwise connected components of $D \setminus Q_0$ such that if $q_1 \neq q_2$, then $f_1$ is the component that contains the segment of $\partial D$ from $q_1$ to $q_2$ in clockwise direction, and if $q_1 = q_2$, then $f_1 = \text{ins}(Q_0)$ is the open disk in $\text{int}(D)$ with boundary $Q_0$. If $q_1 \neq q_2$, we let $Q_1 := sQq_1$ be the segment of $Q$ from $s$ to $q_1$, and $Q_2 := sQq_2$. If $q_1 = q_2$, we let $Q_1$ be the segment of the cycle $Q$ from $s$ to $q_1$ in clockwise direction, and we let $Q_2$ be the segment in anti-clockwise direction. Figures [13.5] (a) and (b) illustrate the two possibilities. From now on, in our figures — but only in the figures, not in the actual proof — we will always assume $q_1 \neq q_2$, that is, the situation of Figure [13.5] (a).

Claim 5. $C_1 \cup C_2 \subseteq H_0$.

Proof. Suppose for contradiction that $C_1 \cup C_2 \subseteq H_0$. As $C$ is nonseparating and chordless in $H$, there is a path $P' \subseteq H \setminus C$ from a vertex $v_{11} \in V(C_1) \cap f_1$ to a vertex $v_{12} \in V(C_1) \cap f_2$.

Figure 13.5.
that is internally disjoint from $V(C_1)$. Then there is a path $P \subseteq G \setminus C$ from $v_{11}$ to $v_{12}$ such that $P^* = P'$. The path $P$ must have a nonempty intersection with $R$, because it connects vertices in two different arcwise connected components of $D \setminus Q$. Hence for $j = 1, 2$, the path $P$ must have a nonempty intersection with $V(C_2) \cap f_j$, because $C_2$ separates $C_1$ from $R$. Let $v_{21}$ be the first vertex of $P$ in $V(C_2)$ and $v_{22}$ the last vertex of $P$ in $V(C_2)$. Then for $j = 1, 2$ we have $v_{2j} \in V(C_2) \cap f_j$ (see Figure 13.6). For $i, j \in [2]$, let $u_{ij}, w_{ij} \in V(C_i)$ be the vertices before and after $v_{ij}$ in clockwise direction (so $u_{ij}v_{ij}, v_{ij}w_{ij} \in E(C_i)$). Let $I_1, I_2, I_3, I_4$ be the segments of the cycles $C_1, C_2$ defined as follows:

- $I_1$ is the segment of $C_1$ from $v_{11}$ to $u_{12}$ in clockwise direction.
- $I_2$ is the segment of $C_1$ from $v_{12}$ to $u_{11}$ in clockwise direction.
- $I_3$ is the segment of $C_2$ from $w_{21}$ to $v_{22}$ in clockwise direction.
- $I_4$ is the segment of $C_2$ from $w_{22}$ to $v_{21}$ in clockwise direction.

Furthermore, let $I'_1$ be the extension of $I_1$ by the edge $u_{12}v_{12}$, and let $I'_2$ be the extension of $I_2$ by the edge $u_{11}v_{11}$. Then $V(I'_1 \cap I'_2) = \{v_{11}, v_{12}\}$ and $I'_1 \cup I'_2 = C_1$. Similarly, let $I'_3$ be the extension of $I_3$ by the edge $v_{21}w_{21}$, and let $I'_4$ be the extension of $I_4$ by the edge $v_{22}w_{22}$. Then $V(I'_3 \cap I'_4) = \{v_{21}, v_{22}\}$ and $I'_3 \cup I'_4 = C_2$. Observe that for $i = 1, \ldots, 4$ we have $I_i \cap Q \neq \emptyset$, because $I'_i$ is a path from a vertex in $f_1$ to a vertex in $f_2$. We now define a family of paths $P_{ij} \subseteq H$, for $1 \leq i < j \leq 4$ (see Figure 13.7):

- $P_{12}$ consists of the single edge $u_{12}v_{12}$.
- $P_{34}$ consists of the single edge $v_{21}w_{21}$.
- $P_{14} := v_{11}P_{v_{21}}$.
- $P_{23} := v_{12}P_{v_{22}}$.
- To define $P_{13}$ and $P_{24}$, let $C_{13} := I'_1 \cup P_{23} \cup I'_3 \cup P_{14}$ and $C_{24} := I'_2 \cup P_{23} \cup I'_4 \cup P_{14}$. Then $C_{13}$ and $C_{24}$ are cycles, and there are disjoint open disks $O_{13}, O_{24} \subseteq D$ such that $C_{13} = bd(O_{13})$ and $C_{24} = bd(O_{24})$. This is easily seen from the Figure 13.7. For $p \in \{13, 24\}$, let $D_p := cl(O_p)$. Note that $P_{14} \subseteq f_1$ and $P_{23} \subseteq f_2$, and as $f_1$ and $f_2$ are
Chapter 13. Almost Planar Graphs

separated by \( Q \), there must be a segment of \( Q \) from \( C_1 \) to \( C_2 \) in \( D_p \). We let \( P_p \) be this segment and \( P_p \subseteq G \) the underlying path in \( G \).

It is straightforward to verify that for \( 1 \leq i < j \leq 4 \), the path \( P_{ij} \) is a path from a vertex in \( I_i \) to a vertex in \( I_j \) that is internally disjoint from \( I_1, I_2, I_3, I_4 \). Moreover, the \( P_{ij} \) are pairwise internally disjoint. As subgraphs of \( Q \cup C_1 \cup C_2 \), the paths \( I_1, \ldots, I_4, P_{12}, P_{34}, P_{13}, P_{14} \) are contained in \( H_0 \subseteq H \). Here we use our assumption that \( C_1 \cup C_2 \subseteq H_0 \). The paths \( P_{14} \) and \( P_{23} \) are not necessarily in \( H \), but their contractions \( P_{14}^* := v_{11}P^*v_{21} \) and \( P_{23}^* := v_{12}P^*v_{22} \) are.

Hence by contracting \( I_1, \ldots, I_4 \) to single vertices and the paths \( P_p \), for \( p \in \{12, 13, \ldots, 34\} \) to single edges, we obtain a \( K_4 \)-minor of \( H \). Moreover, all \( I_j \) have a nonempty intersection with \( C \). Let \( H^+ \) be the graph obtained from \( H \) by adding a new vertex \( v^+ \) and edges from \( v^+ \) to all vertices in \( V(C) \). Then \( H^+ \) is still a planar graph, because \( C \) is a facial cycle of \( H \), and therefore we can extend a planar embedding of \( H \) to one of \( H^+ \) by placing \( v^+ \) and all edges incident with \( v^+ \) into the face bounded by \( C \). However, \( H^+ \) has a \( K_5 \)-minor, because there are edges from \( v^+ \) to the paths \( I_1, \ldots, I_4 \), which can be contracted to the vertices of a \( K_4 \)-minor of \( H \) (cf. Lemma 9.1.25). This is a contradiction.

Recall that our goal is to show that our assumption \((13.3.1)\) leads to a contradiction. To achieve this goal, we shall prove that \( C_1 \subseteq H_0 \) and \( C_2 \subseteq H_0 \), which obviously contradicts Claim 5.

**Claim 6.** \( C_1 \subseteq H_0 \).

**Proof.** Suppose for contradiction that \( C_1 \nsubseteq H_0 \). Let \( s' \in V(C_1) \setminus V(H_0) \subseteq V(F) \). Without loss of generality we may assume that \( s' \in f_1 \). Then there is a unique segment \( I \subseteq C_1 \) with endvertices \( t_1, t_2 \in V(C_1 \cap C) \) such that \( I \setminus \{t_1, t_2\} \subseteq f_1 \). Let \( I_1 := s't_1t_1 \) and \( I_2 := s't_2t_2 \). Then for all \( i \in \{2, \ldots, p+1\} \), the cycle \( C^i \) has a nonempty intersection with \( I_i \), because \( C^i \) separates \( V(F) \) from \( V(C) \). Since the cycle \( C^1 \) separates \( V(F) \) from \( V(C) \) and for \( j \in \{2, \ldots, p+1\} \), the cycle \( C^j \) separates \( V(F) \) from \( V(C^{j-1}) \), there are vertices \( u^1_i \in C^i \cap I_i \), for \( j \in \{p+1\} \), such that the vertices \( s', t_1, t_2, u^1_i, u^2_i \) appear in the following clockwise order on \( I_1 \):

\[
t_1, u^1_1, u^2_1, \ldots, u^{p+1}_1, s', u^{p+1}_2, u^p_2, \ldots, u^1_2, t_2.
\]

Let \( Q^X \subseteq X \) be a path from \( s' \in V(F) \subseteq V(X) \) to a vertex \( r' \in \pi(r) \). Such a path exists because the exterior graph \( X \) is connected and contains a vertex in \( R \) by Claim 1. Let \( Q^X_0 \subseteq

M. Grohe, *Definable Graph Structure Theory*
13.3. Defining the Central Faces

$G_0$ be a path from $s'$ to a vertex $r \in \overline{r}$ such that $\pi(Q_0^X) = Q^X$. Then $Q^X \setminus \{r'\} = Q_0^X \setminus \{r\}$. By Claim 2 it holds that $Q^X \cap C = \emptyset$ and $Q^X \cap C^j = \emptyset$ for all $j \in [p+1]$. As $Q^X \cap C = \emptyset$, we have $Q_0^X \subseteq f_1$ (see Figure 13.8). Let $i' \in [n]$ such that $r = r_{i'}$, and recall that $q_1 = r_{i_1}$ and $q_2 = r_{i_2}$. Then $i_1 < i' < i_2$.

For $j \in [p+1]$, let $P^{2j-1}, P^{2j}$ be the two segments of $C^j$ from $u_1^j$ to $u_2^j$. Then for each $k \in [2(p+1)]$, the path $P^k$ satisfies one of the following two conditions:

(i) There is a segment $P' \subseteq P^k$ such that $P' \subseteq R$, and the endvertices of $P'$ are $\pi(r_{\ell_1}), \pi(r_{\ell_2})$ with $\ell_1 \in [i_1+1, i'-1]$ and $\ell_2 \not\in [i_1+1, i'-1]$.

(ii) There is a segment $P' \subseteq P^k$ with endvertices $w_1 \in V(I_1), w_2 \in V(I_2)$ such that $P' \setminus \{w_1, w_2\} \subseteq \text{int}(D_1)$.

To see that one of these conditions must hold, consider a path $P_k$, where $k \in \{2j-1, 2j\}$ for some $j \in [p+1]$. Remember that $P^k \cap C \subseteq C^j \cap C = \emptyset$ and $P^k \cap Q^X \subseteq C^j \cap X = \emptyset$. Remember that $Q_0$ is the segment of $C$ in $D$ that contains $s$ and observe that every closed curve in $D \setminus (Q_0 \cup Q_0^X)$ from $u_1^j$ to $u_2^j$ goes through $\text{int}(D_1)$. Hence either the path $P^k$ satisfies (i), or it must go through $\text{int}(D_1)$ to get around $Q_0^X$ and thus satisfies (ii). Note that condition (ii) contradicts Claim 3(3). Hence all paths $P^k$ must satisfy (i). However, at most $2p$ of the $2(p+1)$ paths $P^k$ satisfy (i), because $R$ is a $2p$-vortex. This is a contradiction. 

Claim 7. $C_2 \subseteq H_0$. 

Proof. The proof is very similar to the proof of Claim 6, except that the analogue of condition (ii) cannot be ruled out by Claim 3, and hence the final part of the proof gets slightly more complicated. We assume that $s' \in V(C_2) \setminus V(H_0)$. Arguing completely analogously to the proof of Claim 6, we find a segment $I \subseteq C_2$ with endvertices $t_1, t_2 \in V(C_2 \cap C)$ and all internal vertices in $f_1$ (without loss of generality) and $s' \in V(I)$. For $i \in [2], j \in [p+1]$ we find vertices $u_i^j \in V(I \cap C^j)$ such that the vertices appear on $I$ in the following clockwise order:

$t_1, u_1^1, u_1^2, \ldots, u_1^{p+1}, s', u_2^{p+1}, u_2^p, \ldots, u_2^1, t_2$.

Figure 13.8. Proof of Claim 6

Preliminary Version
Furthermore, we find a path $Q^X \subseteq X$ from $s'$ to a vertex $r' \in \pi(\tilde{r})$ and a corresponding path $Q_0^X \subseteq G_0$ from $s'$ to a vertex $r = \pi(\tilde{r})$ with $\pi(r) = r'$ and $i_1 < i' < i_2$. For $j \in [p+1]$, let $P^{2j-1}, P^{2j}$ be the two segments of $C_j$ from $u_1^j$ to $u_2^j$ and observe that for each $k \in [2(p+1)]$ the path $P^k$ satisfies one of the following two conditions:

(i) There is a segment $P' \subseteq P^k$ such that $P' \subseteq R$, and the endvertices of $P'$ are $\pi(r_{\ell_1}), \pi(r_{\ell_2})$ with $\ell_1 \in [i_1+1, i'-1]$ and $\ell_2 \notin [i_1+1, i'-1]$.

(ii) There is a segment $P' \subseteq P^k$ with endvertices $w_1 \in V(I_1), w_2 \in V(I_2)$ and $P' \setminus \{w_1, w_2\} \subseteq \text{int}(D_2)$.

At this point, we deviate from the proof of Claim 6, because condition (ii) is not ruled out by Claim 3. However, we argue that there is at most one $k \in [2(p+1)]$ such that $P^k$ satisfies (ii). Suppose for contradiction that there are distinct $k_1, k_2 \in [2(p+1)]$ such that both $P^{k_1}$ and $P^{k_2}$ satisfy (ii). Let $P^i_1, P^i_2$ be corresponding segments with endvertices on $C_2$ and internal vertices in $\text{int}(D_2)$. By Claim 3(3) we have $P^i_1 \cap \text{int}(D_1) = \emptyset$ for $i = 1, 2$. Let $h \subseteq D_2$ be a simple curve from $s' \in V(C_2)$ to a vertex $u' \in V(C_1)$ such that $h$ is internally disjoint from $G_0$. Such a curve exists by Claim 3(2). As $s'$ appears between the endvertices of the path $P^i$, for $i = 1, 2$, and $P^i \subseteq (f_1 \cap D_2) \setminus \text{int}(D_1)$, we have $P^i_1 \cap h \neq \emptyset$ and hence $u' \in V(P^i_2)$. But then $u' \in V(P^i_1 \cap P^i_2)$, which is a contradiction because the paths $P^{k_1}$ and $P^{k_2}$ are internally disjoint.

Hence at least $2(p+1) - 1 > 2p$ of the paths $P^k$ must satisfy (i). This is impossible because $R$ is a $2p$-vortex.

To complete the proof of the lemma, we just note that Claims 6 and 7 contradict Claim 5.

13.3.2 Straightening the Boundary

In the following, let $p' := 4p + 10$ and $p'' := p' + p + 5$, and let $u_0 \in V(G_0)$ be $p''$-central in $G_0$. Let $C_{-(p+3)}, \hdots, C_{p'+1} \subseteq G_0$ be cycles such that $u_0$ is $p''$-central in $G_0$ within these cycles, and for all $i \in [-(p+3), p'+1]$ and all $v \in V(C_i)$, there is a $G_0$-normal simple curve $g \subseteq D$ from $v$ to a point in $\text{bd}(D)$ with $|g \cap V(G_0)| = p' + 2 - i$. Such cycles exist by Lemma 13.2.4. For $i \in [-(p+3), p'+1]$, let $D_i := \text{clins}(C_i)$ be the closed disk in $D$ bounded by $C_i$. For all $i \in [-(p+3), p']$, the subgraph $\text{Clins}(C_i)$ is 1-central in $G_0$ (within $C_{p'+1}$). Hence $\text{Clins}(C_i) \subseteq G \setminus R$. Let $G'_0 := \text{Clins}(C_{p'}) \subseteq G \setminus R$. Let $R' := G \setminus \text{Ins}(C_{p'})$. Furthermore, let $r' = (r'_1, \hdots, r'_{n'})$ be a tuple of vertices such that $V(C_{p'}) = r'$ and the vertices $r'_1, \hdots, r'_{n'}$ appear on $C_{p'}$ in clockwise order.

Lemma 13.3.6. For all $i, j \in [n']$ with $i \leq j$ there is a set $S' \subseteq V(R') \setminus r'$ with $|S'| \leq 2(p+1)$ such that $S' \cup \{r'_i, r'_j\}$ separates $\{r'_{i+1}, \hdots, r'_{i'}\} \cup \{r'_{j+1}, \hdots, r'_{j'}\}$ from $\{r'_{i'+1}, \hdots, r'_{j'}\}$ in $R'$.

In particular, $(R', r')$ is a $(2(p+2))$-vortex.

Proof. Let $Z := D \setminus \text{int}(D_{p'})$, and note that $R' \cap G_0 \supseteq R' \setminus R$ is embedded in $Z$. For every $i \in [n']$, let $g_i \subseteq D$ be a simple curve from $r'_i$ to a point $x_i \in \text{bd}(D)$ such that $|g_i \cap V(G_0)| = 2$ and all internal vertices of $g_i$ are in the open cylinder $\text{int}(Z)$. Let $s_i$ be the unique vertex in $(g_i \cap V(G_0)) \setminus \{r'_i\}$. Now let $i, j \in [n'+1]$ such that $i \leq j$. Let $I'_i$ be the segment of $C_{p'}$ from $r'_i$ to $r'_j$ in clockwise direction, and let $I'_j$ be the opposite segment.

Claim 1. There is a set $S \subseteq V(R)$ with $|S| \leq 2p$ such that $S \cup \{r'_i, s_i, r'_j, s_j\}$ separates $V(I'_i)$ from $V(I'_j)$ in $R'$.
13.3. Defining the Central Faces

Proof. For \( k = 1, 2 \), let \( A_k \) be the arcwise connected component of \( Z \setminus (g_i \cup g_j) \) with \( I_k \subseteq cl(A_k) \). Clearly, \( A_1 \cap A_2 = \emptyset \). Note that every path in \( P \subseteq R^I \setminus \{r'_i, s_i, r'_j, s_j\} \) from \( V(I'_1) \) to \( V(I'_2) \) is contained in \( A_1 \cup A_2 \cup R \). Hence if either \( A_1 \cap bd(D) = \emptyset \) or \( A_2 \cap bd(D) = \emptyset \), then there is no path from \( V(I'_1) \) to \( V(I'_2) \), and we let \( S = \emptyset \). Otherwise, \( A_1 \cap bd(D) \) and \( A_2 \cap bd(D) \) are the two segments of \( bd(D) \) from \( x_i \) to \( x_j \). As \( R \) is a 2p-vortex, there is a set \( S \subseteq V(R) \) with \( |S| \leq 2p \) that separates \( \pi(A_1 \cap \overline{r}) \) from \( \pi(A_2 \cap \overline{r}) \) in \( R \). The claim follows. \( \square \)

To complete the proof of the lemma, we choose \( S \) as in Claim 1 and let \( S' := S \cup \{s_i, s_j\} \). \( \square \)

Remark 13.3.7. Although we shall never use this explicitly, it is worth noting that \( (G'_0 \setminus E(C_{p'}), \pi', R', \tau') \), where \( \pi' \) is the identity on \( V(G'_0) \), is an injective local \( (p + 2) \)-arrangement of \( G \) in the disk \( D'p \). \( \square \)

We choose a vertex \( u^1 \in V(C_0) \) of minimum distance from \( u_0 \) in the graph \( G \) and let \( Q_1 \) be the set of all shortest paths from \( u_0 \) to \( u^1 \) in \( G \). Then we choose \( u^2 \in V(C_0) \setminus \{u^1\} \) of minimum distance from \( u_0 \) in the graph \( G \setminus \{u^1\} \) and let \( Q_2 \) be the set of all shortest paths from \( u_0 \) to \( u^2 \) in \( G \setminus \{u^1\} \). Then we choose \( u^3 \in V(C_0) \setminus \{u^1, u^2\} \) of minimum distance from \( u_0 \) in the graph \( G \setminus \{u^1, u^2\} \) and let \( Q_3 \) be the set of all shortest paths from \( u_0 \) to \( u^3 \) in \( G \setminus \{u^1, u^2\} \). We let

\[
U := \bigcup_{i=1}^{3} \bigcup_{Q \in Q_i} V(Q).
\]

The reason we take the union of three families of shortest paths rather than just three paths is that these families are \( \text{IFP} \)-definable (see Lemma 13.3.8(3)). The following lemma collects what we need to know about \( U \) later.

Lemma 13.3.8. (1) \( U \subseteq V(G) \cap D_0 \), and \( U \cap V(C_0) = \{u^1, u^2, u^3\} \).

(2) \( G[U] \) is connected and \( G[U] \cup C_0 \) is 2-connected.

(3) There is an \( \text{IFP} \)-formula \( \text{def-U}(x_0, x^1, x^2, x^3, y) \) such that \( \text{def-U}(G, u_0, u^1, u^2, u^3, y) = U \).

As usual, the formula \( \text{def-U}(x_0, x^1, x^2, x^3, y) \) does not depend on the graph \( G \) or any other of the specific choices we made.

Proof of Lemma 13.3.8. To prove (1), suppose for contradiction that there is a path \( Q \in Q_1 \) with \( V(Q) \setminus \{u^1\} \subseteq \text{int}(D_0) \). Let \( v \) be the first vertex on \( Q \) in \( V(G) \setminus \text{int}(D_0) \). Then \( v \in V(C_0) \), and \( \text{dist}(u_0, v) < \text{dist}(u_0, u^1) \), because \( Q \) is a shortest path. This contradicts the choice of \( u^1 \). We can rule out that there is a path \( Q \in Q_2 \cup Q_3 \) with \( V(Q) \setminus \{u^1\} \subseteq bd(D_0) \) by similar arguments.

The proof of (2) is straightforward, and (3) follows from the fact that shortest paths are definable in \( \text{IFP} \). \( \square \)

Without loss of generality we assume that the vertices \( u^1, u^2, u^3 \) appear on \( C_0 \) in clockwise order. For \( jk \in \{12, 23, 31\} \), let \( P_0^{jk} \) be the segment of \( C_0 \) from \( u^j \) to \( u^k \) in clockwise direction. It follows from Lemma 13.3.8(1) that there is an arcwise connected component \( o_0^{jk} \) of \( \text{int}(D_0) \) \( \setminus \) \( G[U] \) with \( \text{bd}(o_0^{jk}) \cap C_0 = P_0^{jk} \). Then \( o_0^{jk} \) is a face of the plane graph \( G[U] \cup C_0 \). It follows from Lemma 13.3.8(2) and Fact 9.1.21 that for all \( jk \in \{12, 23, 31\} \) the space \( o_0^{jk} \) is an open disk and \( o_0^{jk} := \text{cl}(o_0^{jk}) \) is a closed disk.
For later reference, we collect all the relevant choices we made so far and some of the facts we derived in the following assumption, which supersedes Assumption \[13.3.4\] and will remain valid until the end of this section:

**Assumption 13.3.9.** (i) \( p \in \mathbb{N}^+ \) and \( p' := 4p + 10 \) and \( p'' := p' + p + 5 = 5p + 15 \);

(ii) \( G \) is a 3-connected graph that is not planar;

(iii) \( D \) is a closed disk;

(iv) \((G_0, \pi, R, \tau)\) is a local \( p \)-arrangement of \( G \) in \( D \), where \( \tau = (r_1, \ldots, r_n) \) and \( \pi \) is the identity on \( V(G_0) \setminus \tau \);

(v) the orientation of \( D \) is fixed in such a way that the vertices \( r_1, \ldots, r_n \) appear on \( \text{bd}(D) \) in clockwise order;

(vi) \( u_0 \in V(G) \) is \( p'' \)-central in \( G_0 \) within cycles

\[ C_{-(p+3)}, C_{-(p+2)}, \ldots, C_{-1}, C_0, C_1, \ldots, C_{p'+1}, C_{p'+1} \subseteq G_0; \]

(vii) for all \( i \in [-(p+3), p'+1] \) and all \( v \in V(C_i) \), there is a \( G_0 \)-normal simple curve \( g \subseteq D \) from \( v \) to a point in \( \text{bd}(D) \) with \( |g \cap V(G_0)| = p' + 2 - i \);

(viii) for all \( i \in [-(p+3), p'+1] \) we let \( D_i := \text{clins}(C_i) \);

(ix) \( G_0' := \text{Clins}(C_{p'}) \subseteq G \setminus R = G_0 \setminus \tau \) and \( D_{p'} \subseteq \text{int}(D) \);

(x) \( R' := G \setminus \text{Ins}(C_{p'}) \), and \( \tau' = (r_{1}', \ldots, r_{n}') \) such that \( V(C_{p'}) = \tau' \) and the vertices \( r_{1}', \ldots, r_{n}' \) appear on \( C_{p'} = \text{bd}(D_{p'}) \) in clockwise order;

(xi) \( (R', \tau') \) is a \((p+2)\)-vortex,

(xii) \( u^1, u^2, u^3 \) are pairwise distinct vertices in \( V(C_0) \) appearing in clockwise order on \( C_0 \);

(xiii) \( U \subseteq V(G) \cap D_0 \) satisfies the statements of Lemma \[13.3.8\];

(xiv) for \( jk \in \{12, 23, 31\} \), the path \( P_{0k}^j \) is the segment of \( C_0 \) from \( u^j \) to \( u^k \) in clockwise direction, \( d_{0k}^j \subseteq D_0 \) is a closed disk with \( \text{bd}(d_{0k}^j) \cap C_0 = P_{0k}^j \) such that \( o_{0k}^j := \text{int}(d_{0k}^j) \) is an arcwise connected component of \( \text{int}(D_0) \setminus G[U] \).

Our next goal is to define a minor \( H \) of \( G \) such that \( H \) is a 3-connected planar graph and \( C_{-(p+3)}, \ldots, C_0 \subseteq V(H) \) and \( u_0 \in V(H) \) is \((p+4)\)-central within these cycles in \( H \). This implies that all facial cycles of \( H \) that contain \( u_0 \) will be \((p+3)\)-central within \( C_{-(p+2)}, \ldots, C_0 \). We will complete the proof of Lemma \[13.3.3\] by an application of Lemma \[13.3.5\]. To define \( H \), in the next subsection we first consider a special case, before we proceed to the general case in Subsection \[13.3.4\]. The construction in the special case contains all important ideas appearing in the general case later, but these ideas are not buried under technical complications that we will have to deal with in the general case.

M. Grohe, Definable Graph Structure Theory
13.3.3 A Special Case: Disjoint Curves

In this case, we make the following assumption (in addition to Assumption 13.3.9):

**Assumption 13.3.10.** For $j \in [3]$, there is a $G_0$-normal simple curve $g^j \subseteq D_{p'}$ from $w^j$ to a vertex $x^j \in V(C_{p'}) \subseteq bd(D_{p'})$ such that $|g^j \cap V(G_{0})| = p' + 1$ and the curves $g^1, g^2, g^3$ are mutually disjoint.

Note that the crucial restriction in this assumption is that the three curves $g^1, g^2, g^3$ be mutually disjoint; the rest follows already from Assumption 13.3.9. In the following, we fix curves $g^1, g^2, g^3$ satisfying the conditions of Assumption 13.3.10 and let $x^1, x^2, x^3$ be their endpoints in $V(C_{p'})$. For $j \in [3]$, we let $t^j_0, \ldots, t^j_{p'}$ be the vertices in $g^j \cap V(G_{0})$ in the order in which they appear on the curve $g^j$, starting at $w^j$. Then $t^j_0 = w^j, t^j_{p'} = x^j$, and for all $i \in [0, p']$ we have $g^j \cap C_i = \{t^j_i\}$. Let $T^j := \{t^j_i \mid i \in [0, p']\}$ and

$$T := T^1 \cup T^2 \cup T^3.$$  

For $i \in [0, p']$, let $T^j_i := \{t^j_i, t^j_{i+1}, \ldots, t^j_{p'}\}$. As $u^1, u^2, u^3$ appear on $C_0$ in clockwise order, for every $i \in [0, p' - 1]$, the vertices $t^1_i, t^2_i, t^3_i$ appear on $C_i$ in clockwise order. For $jk \in \{12, 23, 31\}$, let $P^{jk}_i$ be the segment of $C_i$ from $u^j$ to $u^k$ in clockwise direction.

Note that the curves $g^1, g^2, g^3$ cut the cylinder $Z := D_{p'} \setminus \text{int}(D_0)$ into three pieces. For $jk \in \{12, 23, 31\}$, let $o^{jk}$ be the connected component of $\text{int}(Z) \setminus (g^1 \cup g^2 \cup g^3)$ with $bd(o^{jk}) = g^j \cup g^k \cup P^{0k}_i \cup P^{jk}_{p'}$. Let $d^{jk} := \text{cl}(o^{jk})$. Then $d^{jk} \subseteq Z$ is a closed disk, and $d^{jk}_0 \cap d^{jk} = P^{jk}_o$. Let $D^{jk} := d^{jk}_0 \cup d^{jk}.$

Then $D^{jk}$ is a closed disk by Fact 9.1.9. Furthermore, for every $i \in [0, p']$, it holds that $D^{jk} \cap C_i = P^{jk}_i$, and for $i \in [0, p' - 1]$ it holds that $\text{int}(D^{jk}) \cap C_i = P^{jk}_{i+1} \setminus \{t^j_{i+1}, t^j_{i+2}\}$. Let $S^{jk} \subseteq V(R')$ with $|S^{jk}| \leq 2(p + 1)$ such that $S^{jk} \cup \{t^j_{p'}, t^{j'}_{p'}\}$ separates $V(P^{jk}_{p'}) = D^{jk} \cap \overline{r'}$ from $V(C_{p'} \setminus P^{jk}_{p'}) = \overline{r'} \setminus D^{jk}$ in the graph $R'$. Such a set exists by Lemma 13.3.6. Let

$$S := S^{12} \cup S^{23} \cup S^{31}.$$  

Then $|S| \leq 6p + 6$. Observe that $(S \cup T \cup U) \cap \text{int}(D^{jk}) = \emptyset$.

**Lemma 13.3.11.** Let $j, k, \ell \in [3]$ such that $k = j + 1 \pmod 3$ and $\ell = j + 2 \pmod 3$. Then $(S \cup T \cup U)$ separates $V(G) \cap D^{jk}$ from $V(G_{0}) \setminus \text{int}(D^{jk})$ in $G$.

**Proof.** Observe first that $V(G_{0}) \setminus bd(D^{jk}) \subseteq U \cup T \cup \overline{r'}$. Hence every path $P \subseteq G \setminus (T \cup U)$ from a vertex in $V(G) \cap D^{jk}$ to a vertex in $V(G \setminus (D^{kj} \cup D^{lj}))$ contains a segment $P' \subseteq R'$ from a vertex in $\overline{r'} \cap D^{jk}$ to a vertex in $\overline{r'} \setminus D^{jk}$. However, by the choice of $S$, each such path $P'$ contains a vertex of $S$. Now observe

For $i \in [0, p']$ and $j \in [3]$, we let $v^j_i$ be the clockwise and $w^j_i$ the anticlockwise neighbour of $t^j_i$ on $C_i$. Let $jk \in \{12, 23, 31\}$. Note that $v^j_i$ and $w^j_i$ are the first and last internal vertices of the path $P^{jk}_i$ unless the path $P^{jk}_i$ has length 1, in which case $v^j_i = t^j_i$ and $w^j_i = t^j_{i+1}$. Hence if the path $P^{jk}_i$ has length greater than 1, it holds that $v^j_i, w^j_i \in \text{int}(D^{jk}) \subseteq D^{jk} \setminus (S \cup T \cup U).$
In this case, there is an \((S \cup T \cup U)\)-bridge \(B_{ik}^j\) with \(P_{ik}^j \subseteq B_{ik}^j\). If the path \(P_{ik}^j\) has length 1, then we let \(e_{ik}^j\) be the edge from \(t_{i}^j\) to \(t_{i}^k\) and \(B_i^j := P_{ik}^j \cup \{t_{i}^j, t_{i}^k, (e_{ik}^j)\}\). We let

\[
B_{ik}^j := \bigcup_{i=0}^{p'} B_i^j.
\]

For \(i \in [0, p' - 1]\), let \(A_i^j := B_i^j \setminus (S \cup T \cup U)\). That is, if the path \(P_{ik}^j\) has length greater than 1 then \(A_i^j\) is the connected component of \(G \setminus (S \cup T \cup U)\) contained in the bridge \(B_i^j\). Otherwise, \(A_i^j = \emptyset\). Let

\[
A^j := \bigcup_{i=0}^{p'} A_i^j.
\]

Note that for all \(i \in [0, p']\) the path \(P_{ik}^j\) is contained in \(B^j\), and if the path \(P_{ik}^j\) has length greater than 1 then the segment \(v_i^j P_{ik}^j w_k^j\) is contained in \(A^j\).

**Lemma 13.3.12.** For \(jk \in \{12, 23, 31\}\), let \(A_{0}^j := A^j \cap G'\) and \(B_{0}^j := B^j \cap G'\). Then \(A_{0}^j \subseteq D_{0}^j \setminus (S \cup T \cup U)\) and \(B_{0}^j \subseteq D_{0}^j\).

**Proof.** Let \(A\) be a connected component of \(A^j\). Then there is an \(i \in [0, p']\) such that \(v_i^j P_{ik}^j w_k^j \subseteq A\). As \(v_i^j \in D^j\) and \(V(A) \cap (S \cup T \cup U) = \emptyset\) and \((S \cup T \cup U)\) separates \(V(G) \cap D^j\) from \(V(G_0) \setminus D^j\), we have \(V(A \cap G_0) \subseteq D^j \setminus (S \cup T \cup U)\). It follows from the definition of \(D^j\) that for all edges \(e \in E(G_0)\), either \(e \subseteq bd(D^j)\) or \(e \cap bd(D^j)\) is a vertex in \(S \cup T \cup U\). Thus \(e \subseteq D^j \setminus (S \cup T \cup U)\) for all edges in \(E(A \cap G_0)\). Letting \(A_0 := A \cap G_0\), this implies \(A_0 \subseteq D_0^j \setminus (S \cup T \cup U)\). Furthermore, if \(B\) is the bridge that contains the component \(A\) and \(B_0 := B \cap G_0\), we have \(B_0 \subseteq D^j\).

As \(A\) is an arbitrary component of \(A^j\), it follows that \(A_{0}^j \subseteq D_{0}^j \setminus (S \cup T \cup U)\). It also follows that for all bridges \(B = B_{ik}^j\) where \(P_{ik}^j\) has length greater than 1 it holds that \(B_{0}^j \subseteq D_{0}^j\). For the remaining bridges in \(B^j\), that is, those where \(P_{ik}^j\) has length 1, it holds that \(B_{0}^j = P_{ik}^j \subseteq D^j\). Hence \(B_{0}^j \subseteq D^j\).

**Lemma 13.3.13.** For all \(jk, j'k' \in \{12, 23, 31\}\) with \(jk \neq j'k'\) it holds that \(V(B_{ik}^j \cap B_{i'k'}^{j'}) \subseteq S \cup T \cup U\) and \(E(B_{ik}^j \cap B_{i'k'}^{j'}) = \emptyset\).

**Proof.** Note that \(D^j \cap D^{j'k} \cap V(G) \subseteq (S \cup T \cup U)\). Hence \(V(B_{ik}^j \cap B_{i'k'}^{j'}) \setminus (S \cup T \cup U) = V(A_{0}^j \cap A_{0}^{j'}) \setminus (S \cup T \cup U) \subseteq (D_{0}^j \cap D_{0}^{j'k'} \cap V(G)) \setminus (S \cup T \cup U) = \emptyset\). The only edges in \(B^j\) with both endvertices in \(S \cup T \cup U\) are those of the form \(e_{ik}^j\). However, as \(jk \neq j'k'\) we have \(e_{ik}^j \neq e_{i'k'}^{j'}\) for all \(i, i' \in [0, p']\) such that \(e_{ik}^j\) and \(e_{i'k'}^{j'}\) are defined. Hence \(E(B_{ik}^j \cap B_{i'k'}^{j'}) = \emptyset\).

We define a family of subsets \(W_n^j \subseteq V(A^j)\), for \(n \in \mathbb{N}\), as follows.

- If the length of \(P_{0}^j\) is 1, then we let \(W_n^j = \emptyset\) for all \(n \in \mathbb{N}\).

Assume that the length of \(P_{0}^j\) is greater than 1. Let \(p'' := p' - 2p - 4\) and note that

\[
|T_{p'}^j| = |\{t_{p'}, t_{p'} + 1, \ldots, t_{p'} + p''\}| = p' - p'' + 1 = 2(p + 2) + 1
\]

and similarly \(|T_{p''}^j| = 2(p + 2) + 1\). We define

M. Grohe, *Definable Graph Structure Theory*
Observe that for every $G$

13.3. Defining the Central Faces

303

Proof. The proof is by induction on $\{v^j_0\}$;

• $W^j_{0} := \emptyset$;

• $W^j_{1} := \{v^j_0\}$;

• for $n \in \mathbb{N}^+$, the set $W^j_{n+1}$ is the union of $W^j_n$ with the set of all $w \in V(A^j_k)$ with the following property: $w$ is adjacent to a vertex in $W^j_n$ and there is no $S' \subseteq V(B^{jk})$ such that $|S'| \leq 2p + 3$ and $S' \cup \{w\}$ is a minimal $(T^j_{p''}, T^k_{p''})$-separator in the graph $B^{jk} \setminus W^j_n$.

Figure 13.9. Proof of Lemma 13.3.14

\[ \begin{array}{c}
\text{Figure 13.9. Proof of Lemma 13.3.14} \\
\end{array} \]

Observe that for every $n \geq 1$ and every vertex $w \in W^j_n$ there is a path $Q$ from $v^j_0$ to $w$ in $G[W^j_n]$. Hence the graph $G[W^j_n]$ is connected. Also note that $W^j_n \cap V(G'_0) \subseteq V(A^j_k \cap G'_0) \subseteq D^{jk} \setminus (S \cup T \cup U)$ by Lemma 13.3.12.

Lemma 13.3.14. For all $n \in \mathbb{N}$ it holds that $W^j_n \subseteq \text{int}(D^j_{p'})$.

Proof. The proof is by induction on $n$. The induction basis $n \leq 1$ is trivial. For the inductive step, let $n \geq 1$ and suppose that $W^j_n \subseteq \text{int}(D^j_{p'})$. Let $w \in W^j_{n+1} \setminus W^j_n$. Suppose for contradiction that $w \notin \text{int}(D^j_{p'})$. Then $w \in V(A^j_k) \cap \text{bd}(D^j_{p'}) \subseteq V(C^j_{p'})$. Let $Q$ be a path from $t^j_0$ through $v^j_0 \in W^j_1$ to $w$ with all internal vertices in $W^j_n$ (see Figure 13.9). Then $Q$ is a simple curve from $t^j_0 \in \text{bd}(D^j_k)$ to $w \in \text{bd}(D^j_k)$ with interior in $\text{int}(D^j_k)$. This curve separates $T^j_{p''}$ and $T^k_{p''}$ in $D^j_k$, and therefore the set $V(Q)$ separates $T^j_{p''}$ and $T^k_{p''}$ in $B^{jk} \cap G'_0$. Hence $\{w\}$ separates $T^j_{p''}$ from $T^k_{p''}$ in $(B^{jk} \cap G'_0) \setminus W^j_n$.

Let $I$ be the segment of $C^j_{p'}$ from $t^j_{p'}$ to $w$ and $J := C^j_{p'} \setminus I$. Let $w' := w^j_{p'}$, the anticlockwise neighbour of $t^j_{p'}$ on $C^j_{p'}$. Then $w' \in V(J)$. By Lemma 13.3.6 there exists a set $S_1 \subseteq V(R') \setminus t^j_{p'} = V(R') \setminus V(C^j_{p'})$ of size $|S_1| \leq 2(p + 1)$ such that $S_1 \cup \{w, w'\}$ separates $V(I)$ from $V(J)$ in $R'$. Then $(S_1 \cup \{w, w'\}) \cap V(B^{jk})$ separates $T^j_{p''}$ from $T^k_{p''}$ in $B^{jk} \setminus W^j_n$. Let $S_2 \subseteq (S_1 \cup \{w, w'\}) \cap V(B^{jk})$ be a minimal $(T^j_{p''}, T^k_{p''})$-separator in $B^{jk} \setminus W^j_n$. Then $w \in S_2$, Preliminary Version
Lemma 13.3.16. 

By Lemma 13.3.12 we have 

Proof.

For all 

Corollary 13.3.15.

Because \( (S_1 \cup \{w, w'\}) \cap P_{p'}^{jk} = \{w\} \), and \( S_2 \) must contain a vertex of the path \( P_{p'}^{jk} \subseteq B^{jk} \) from \( t_{p'}^j \in T_{p''}^{jk} \) to \( t_{p'}^k \in T_{p''}^{jk} \). However, this contradicts \( w \) being in \( W_{n+1}^{jk} \).

\[ \square \]

Corollary 13.3.15. For all \( n \in \mathbb{N} \) it holds that \( W_{n}^{jk} \subseteq int(D^{jk}) \).

Proof. By Lemma 13.3.12 we have \( W_{n}^{jk} \cap D_{p'} \subseteq V(A^{jk} \cap G_0^j) \subseteq D^{jk} \setminus (S \cup T \cup U) \). By Lemma 13.3.14 we have \( W_{n}^{jk} \subseteq int(D_{p'}) \) and hence

\[ W_{n}^{jk} \subseteq D^{jk} \setminus (S \cup T \cup U) \cap int(D_{p'}) \subseteq D^{jk} \setminus (S \cup T \cup U \cup C_{p'}) \]

and as \( V(G) \cap bd(D^{jk}) \subseteq S \cup T \cup U \cup P_{p'}^{jk} \subseteq S \cup T \cup U \cup C_{p'} \), it follows that \( W_{n}^{jk} \subseteq int(D^{jk}) \).

\[ \square \]

Lemma 13.3.16. There is an \( n \in \mathbb{N}^+ \) such that \( V(P_0^{jk}) \subseteq W_{n+1}^{jk} \cup \{i_1^j, t_0^k\} \).

Proof. The statement is trivial if \( P_0^{jk} \) has length 1. Otherwise, let \( n \geq 1 \), and let \( w \) be the first internal vertex on \( P_0^{jk} \) not in \( W_{n+1}^{jk} \). I claim that \( w \in W_n^{jk} \). Once we have proved this, we are done, because it implies that \( V(P_0^{jk}) \subseteq W_n^{jk} \cup \{i_1^j, t_0^k\} \). Figures 13.10 and 13.11 illustrate the proof.

Suppose for contradiction that \( w \not\in W_n^{jk} \). As \( w \) is adjacent to a vertex in \( W_n^{jk} \), this means that there is an \( S' \subseteq V(B^{jk}) \) such that \( |S'| \leq 2p + 3 \) and \( S' \cup \{w\} \) is a minimal \( (T_{p''}^{jk}, T_{p''}^{jk}) \)-separator in the graph \( B^{jk} \setminus (W_n^{jk}) \). Choose such an \( S' \). By the minimality of \( S' \cup \{w\} \), there is a path from \( T_{p''}^{jk} \) to \( T_{p''}^{jk} \) in \( B^{jk} \setminus (W_n^{jk} \cup S') \). Let \( Q \) be such a path. Then \( w \in V(Q) \). Let \( x \in T_{p''}^{jk} \) and \( y \in T_{p''}^{jk} \) be the endvertices of \( Q \). As \( |S'| \leq 2p + 3 \) and \( p' > 4p + 9 \) and therefore \( p'' = 1 = p' - 2p - 5 > 2p + 3 \), there is an \( i_0 \in [p'' - 1] \) such that \( S' \cap V(C_{i_0}) = \emptyset \). We fix such an \( i_0 \) and let \( P := P_{i_0}^{jk} \). As \( w \in int(D_{i_0}) \) and \( x, y \in D_{p'} \setminus int(D_{i_0}) \), the two segments

Figure 13.10. Proof of Lemma 13.3.16

M. Grohe, Definable Graph Structure Theory
Let $A \subseteq \ell$ with $Q = W \cap P$ because $Q \cap G_0 \subseteq B^{jk} \cap G_0 \subseteq D^{jk}$ and $D^{jk} \cap C_{i_0} = P$.

We shall prove that $Q \cup P$ contains a path from $x$ to $y$ that has an empty intersection with $W_n^{jk} \cup \{w\} \cup S'$. This contradicts $S' \cup \{w\}$ being a $(T_{jk}^p, T_{jk}^p)$ separator in $B^{jk} \setminus (W_n^{jk})$.

Let $D^* := D^{jk} \cup D_{i_0}$. Then $D^*$ is a closed disk by Fact 9.1.9. Note that

$$bd(D^*) = (C_{i_0} \setminus P) \cup (t_{i_0}^j \ell t_{p'}^j) \cup P^{jk} \cup (t_{i_0}^k g^k t_{p'}^k),$$

where for $\ell \in \{i, k\}$, by $t_{i_0}^\ell g^\ell t_{p'}^\ell$ we denote the segment of the curve $g^\ell$ from $t_{i_0}^\ell$ to $t_{p'}^\ell$.

Let $A_0 := \text{int}(D_{i_0})$ and $A'_0 := \text{int}(D^{jk}) \setminus D_{i_0}$ (cf. Figure 13.11(a)). Then $A_0, A'_0$ are the two arcwise connected components of $\text{int}(D^*) \setminus P$, and we have $bd(A_0) = C_{i_0}$ and $bd(A'_0) = P \cup (t_{i_0}^j \ell t_{p'}^j) \cup P^{jk} \cup (t_{i_0}^k g^k t_{p'}^k)$. Furthermore,

$$bd(D^*) \cap V(B^{jk}) \subseteq bd(D^*) \cap D^{jk} \subseteq bd(A'_0).$$

Note the two endvertices $x, y$ of $Q$ are in $bd(D^*)$, and the internal vertex $w \in V(Q)$ is in $\text{int}(D^*)$. Hence there is a (unique) segment $Q'$ of $Q$ such that $w \in V(Q')$, the endvertices of $Q'$ are in $bd(D^*)$, and all interior vertices are in $\text{int}(D^*)$. Let $x', y'$ be the two endvertices of $Q'$ such that $x'$ is between $x$ and $w$ on $Q$ and $y'$ is between $y$ and $w$. Since $Q \subseteq B^{jk}$, we have $x', y' \in bd(D^*) \cap V(B^{jk}) \subseteq bd(A'_0)$. Let $A_1$ and $A'_1$ be the two arcwise connected components of $\text{int}(D^*) \setminus Q'$ (cf. Figure 13.11(b)). Since $G[W_n^{jk}]$ is a connected graph that has an empty intersection with $Q$ and $W_n^{jk} \subseteq \text{int}(D^{jk})$ by Corollary 13.3.15, the set $W_n^{jk}$ is fully contained in one of these components. Say, $W_n^{jk} \subseteq A_1$. As $v_0^j \in W_n^{jk}$ and the edge $e$ from $v_0^j$ to $t_{i_0}^j$ is a simple curve that has an empty intersection with $Q'$, we have $t_{i_0}^j \in A_1$. For every $z \in C_{i_0} \setminus P$, we can find a simple curve $g_z$ from $t_{i_0}^j$ to $z$ such that $g_z \setminus \{z\} \subseteq \text{int}(D_{i_0}) \setminus D^{jk}$. As $Q' \subseteq D^{jk}$, we have $g_z \cap Q' = \emptyset$, and thus $g_z \setminus \{z\} \subseteq A_1$. This implies that $C_{i_0} \setminus P \subseteq bd(A_1)$.

Let $h := bd(A_1) \cap bd(D^*)$ and $h' := bd(A'_1) \cap bd(D^*)$. Then $C_{i_0} \setminus P \subseteq h$. Note that $h, h'$ are the two segments of $bd(D^*)$ from $x'$ to $y'$. As $x', y' \in bd(A'_0)$, one of these two segments is contained in $bd(A'_0)$. As $C_{i_0} \setminus P \subseteq bd(A_0)$, we have $h' \subseteq bd(A'_0)$ and hence $h' \subseteq bd(A'_0) \cap bd(A'_1)$. Thus there is an arcwise connected component $A'$ of $A'_0 \cap A'_1$ with $h' \subseteq bd(A')$ (cf. Figure 13.11(c)). Then $A'$ is an open disc, because we may view it as a face of a 2-connected plane graph $H$ with $H = bd(D^*) \cup P \cup Q'$. Furthermore, $bd(A')$ is a simple closed curve and hence $bd(A') \setminus (h' \setminus \{x', y'\})$ is a simple curve $P'$ contained in
Lemma 13.3.20. \( P \cup Q' \). It corresponds to a path \( P' \subseteq P \cup Q' \) from \( x' \) to \( y' \). As \( w \in A_0 \) and \( W^{jk} \subseteq A_1 \) and \( bd(A') \cap (A_0 \cup A_1) \subseteq (cl(A_0) \cap cl(A')) \cap (A_0 \cup A_1) = \emptyset \), we have \( V(P') \cap (W^{jk} \cup \{ w \}) = \emptyset \). We also have \( V(P') \cap S' \subseteq V(P \cup Q) \cap S' = \emptyset \). Let \( Q' := xQ'x'y'Q'y. \) Then \( Q' \) is a path from \( x \in T^{j'\prime}_p \), to \( y \in T^{j''}_p \) with \( V(Q') \cap (W^{jk} \cup S' \cup \{ w \}) = \emptyset \), which is a contradiction. \( \square \)

Remark 13.3.17. The reader may wonder why in the previous proof we worked with the disk \( D^* \) instead of the disk \( D^{jk} \). The reason is that this allowed us to guarantee that the first vertices \( x', y' \) on the intersection of \( Q \) with the boundary of the disc on each side of \( w \) are outside the disk \( D_0 \).

For \( jk \in \{12, 23, 31\} \), let \( W^{jk}_\infty := \bigcup_{n \in \mathbb{N}^+} W^{jk}_n \), and let \( W^{jk} \) be the set of all vertices in \( W^{jk}_\infty \) that appear on a path from \( u^j = u^j_i \) to \( u^k = t^k_0 \) with all internal vertices in \( W^{jk}_n \). Let \( \tilde{V} := W^{12} \cup W^{23} \cup W^{31} \cup \{ t^1_i, t^2_i, t^3_i \} \). It follows from Lemma 13.3.16 that \( V(C_0) \subseteq \tilde{V} \). Let \( H_0 \subseteq G \) be the union of \( G[U \cup W] \) with all \( (U \cup W) \)-bridges in \( G \) that have a vertex of attachment in \( U \setminus W \). By Lemma 13.3.14, we have \( H_0 \subseteq int(D_{p'}) \) and thus \( H_0 \subseteq G' \). Let \( A_1, \ldots, A_m \) be the connected components of \( G \setminus H_0 \) and \( H := G/A_1/\ldots/A_m \).

Without loss of generality we may assume that \( S \neq \emptyset \). We let \( \bar{s} = (s_1, \ldots, s_{6p+6}) \) be a tuple of vertices of \( G \) such that \( \bar{s} = S \). Furthermore, we let

\[
\bar{a} := (u_0, t^1_0, t^2_0, t^3_0, v^1_0, v^2_0, v^3_0, w^1_0, w^2_0, w^3_0, s_1, \ldots, s_{6p+6}).
\]

Then \( |ar{s}| = 1 + 9(p' + 1) + 6p = 42p + 106 \).

Lemma 13.3.18. There is an \( \text{IFP} \)-formula \( \text{def}-H_0[\bar{x}, y] \), where \( \bar{x} = (x_1, \ldots, x_{42p+106}) \) such that

\[
\text{def}-H_0[G, \bar{x}, y] = V(H_0).
\]

Proof. Recall that the set \( U \) is definable in terms of \( u_0, u^1, u^2, u^3 \in \bar{a} \) by Lemma 13.3.8 (3). It is easy to see that the set \( W \) is \( \text{IFP} \)-definable in terms of the set \( U \), the vertices \( t^i_1, v^i_1, w^i_1 \in \bar{a} \), and the sets \( S^{jk} \subseteq \bar{a} \) and that the set \( V(H_0) \) is \( \text{IFP} \)-definable in terms of the sets \( U \) and \( W \). Combining these definitions, we obtain an \( \text{IFP} \)-definition of \( V(H_0) \) in terms of \( \bar{a} \). \( \square \)

Corollary 13.3.19. There is a 1-dimensional \( \text{IFP} \)-graph transduction \( \Theta(\bar{x}) \) such that \( \Theta[G, \bar{x}] = H \).

13.3.4 The General Case

We drop Assumption 13.3.10 that there are disjoint curves \( g^1, g^2, g^3 \) and generalise the arguments to the unrestricted setting of Assumption 13.3.9. For every \( i \in \{p'\} \), we let \( Z_i := D_i \setminus \text{int}(D_{i-1}) \). Then \( Z_i \) is a closed cylinder with \( \text{bd}(Z_i) = C_{i-1} \cup C_i \).

Lemma 13.3.20. Let \( i \in \{p'\} \), and let \( v_1, v_2, v_3 \in V(C_{i-1}) \). Then for \( j \in \{ 3 \} \) there is a \( G_0 \)-normal simple curve \( \bar{h}_j \subseteq Z_i \) from \( v_j \) to \( v_3 \in V(C_i) \) such that \( \text{such that} \; \bar{h}_j \cap G_0' = \{ v_j, w_j \} \) (and thus \( \bar{h}_j \setminus \{ v_j, w_j \} \subseteq \text{int}(Z_i) \)) and the curves \( \bar{h}_1, \bar{h}_2, \bar{h}_3 \) are internally disjoint.

M. Grohe, Definable Graph Structure Theory
13.3. Defining the Central Faces

Note that we do not (and cannot) rule out the endpoints of the curves \( h_1, h_2, h_3 \) on \( C_i \) to be equal.

**Proof of Lemma 13.3.2.** For \( j \in [3] \), let \( g_j \) be a \( G_0' \)-normal simple curve from \( v_j \) to a vertex in \( C_{p'} \) with \( |g_j \cap V(G_0)| = p' - i + 2 \). Let \( w_j \) be the unique vertex in \( |g_j \cap V(C_i)| \), and let \( h_j' \) be the segment of \( g_j \) from \( v_j \) to \( w_j \). Then \( h_j' \cap G_0' = \{ v_j, w_j \} \). Unfortunately, the curves \( h_1', h_2', h_3' \) are not necessarily internally disjoint. The interior of the three curves is contained in \( \text{int}(Z_i) \setminus G_0' \). For \( j \in [3] \), let \( f_j \subseteq Z \) be the face of \( G_0' \) such that \( h_j' \setminus \{ v_j, w_j \} \subseteq f_j \).

If the three faces \( f_1, f_2, f_3 \) are pairwise disjoint (or, equivalently, pairwise distinct), then the curves \( h_1', h_2', h_3' \) are pairwise internally disjoint, and we are done.

If two of the three faces are equal and disjoint from the third one, say, \( f_1 = f_2 \neq f_3 \), we let \( h_3 := h_3' \). Furthermore, we can choose two internally disjoint simple curves \( h_1, h_2 \) in the disc \( \text{cl}(f_1) \) such that the endpoints of \( h_1 \) are \( v_1 \) and one of the points \( w_1, w_2 \), and the endpoints of \( h_2 \) are \( w_1, w_2 \) and the remaining point in \( w_1, w_2 \). (It is easy to see that if we have four distinct points \( x_1, x_2, y_1, y_2 \) on the boundary of a closed disk, then we can find two disjoint simple curves from \( \{ x_1, x_2 \} \) to \( \{ y_1, y_2 \} \) whose interior is in the interior of the disk. If the points are not pairwise distinct, we can slightly perturb them first.)

If \( f_1 = f_2 = f_3 \), then we can find three internally disjoint simple curves \( h_1, h_2, h_3 \) in the disc \( \text{cl}(f_1) \) such that for \( j \in [3] \) the endpoints of \( h_j \) are \( v_j \) and one of the points \( w_1, w_2, w_3 \).

To see that this is possible, we need to argue that if we have six points \( x_1, x_2, x_3, y_1, y_2, y_3 \) on the boundary of a closed disk \( d \), then we can find three internally disjoint simple curves from \( \{ x_1, x_2, x_3 \} \) to \( \{ y_1, y_2, y_3 \} \) whose interior is in the interior of the disk. To find the curves, we pick a segment \( s \subseteq bd(d) \) with one endpoint \( x_k \) and one endpoint \( y_\ell \) and no interior points in \( \{ x_1, x_2, x_3, y_1, y_2, y_3 \} \). Clearly, this is possible. Now we let \( h_k \) be a simple curve from \( x_k \) to \( y_\ell \) with interior in \( d \). We let \( d' \subseteq d \) be the disk with boundary \( h_k \cup (bd(d) \setminus s) \). Note that \( \{ x_1, x_2, x_3, y_1, y_2, y_3 \} \setminus \{ x_k, y_\ell \} \subseteq bd(d') \). We continue inductively.

For all \( i \in [p'] \) and \( j \in [3] \), we shall define vertices \( t_{i-1}^j \in V(C_{i-1}) \) and \( (t_i^j)' \in V(C_i) \), a simple curve \( h_i^j \subseteq Z_i \) from \( t_{i-1}^j \) to \( (t_i^j)' \) with interior in \( \text{int}(Z_i) \), and a simple curve \( g_i^j \subseteq D_i \setminus \text{int}(D_0) \) from \( t_{i-1}^j \) to \( t_i^j \). The curves \( h_i^j, h_{i+1}^j, h_i^j \) will be pairwise internally disjoint. The vertices \( t_{i-1}^j, t_{i-1}^j, t_{i+1}^j, t_{i-1}^j \) will be pairwise distinct and appear in clockwise order on \( C_{i-1} \). The vertices \( (t_i^j)'(t_i^j)'(t_i^j)' \) may be equal. For all \( i \in [p'] \) and \( jk \in \{12, 23, 31\} \), we shall define open disks \( o_i^j \subseteq \text{int}(Z_i) \) and closed disks \( d_i^j \subseteq Z_i \).

The definition is by induction on \( i \). For \( j \in [3] \) we let \( t_i^j := u_i \). For the inductive step, let \( i \geq 1 \) and assume that the points \( t_{i-1}^1, t_{i-1}^2, t_{i-1}^3 \) are defined. For \( j \in [3] \), we let \( h_i^j \) be a \( G_0' \)-normal simple curve from \( t_{i-1}^j \) to \( V(C_i) \) with interior in \( \text{int}(Z_i) \setminus G_0' \) such that the three curves \( h_i^1, h_i^2, h_i^3 \) are internally disjoint. For \( j \in [3] \), let \( (t_i^j)' \) be the endpoint of \( h_i^j \) in \( V(C_i) \). Then \( \text{int}(Z_i) \setminus (h_{i+1}^j \cup h_i^j \cup h_i^j) \) has three arcwise connected components \( o_i^j, o_i^j, o_i^j \), where \( o_i^j \) is the component with \( h_i^j \cup o_i^j \subseteq \text{bd}(o_i^j) \subseteq C_i \cup C_{i+1} \cup h_i^j \cup h_i^j \).

We let \( d_i^j := cl(o_i^j) \). Then \( d_i^j \) is a closed disk. If \( i < p' \), we define the vertices \( t_i^1, t_i^2, t_i^3 \) as follows.

**Case 1:** \( (t_i^1)', (t_i^2)', (t_i^3)' \) are pairwise distinct.

Then for \( j \in [3] \) we let \( t_i^j := (t_i^j)' \).
Chapter 13. Almost Planar Graphs

Two of the three vertices \((t_1^j)'\), \((t_2^j)'\), \((t_3^j)'\) are equal and distinct from the third. Say, \((t_1^j)' = (t_2^j)' \neq (t_3^j)'\), where \(k = j + 1\) (mod 3) and \(\ell = j + 2\) (mod 3). In this case, we let \(t_i^j := (t_i^j)'\). Let \(t\) be the anti-clockwise neighbour of \((t_i^j)'\) on \(C\) and \(t'\) the clockwise neighbour. Note that \(t \neq t'\), because \(|C| \geq 3\). If \(t_i^j \neq t'\), we let \(t_i^j := (t_i^j)'\) and \(t_k^j := t'\) (see Figure 13.12(a)). If \(t_i^j = t'\), we let \(t_i^j := t\) and \(t_k^j := (t_i^j)'\) (see Figure 13.12(b)).

**Case 3**: \((t_i^j)' = (t_2^j)' = (t_2^j)'\).

Then there is a (unique) \(\ell \in \{12, 23, 31\}\) such that \(C_i \subseteq bd(d^j)\). Let \(j, k, \ell \in \{3\}\) such that \(k = j + 1\) (mod 3) and \(\ell = j + 2\) (mod 3) and \(C_i \subseteq bd(d^j)\) (see Figure 13.12(c)). Then the curve \(h_j^k\) enters \((t_i^j)'\) between \(h_j^j\) and \(h_j^j\). Let \(t\) be the anti-clockwise neighbour of \((t_i^j)'\) and \(t'\) the clockwise neighbour. We let \(t_i^j := t\), \(t_i^j := (t_i^j)'\), and \(t_i^j := t'\).

Note that for \(j \in \{3\}\) either \(t_i^j = (t_i^j)'\) or there is an edge \(e_i^j \in E(C_i)\) from \(t_i^j\) to \(t_i^j\). If \(t_i^j = (t_i^j)'\), we let \(g_i^j := g_i^j \cup h_j^j\). Otherwise, we let \(g_i^j := g_i^j \cup h_j^j \cup e_i^j\). This completes the inductive definition. To unify the notation, for \(j \in \{3\}\) we let \((t_0^j)' := t_0^j\) and \((t_j^j)' := (t_j^j)'\) and \(g_j^j := g_j^j \cup h_j^j\). We need to be slightly careful because, as opposed to \(t_i^j, t_i^j, t_i^j\) for \(i \neq p\), the vertices \((t_0^j)'\), \((t_j^j)'\), \((t_j^j)'\) are not necessarily distinct. Observe that for all \(i \in [0, p']\) we have \(\{(t_i^j)', (t_i^j)', (t_i^j)\} \subseteq \{t_i^j, t_i^j, t_i^j\}\).

Let \(g_j^j := g_j^j\). Then

\[g_j^j \cap V(G_0^j) = \{t_0^j, (t_1^j)', (t_1^j)', (t_j^j)', (t_j^j)', \ldots, (t_j^j)'\},\]

and the vertices \(t_0^j, (t_1^j)', (t_1^j)', (t_j^j)', (t_j^j)', \ldots, (t_j^j)'\) appear on \(g_j^j\) in this order, though some may be equal. For all \(i \in [0, p']\) we have \(|g_j^j \cap V(C_i)| = (t_i^j)'\). Furthermore, either \(t_i^j = (t_i^j)'\) or \(t_i^j = (t_i^j)'\) are adjacent on \(C_i\) and the edge \(e_i^j \in E(C_i)\) from \((t_i^j)'\) to \(t_i^j\) is contained in \(g_j^j\). The curve \(g_j^j\) is not necessarily \(G_0^j\)-normal, but it has an empty intersection with the interior of all

M. Grohe, *Definable Graph Structure Theory*
edges except the edges $e^j_i$ it contains. For $j \in [3]$ and $i \in [0,p']$, let $T^j_i := \{t^j_i, t^j_{i+1}, \ldots, t^j_{p'}\}$ and $(T^j_i)' := \{(t^j_i)', (t^j_{i+1})', \ldots, (t^j_{p'})'\}$. Let $T^j := T^0_0$ and $(T^j)' := (T^j_0)'$ and $T := \bigcup_{j=1}^3 (T^j \cup (T^j)')$.

For $i \in [0,p'-1]$ and $jk \in \{12, 23, 31\}$, we let $P^j_{ik}$ be the segment of $C_i$ from $t^j_i$ to $t^k_i$ in clockwise direction. Note that $P^j_{ik}$ is a path of positive length, because $t^j_i \neq t^k_i$. To unify the notation, we define $P^j_{ik}$ in such a way that $bd(d^j_{ik}) = P^j_{ik}$. Then $P^j_{ik}$ is not necessarily a path of positive length, but it may be a single vertex (a path of length 0), or it may be the whole cycle $C_{ip'}$. However, for all $i \in [0,p']$ we have $C_i = P^0_{1^2} \cup P^0_{2^3} \cup P^0_{3^1}$ and $P^j_{ik} \cap P^j_{ik'} = \{i^j_k\}$.

Recall that, for all $i \in [p']$,

$$h^i_j \cup h^i_k \subseteq bd(d^j_{ik}) \subseteq h^j_i \cup h^k_i \cup C_{i-1} \cup C_i. \quad (13.3.2)$$

Observe, furthermore, that

$$bd(d^j_{ik}) \cap C_{i-1} = P^j_{i-1} \quad (13.3.3)$$

and

$$bd(d^j_{ik}) \cap C_i = \begin{cases} 
\{(t^j_i)'\} & \text{or } P^j_{ik} \\
\text{or } P^j_{ik} \cup e^j_{i+1} & \\
\text{or } P^j_{ik} \cup e^k_{i+1} & \\
\text{or } P^j_{ik} \cup e^j_i \cup e^k_i & 
\end{cases} \quad (13.3.4)$$

Let $S^j_{ik} \subseteq V(R')$ with $|S^j_{ik}| \leq 2(p+1)$ such that $S^j_{ik} \cup \{t^j_{ip'}, t^k_{ip'}\}$ separates $\bar{r'} \cap P^j_{ip'}$ from $\bar{r'} \setminus P^j_{ip'}$ in $R'$. Such a set exists by Lemma 13.3.6. Let $S := S^0_{1^2} \cup S^2_{2^3} \cup S^3_{3^1}$ and note that $|S| \leq 6p + 6$.

For $jk \in \{12, 23, 31\}$, let

$$R^j_{ik} := \bigcup_{i=0}^{p'} d^j_{ik}$$

Then $R^j_{ik}$ is arcwise connected, but its interior may not be. If $\ell \in [3]$ with $\ell = k+1 \pmod{3}$, then $R^j_{ik}$ and $R^{j\ell}_{ik}$ have disjoint interiors, and we have

$$(R^j_{ik} \cap R^{j\ell}_{ik}) \setminus \text{int}(D_0) = g^k.$$  

Furthermore, for every $i \in [p']$ we have $Z_i = d^1_{1^2} \cup d^2_{2^3} \cup d^3_{3^1}$. Thus

$$D_{p'} \setminus \text{int}(D_0) \subseteq R^0_{1^2} \cup R^2_{2^3} \cup R^3_{3^1}.$$  

**Lemma 13.3.21.** Let $j, k, \ell \in [3]$ such that $k = j+1 \pmod{3}$ and $\ell = j+2 \pmod{3}$. Then the set $(S \cup T \cup U)$ separates $V(G) \cap R^j_{ik}$ from $V(G') \setminus R^j_{ik}$.

**Proof.** Similar to the proof of Lemma 13.3.11. \hfill \Box

**Lemma 13.3.22.** Let $jk \in \{12, 23, 31\}$. Then for all $i \in [0,p']$ it holds that $P^j_{ik} \subseteq R^j_{ik} \cap C_i$ and

$$V(P^j_{ik}) \cap (S \cup T \cup U) = \{t^j_i, t^k_i\}.\quad (13.3.22)$$

Furthermore, if $i < p'$ then $V(P^j_{ik}) \cap bd(R^j_{ik}) = \{t^j_i, t^k_i\}$. 

Preliminary Version
Proof. $P^i_{jk} \subseteq R^j_k \cap C_i$ follows from (13.3.3) and (13.3.4) and the definition of $P^i_{jk}$ for $i = p'$.

To prove $V(P^i_{jk}) \cap (S \cup T \cup U) = \{t^i_{jk}, t^i_{k}\}$, note that $V(P^i_{jk}) \cap (S \cup T \cup U) \subseteq C_i \cap (S \cup T \cup U) = \{t^i_{jk}, t^i_{k}\}$, it follows from the definition of the vertices $t^i_{jk}, t^i_{k}$ (by a simple case distinction) that $V(P^i_{jk}) \cap \{t^i_{jk}, t^i_{k}\} \subseteq \{t^i_{jk}, t^i_{k}\}$. This proves $V(P^i_{jk}) \cap (S \cup T \cup U) = \{t^i_{jk}, t^i_{k}\}$.

Furthermore, if $i < p'$ then $bd(R^i_{jk}) \cap V(C_i) = (g^i_j \cup g^i_k) \cap V(C_i) \cup \{t^i_{jk}, t^i_{k}\}$ and $V(C_i) \subseteq \{t^i_{jk}, t^i_{k}\}$, it also follows that $V(P^i_{jk}) \cap bd(R^i_{jk}) = \{t^i_{jk}, t^i_{k}\}$. □

Let $jk \in \{12, 23, 31\}$. So far, $R^i_{jk}$ plays the role the disk $D^i_{jk}$ played in the previous subsection. However, for some of the arguments of the previous subsection we really need to work with a disk. It turns out that the “component” of $R^i_{jk}$ that contains $P^i_{0}$ can serve as this disk. The precise definitions are as follows. Note that $P^i_{jk} \subseteq d^i_{0} \cap d^i_{jk} \subseteq R^i_{jk}$. Hence

$$P^i_{0} \setminus \{t^i_{0}, t^i_{k}\} \subseteq \text{int}(d^i_{0} \cup d^i_{jk}) \subseteq \text{int}(R^i_{jk}).$$

Let $O^i_{jk}$ be the arcwise connected component of $\text{int}(R^i_{jk})$ that contains $P^i_{0} \setminus \{t^i_{0}, t^i_{k}\}$, and let $D^i_{jk} := \text{cl}(O^i_{jk})$. Then $P^i_{0} \subseteq D^i_{jk}$. Let

$$p^i_{jk} := \min \{\{i \in [p'] \mid D^i_{jk} \cap C_i = \{(t^i_{jk})'\} \cup \{p'+1\}\}.$$

Then

$$D^i_{jk} = \bigcup_{i=0}^{\min\{p^i_{jk}, p'\}} d^i_{jk}. \quad (13.3.5)$$

Lemma 13.3.23. Let $jk \in \{12, 23, 31\}$.

(1) $D^i_{jk}$ is a closed disk.

(2) For all $i \in [0, p']$ it holds that $D^i_{jk} \cap D_i$ is closed disk.

(3) For all $i \in [0, p^i_{jk} - 1]$ it holds that $D^i_{jk} \cap D_i = R^i_{jk} \cap D_i$.

(4) If $p^i_{jk} \leq p'$, then $D^i_{jk} \cap \text{int}(D_{p^i_{jk}}) = R^i_{jk} \cap \text{int}(D_{p^i_{jk}})$ and $D^i_{jk} \cap C_{p^i_{jk}} = \{(t^i_{p^i_{jk}})'\} = \{(t^i_{p^i_{jk}})\}'$.

(5) For all $i \in [p^i_{jk} + 1, p']$ it holds that $D^i_{jk} \cap D_i = D^i_{jk} \cap D_{p^i_{jk}} = D^i_{jk}$.

(6) If $p^i_{jk} \leq p'$, then $V(G) \cap bd(D^i_{jk}) \subseteq T \cup U$. Otherwise, $V(G) \cap bd(D^i_{jk}) \subseteq T \cup U \cup V(P^i_{p'})$.

Proof. All statements follow easily from (13.3.3)–(13.3.5). □

From now on, we proceed analogously to the case of disjoint curves $g^1, g^2, g^3$, except that in some places we need to make a distinction between the disks $D^i_{jk}$ and the larger spaces $R^i_{jk}$. For $i \in [0, p']$ and $j \in [3]$, we let $v^i_j$ be the clockwise and $w^i_j$ the anticlockwise neighbour of $t^i_j$ on $C_i$. Now let $jk \in \{12, 23, 31\}$ and $i \leq \min\{p', p^i_{jk}\}$. Note that $v^i_j$ and $w^i_j$ are the first and last internal vertex of the path $P^i_{jk}$ unless the path $P^i_{jk}$ has length 1, in which case $v^i_j = t^i_j$ and $w^i_j = t^i_j$. By Lemmas 13.3.22 and 13.3.23 we have $P^i_{jk} \setminus \{t^i_j, t^i_k\} \subseteq D^i_{jk} \setminus (S \cup T \cup U)$. Hence if the path $P^i_{jk}$ has length greater than 1, there is an $(S \cup T \cup U)$-bridge $B^i_{jk}$ with
For all \( i \), if the path \( P_{ij}^k \) has length 1, then we let \( e_{ij}^k \) be the edge from \( t_{ij}^1 \) to \( t_{ij}^k \) and \( B_{ij}^k := P_{ij}^k \) \( = \) \( \{(t_{ij}^1, t_{ij}^k), (e_{ij}^k)\} \). We let

\[
B_{ik}^j := \bigcup_{i=0}^{\min\{p', p^j\}} B_{i}^{jk}.
\]

For \( i \in \{0, \min\{p', p^j\}\} \), let \( A_{ij}^k := B_{i}^{jk} \setminus (S \cup T \cup U) \). Let

\[
A_{ik}^j := \bigcup_{i=0}^{p'-1} A_{i}^{jk}.
\]

Note that for all \( i \in \{0, \min\{p', p^j\}\} \) the path \( P_{ij}^{jk} \) is contained in the bridge \( B_{ij}^k \), and if the path \( P_{ij}^{jk} \) has length greater than 1 then the segment \( v_i^j P_{ij}^k w_i^k \) is contained in \( A_{ij}^k \).

**Lemma 13.3.24.** For \( jk \in \{12, 23, 31\} \), let \( A_{ij}^k := A_{ij}^k \cap G'_{0} \) and \( B_{ij}^k := B_{ij}^k \cap G'_{0} \). Then \( A_{0}^{jk} \subseteq R_{ij}^k \setminus (S \cup T \cup U) \) and \( B_{0}^{jk} \subseteq R_{ij}^k \).

**Proof.** Analogous to the proof of Lemma 13.3.12.

**Lemma 13.3.25.** For all \( jk, j'k' \in \{12, 23, 31\} \) with \( jk \neq j'k' \) it holds that \( V(B_{ij}^{jk} \cap B_{ij}^{j'k'}) \subseteq S \cup T \cup U \) and \( E(B_{ij}^{jk} \cap B_{ij}^{j'k'}) = \emptyset \).

**Proof.** Analogous to the proof of Lemma 13.3.13.

We define sets \( W_{ij}^{jk} \subseteq V(A_{ij}^{jk}) \), for \( n \in \mathbb{N} \), as follows.

- If the length of \( P_{0}^{jk} \) is one, then we let \( W_{ij}^{jk} = \emptyset \) for all \( n \in \mathbb{N} \).

Assume that the length of \( P_{0}^{jk} \) is greater than one. Again, we let \( p'' := p' - 2p - 4 \). We define

- \( W_{0}^{jk} := \emptyset \);
- \( W_{1}^{jk} := \{v_{ij}^j\} \);
- for \( n \geq 0 \), the set \( W_{n+1}^{jk} \) is the union of \( W_{n}^{jk} \) with the set of all \( w \in V(A_{ij}^{jk}) \) with the following property: \( w \) is adjacent to a vertex in \( W_{n}^{jk} \) and there is no \( S' \subseteq V(B_{ij}^{jk}) \) such that \( |S'| \leq 2p+3 \) and \( S' \cup \{w\} \) is a minimal \( (T_{p''}, T_{p''}) \)-separator in the graph \( B_{ij}^{jk} \setminus W_{n}^{jk} \).

Observe that for every \( n \geq 1 \) and every vertex \( w \in W_{n}^{jk} \) there is a path \( Q \) from \( v_{ij}^j \) to \( w \) in \( G[W_{n}^{jk}] \). Hence the graph \( G[W_{n}^{jk}] \) is connected. The following lemma was trivial in the case of disjoint paths, but is not entirely obvious here.

**Lemma 13.3.26.** For all \( n \in \mathbb{N} \), it holds that \( W_{n}^{jk} \cap V(G'_{0}) \subseteq D_{ij}^{jk} \setminus (S \cup T \cup U) \). Furthermore, if \( p^{jk} \leq p' \) then \( W_{n}^{jk} \subseteq \text{int}(D_{ij}^{jk}) \).

**Proof.** Let \( n \geq 1 \). By Lemma 13.3.24, we have

\[
W_{n}^{jk} \cap V(G'_{0}) \subseteq V(A_{ij}^{jk} \cap G'_{0}) \subseteq R_{ij}^{jk} \setminus (S \cup T \cup U).
\]

If \( p^{jk} = p' + 1 \), we have \( D_{ij}^{jk} = R_{ij}^{jk} \), and hence \( W_{n}^{jk} \cap V(G'_{0}) \subseteq D_{ij}^{jk} \setminus (S \cup T \cup U) \). In the following, we assume \( p^{jk} \leq p' \). Then by Lemma 13.3.23 \( (6) \) we have \( \text{bd}(D_{ij}^{jk}) \cap V(G) \subseteq T \cup U \). As \( W_{n}^{jk} \cap (T \cup U) = \emptyset \), the vertex \( v_{ij}^j \in \text{int}(D_{ij}^{jk}) \) is contained in \( W_{n}^{jk} \), and the graph \( G[W_{n}^{jk}] \) is connected, we have \( W_{n}^{jk} \subseteq \text{int}(D_{ij}^{jk}) \subseteq D_{ij}^{jk} \setminus (S \cup T \cup U) \).
Lemma 13.3.27. For all \( n \in \mathbb{N}^+ \) it holds that \( W_n^{jk} \subseteq \text{int}(D_{p'}) \).

**Proof.** If \( p^j \leq p' \), then by Lemma 13.3.26 we have \( W_n^{jk} \subseteq \text{int}(D^{jk}) \subseteq \text{int}(D_{p'}) \). For the case \( p^j = p' + 1 \), the proof is completely analogous to the proof of Lemma 13.3.14. \( \square \)

Corollary 13.3.28. For all \( n \in \mathbb{N} \) it holds that \( W_n^{jk} \subseteq \text{int}(D^{jk}) \).

Lemma 13.3.29. There is an \( n \in \mathbb{N}^+ \) such that \( V(P_n^{jk}) \subseteq W_n^{jk} \cup \{t_0, t_0'\} \).

**Proof.** The proof is very similar to the proof of Lemma 13.3.16. The only thing to be noted is that if \( Q \subseteq B^{jk} \) is a path from \( x \in \{t_1, (t_1')^j\} \) to \( y \in \{t_2, (t_2')^j\} \) then \( p^j \geq \max\{i, i'\} \). \( \square \)

For \( jk \in \{12, 23, 31\} \), let \( W_{\infty}^{jk} := \bigcup_{n \in \mathbb{N}^+} W_n^{jk} \), and let \( W^{jk} \) be the set of all vertices in \( W_{\infty}^{jk} \) that appear on a path from \( w^j = t_0 \) to \( u^k = t_0' \) with all internal vertices in \( W_{\infty}^{jk} \). Let \( W := W^{12} \cup W^{23} \cup W^{31} \cup \{u^1, u^2, u^3\} \). We let \( H_0 \subseteq G \) be the union of \( G[U \cup W] \) with all \((U \cup W)\)-bridges in \( G \) that have a vertex of attachment in \( U \setminus W \). By Lemmas 13.3.27 and 13.3.29 we have \( C_0 \subseteq H_0 \subseteq G' \) and \( H_0 \subseteq \text{int}(D_{p'}) \). Let \( A_1, \ldots, A_m \) be the connected components of \( G \setminus H_0 \) and \( H := G/A_1/ \ldots/A_m \).

Without loss of generality we may assume that \( S \neq \emptyset \). We let \( \bar{s} = (s_1, \ldots, s_{6p+6}) \) be a tuple of vertices of \( G \) such that \( \bar{s} = S \). Furthermore, we let

\[
\bar{a} := (u_0, t_0^1, t_0^1, t_0^2, \ldots, t_0^2, t_0^3, \ldots, t_0^3, (t_0^3)', (t_0^3)', \ldots, (t_0^3)', \ldots, (t_0^3)', (t_0^3)', \ldots, (t_0^3)', v_0^1, \ldots, v_0^1, v_0^2, \ldots, v_0^2, v_0^3, \ldots, v_0^3, w_0^1, w_0^1, w_0^2, \ldots, w_0^2, w_0^3, \ldots, w_0^3, s_1, \ldots, s_{6p+6}).
\]

Then \( |\bar{a}| = 1 + 12(p' + 1) + 6p + 6 = 54p + 139 \).

Lemma 13.3.30. There is an \( \text{IFP} \)-formula \( \text{def-}H_0(\bar{x}, y) \), where \( \bar{x} = (x_1, \ldots, x_{54p+139}) \) such that

\[
\text{def-}H_0[G, \bar{a}, y] = V(H_0).
\]

**Proof.** Analogous to the proof of Lemma 13.3.18. \( \square \)

Corollary 13.3.31. There is a 1-dimensional \( \text{IFP} \)-graph transduction \( \Theta(\bar{x}) \) such that \( \Theta[G, \bar{a}] = H \).

### 13.3.5 Proof of Lemma 13.3.3

Recall the definition of the graphs \( H_0 \) and \( H \) (given after Lemma 13.3.29 in the general case). The graph \( H_0 \) is guaranteed to be a subgraph of \( G'_0 \), in fact \( G'_0 \setminus C_{p'} \), and as such is embedded in the interior of the disk \( D_{p'} \). Furthermore, \( H_0 \) is guaranteed to contain the set \( U \) and all bridges attached to \( U \) and thus the whole “inner part” of \( G'_0 \), which is embedded in the disk \( D_0 \). Observe that

\[
\partial'^G(H_0) = \partial^H(H_0) \subseteq W. \tag{13.3.7}
\]
The equality \( \partial^G(H_0) = \partial^H(H_0) \) follows immediately from the definition of \( H \), and the inclusion \( \partial^G(H_0) \subseteq W \) follows from the fact all \( U \cup W \)-bridges that have a vertex of attachment in \( U \) are included in \( H_0 \).

Before we actually prove Lemma 13.3.32, we still need a few more lemmas.

**Lemma 13.3.32.** \( H \) is a 3-connected planar graph. Furthermore, if \( S \supseteq D_{r'} \) is a sphere then the embedding of \( H_0 \) in the disk \( D_{r'} \) can be extended to an embedding of \( H \) into \( S \) that strongly coincides with the embedding of \( H_0 \).

**Proof.** The crucial fact we need to establish is the following.

**Claim 1.** \( H_0 \) is 2-connected.

**Proof.** Let \( v, w_1, w_2 \in V(H_0) \) and suppose for contradiction that \( \{v\} \) separates \( w_1 \) from \( w_2 \) in \( H_0 \). Let \( P \) be a path from \( w_1 \) to \( w_2 \) in \( G' \setminus \{v\} \). Let \( x'_1 \) be the first vertex of \( P \) in \( V(G) \setminus V(H_0) \), and let \( x_1 \) be the predecessor of \( x'_1 \) on \( P \). Let \( x'_2 \) be the last vertex of \( P \) in \( V(G) \setminus V(H_0) \), and let \( x_2 \) be the successor of \( x'_2 \) on \( P \). Then \( x_1, x_2 \in W \). Say, \( x_1 \in W_{j_1k_1} \) and \( x_2 \in W_{j_2k_2} \) for \( j_1k_1, j_2k_2 \in \{12, 23, 31\} \). For \( i = 1, 2 \), let \( Q_i \) be a path from \( u^{i_1} \) to \( u^{i_2} \) in \( G[W] \) such that \( x_1 \in V(Q_i) \). Such a path exists by the definition of \( W_{j_1k_1} \). Then \( Q_1 \cup Q_2 \cup C_0 \) is a 2-connected subgraph of \( H \) that contains \( x_1, x_2 \). Hence there is a path \( Q \) from \( x_1 \) to \( x_2 \) in \( (Q_1 \cup Q_2 \cup C_0) \setminus \{v\} \subseteq H_0 \setminus \{v\} \). Then \( w_1Pw_1 \cup x_1Qx_2 \cup x_2Pw_2 \) contains a path from \( w_1 \) to \( w_2 \) in \( H_0 \setminus \{v\} \), which is a contradiction. \( \blacksquare \)

Now it follows from Lemma 13.2.7 that \( H \) is 3-connected. It remains to prove that the embedding of \( H_0 \) in the disk \( D_{r'} \) can be extended to an embedding of \( H \) in a sphere \( S \supseteq D_{r'} \) that strongly coincides with the embedding of \( H_0 \). Let \( S \supseteq D_{r'} \) be a sphere. Let \( A := G' \setminus \text{Ins}(C_{r'}) \).

**Claim 2.** \( A \) is connected.

**Proof.** We prove that for every vertex \( v \in V(A) \) there is a path \( Q \subseteq A \) from \( v \) to a vertex in \( V(C_{r'}) \). Let \( v \in V(A) \setminus V(C_{r'}) \). Let \( Q \subseteq G \) be a path from \( v \) to \( V(C_{r'}) \). Then \( V(Q) \cap \text{Ins}(C_{r'}) = \emptyset \), because \( V(C_{r'}) \) separates \( V(G') \setminus \text{Ins}(C_{r'}) \) from \( \text{Ins}(C_{r'}) \). Hence \( Q \subseteq A \). \( \blacksquare \)

We can extend the embedding of \( \text{Ins}(C_{r'}) \) in the disk \( D_{r'} \) to an embedding of \( G/A \) in \( S \) as follows. We place the vertex \( a \) of \( G/A \) corresponding to \( A \) into \( S \setminus D \), and for every edge \( e \) of \( G_0 \) from a vertex \( v \in V(\text{Ins}(C_{r'})) \) to a vertex in \( V(C_{r'}) \) we draw the corresponding edge \( e' \) of \( G/A \) from \( v \) to \( a \) in such a way that \( e' \cap D_{r'} = e \). Note that this embedding of \( G/A \) into \( S \) strongly coincides with the embedding of \( \text{Ins}(C_{r'}) \) in \( S \).

Observe next that \( H_0 \subseteq \text{ins}(C_{r'}) \) and that \( H \) is a minor of \( G/A \). Hence it follows from Corollary 0.1.19 that the embedding of \( H_0 \) can be extended to an embedding of \( H \) into \( S \) that strongly coincides with the embedding of \( H_0 \). \( \square \)

**Corollary 13.3.33.** Let \( u, u' \in N^G(u_0) \). Then \((u, u_0, u')\) is an angle of \( G_0 \) if and only if it is an angle of \( H \).

**Proof.** For the proof, just note that \( u, u' \in V(H_0) \subseteq V(G_0) \cap V(H) \) and that the embeddings of \( G_0 \) and \( H_0 \) strongly coincide on \( H_0 \). \( \square \)

Our next step is to strengthen the previous corollary in such a way that is becomes largely independent of the choices we made in defining the graphs \( G_0 \) and \( H \). Let \( k \geq 1 \) and \( u'_0 \in V(G) \). Let \( H'_0 \subseteq G \) be an induced subgraph with \( u'_0 \in V(H'_0) \), and let \( A'_1, \ldots, A'_{m'} \) be the connected components of \( G \setminus H'_0 \). Let \( H' := G/A'_1/\cdots/A'_{m'} \). Then we call \( H'_0 \) a \( k \)-horizon for \( u'_0 \) if the following conditions are satisfied:
(H.1) $H'$ is a 3-connected planar graph.
(H.2) $u'_0$ is $k$-central in $H'$ within cycles $C'_1, \ldots, C'_k \subseteq H'_0$.
(H.3) $V(C'_k)$ separates $u'_0$ from $\bigcup_{i=1}^{m'} V(A'_i)$ in $G$.

**Corollary 13.3.34.** $H_0$ is a $(p + 4)$-horizon for $u_0$.

*Proof.* This follows from Lemma 13.3.32 letting $C'_i := C_{-(p+4)+i}$ for $i \in [p + 4]$.  

For the next lemma, recall our notation for angles from Section 9.2. Note that in (i) and (v) we refer to the angles of an embedded graph, in (ii) and (vi) we refer to the angles of an embedding, and in (iii), (iv), (vii), (viii) we refer to the angles of 3-connected planar graphs.

**Lemma 13.3.35.** Let $(G_0'', \pi'', R'', \tau'')$ be a local $p$-arrangement of $G$ in a disk $D''$ such that $u_0$ is 3-central in $G_0''$. Let $H_0''$ be a $(p + 4)$-horizon for $u_0$, and let $H'$ be the graph obtained from $G$ by contracting all connected components of $G \setminus H_0''$ into single vertices. Then for all $u \in N^G(u_0)$ and $u', u'' \in N^G(u)$ the following four statements are equivalent:

(i) $(u', u, u'') \in \angle(G_0)$.

(ii) $(u', u, u'') \in \angle(\Pi'')$, where $\Pi''$ denotes the embedding of $G \setminus R''$ defined by $\Pi''(\pi''(v)) = v$ for $v \in V(G_0'') \setminus \tau''$.

(iii) $(u', u, u'') \in \angle(H)$.

(iv) $(u', u, u'') \in \angle(H')$.

Furthermore, for all $u, u' \in N^G(u_0)$ and $u'' \in N^G(u')$ the following four statements are equivalent:

(v) $(u, u_0, u') \sim^{G_0} (u_0, u', u'')$.

(vi) $(u, u_0, u') \sim^{\Pi''} (u_0, u', u'')$.

(vii) $(u, u_0, u') \sim^H (u_0, u', u''$).

(viii) $(u, u_0, u') \sim^{H'} (u_0, u', u'')$.

*Proof.* To prove that (i)–(iv) are equivalent, it suffices to prove that (ii) and (iv) are equivalent. With $(D'', G_0'', \pi'', R'', \tau'') = (D, G_0, \pi, R, \tau)$ and/or $H' = H$, this implies the other equivalences.

To prove that (iv) implies (ii), we apply Lemma 13.3.5. Let $H_0', A'_1, \ldots, A'_m, C'_1, \ldots, C'_{p+4}$ satisfy (H.1) and (H.3). Let $C' := C'_{p+4}$ and $F := \bigcup_{i=1}^{m'} E(A'_i)$, and let $A$ be the connected component of $G \setminus C'$ that contains $u_0$. Note that $u, u', u'' \in V(A)$ and that $u$ is $(p + 3)$-central within $C'_2, \ldots, C'_{p+4}$. Then $F \subseteq E(G \setminus (A \cup C'))$ by (H.3). Let $C$ be the facial cycle of $H'$ with $(u', u, u'') \in \angle(C)$. Then $C$ is $(p + 2)$-central in $H'$ within the cycles $C'_3, \ldots, C'_{p+4}$. Hence, $C, C', A, F$ satisfy all conditions of Lemma 13.3.5 applied to the arrangement $(D'', G_0'', \pi'', R'', \tau'')$ of $G$. Thus $C$ is a facial cycle of $\Pi''$ and therefore $(u', u, u'') \in \angle(\Pi'')$.

To prove the converse, (ii) implies (iv), let $u_1, \ldots, u_m$ be an enumeration of $N^G(u)$ such that for all $i \in [m]$ it holds that $(u_i, u, u_{i+1}) \in \angle(H')$, where we let $u_{m+1} := u_1$. Hence $(u_1, \ldots, u_m)$ is the cyclic permutation of $N^G(u)$ induced by the unique plane embedding of $H'$. By the reverse implication (iv) $\Rightarrow$ (ii), we have $(u_i, u, u_{i+1}) \in \angle(\Pi'')$. As $u_1, \ldots, u_m$ are induced by the unique plane embedding of $H'$, the proof is complete.
is a cyclic permutation of \(N^G(u)\), this implies that all angles \((v, u, v')\) of \(\Pi''\) at \(u\) are of the form \((u_i, u, u_{i+1}) \in \angle(H')\) or \((u_{i+1}, u, u_i) \in \angle(H')\) for some \(i \in [m']\). Hence in particular, for some \(i \in [m']\) it holds that \((u, u, u') = (u_i, u, u_{i+1})\) or \((u, u, u') = (u_{i+1}, u, u_i)\). Thus \((u, u, u') \in \angle(H')\).

To prove that (v)–(viii) are equivalent, it suffices to prove that (vi) and (viii) are equivalent. The implication (viii) \(\implies\) (vi) is proved similarly to the implication (iv) \(\implies\) (ii). Let \(H'_0, A'_1, \ldots, A'_m, C'_1, \ldots, C'_{p+4}\) satisfy (H.1) \[\text{(H.3)}\] Let \(C' := C'_{p+4}\) and \(F := \bigcup_{i=1}^{m'} E(A'_i)\), and let \(A\) be the connected component of \(G \setminus C'\) that contains \(u_0\). Then \(u, u', u'' \in \Gamma(A)\) and \(F \subseteq E(G \setminus (A \cup C'))\) by (H.3). Let \(C\) be the facial cycle of \(H'\) with \((u, u_0, u), (u_0, u', u'') \in \angle(H')\). Such a cycle exists by assumption (viii). Then \(C\) is \((p+2)\)-central in \(H\) within the cycles \(C'_3, \ldots, C'_{p+4}\). Hence \(C, C', A, F\) satisfy all conditions of Lemma 13.3.5 applied to the arrangement \((\mathcal{D}'', C_0'', \pi'', R'', \mathcal{F}'')\) of \(G\). Thus \(C\) is a facial cycle of \(\Pi''\) and therefore \((u, u_0, u), (u_0, u', u'')\) are aligned angles of \(\Pi''\).

It remains to prove that (vi) implies (viii). Suppose that \((u, u_0, u'), (u_0, u', u'')\) are aligned angles of \(\Pi''\). By the implication (ii) \(\implies\) (iv) we have \((u, u_0, u') \in \angle(H')\). Let \(w'' \in N^G(u')\) such that \((u_0, u', w'')\) is aligned to \((u, u_0, u')\). By the implication (viii) \(\implies\) (vi), \((u_0, u', w'')\) is aligned to \((u, u_0, u')\) with respect to \(\Pi''\). Hence \(w'' = u''\) by Lemma 13.1.13(3). \[\square\]

**Lemma 13.3.36.** For every \(k \in \mathbb{N}^+\) there is an IFP-formula protected\(_k\)(Z, z) such that for all 3-connected planar graphs \(H'\), all \(u' \in V(J)\) and all \(U' \subseteq V(H')\) it holds that

\[
H' \models \text{protected\(_k\)[U', u']} \iff u' \text{ is } k\text{-central in } H' \text{ within cycles } C_1, \ldots, C_k \subseteq G[U'] \text{ such that } V(C_k) \text{ separates } u' \text{ from } V(H') \setminus U'.
\]

**Proof.** Let \(k \in \mathbb{N}\). For a 3-connected planar graph \(H'\) and \(u' \in V(H'), U' \subseteq V(H')\), let us say that \(u'\) is \(k\)-protected by \(U'\) in \(H'\) if \(u'\) is \(k\)-central in \(H'\) within cycles \(C_1, \ldots, C_k \subseteq G[U']\) such that \(V(C_k)\) separates \(u'\) from \(V(H') \setminus U'\). We shall prove that for every \(k \in \mathbb{N}^+\) there is a polynomial time algorithm that, given a 3-connected planar graph \(H'\), a vertex \(u' \in V(H')\), and a set \(U' \subseteq V(H')\), decides whether \(u'\) is \(k\)-protected by \(U'\) in \(H'\). As the class of 3-connected planar graphs is also decidable in polynomial time, this means that the class

\[
\mathcal{I} := \{(G, U', u') \mid G \text{ is a 3-connected planar graph and } u' \in V(H') \text{ is } k\text{-protected by } U' \subseteq V(H')\}
\]

of graph interpretations of signature \((Z, z)\) is decidable in polynomial time. As the class of 3-connected planar graphs admits IFP-definable orders (by Theorem 9.3.11), it follows from Lemma 3.2.8 that \(\mathcal{I}\) is IFP-definable. This proves the lemma.

To explain the algorithm, let \(H'\) be a 3-connected plane graph embedded in a sphere \(S\) and \(u' \in V(H'), U' \subseteq V(H')\). Suppose that \(u' \in U'\), otherwise clearly \(u'\) is not \(k\)-protected by \(U'\) in \(H'\). For every cycle \(C \subseteq H' \setminus \{u'\}\), let \(\text{ins}(C)\) be the arcwise connected component of \(S \setminus C\) that contains \(u'\). Furthermore, let \(\text{Clins}(C) := H' \setminus \text{cl}\{\text{ins}(C)\}\), and let \(\text{Ins}(C) := \text{Clins}(C) \setminus C\).

For \(i \geq 0\), we inductively define sets \(F_i \subseteq F(H')\) of faces and subgraphs \(L_i \subseteq H'\) as follows: \(F_0 := \emptyset\) and \(L_0 := \{(u'), \emptyset\}\). For \(i \geq 0\), we let \(F_{i+1}\) be the set of all faces incident to a vertex in \(V(L_i)\), and we let \(L_{i+1} := \bigcup_{f \in F_{i+1}} \text{Bd}(f)\). Then for every \(i \in \mathbb{N}^+\), the graph \(L_i\) is 2-connected, and hence all facial subgraphs are cycles. A face of \(L_i\) is external if it is not

Preliminary Version
contained in \( F_i \), and a cycle \( C \subseteq L_i \) is external if it is the facial cycle of some external face. Note that for every external cycle \( C \subseteq L_i \) it holds that \( L_i \subseteq \text{Clins}(C) \) and \( L_{i-1} \subseteq \text{Ins}(C) \).

**Claim 1.** Let \( C_i \) be an external cycle of \( L_i \). Then there are cycles \( C_1, \ldots, C_{i-1} \) such that:

1. \( u' \) is \( i \)-central within \( C_1, \ldots, C_i \).
2. For each \( j \in [i] \), the cycle \( C_j \) is an external cycle of \( L_j \).

**Proof.** The proof is by induction on \( i \). For \( i = 1 \), let \( C_1 \) be an external cycle of \( L_1 \). Then \( u' \in \text{Ins}(C_1) \) is 1-central in \( C_1 \). For the inductive step, let \( C_{i+1} \) be an external cycle of \( L_{i+1} \). Then \( C_{i+1} \subseteq f \) for some external face \( f \) of \( L_i \). We let \( C_i := \text{Bd}(f) \) and choose \( C_1, \ldots, C_{i-1} \) according to the inductive hypothesis.

**Claim 2.** Suppose that \( u' \) is \( i \)-central in \( H' \) within cycles \( C'_1, \ldots, C'_i \). Then there are cycles \( C'_1, \ldots, C'_i \subseteq H' \) such that:

1. \( u' \) is \( k \)-central within \( C_1, \ldots, C_i \).
2. For all \( j \in [i] \), the cycle \( C_j \) is an external cycle of \( L_j \).
3. For all \( j \in [i] \), it holds that \( \text{Clins}(C_j) \subseteq \text{Clins}(C_j') \).

**Proof.** Again, the proof is by induction on \( i \). For the induction basis \( i = 1 \), let \( C'_1 \) be an arbitrary cycle of \( H' \setminus \{u'\} \). Then all faces in \( F_1 \) are contained in \( \text{ins}(C'_1) \). Hence \( L_1 \subseteq \text{Clins}(C'_1) \), and there is some external face \( f \) of \( L_1 \) such that \( S_0 \setminus \text{cl}(\text{ins}(C'_1)) \subseteq \text{cl}(f) \).

Let \( C_1 := \text{Bd}(f) \). Then (i)–(iii) are clearly satisfied.

For the inductive step, suppose that \( u' \) is \((i+1)\)-central in \( H' \) within cycles \( C'_1, \ldots, C'_{i+1} \). By the induction hypothesis, there are cycles \( C_1, \ldots, C_i \) such that (i)–(iii) are satisfied, and we have \( \text{Clins}(C_1) \subseteq \ldots \subseteq \text{Clins}(C_i) \subseteq \text{Clins}(C'_i) \subseteq \text{Clins}(C'_{i+1}) \). As \( L_i \subseteq \text{Clins}(C_1) \), all faces in \( F_i \) must be contained in \( C'_{i+1} \), and we can argue similarly as in the inductive basis to obtain an external cycle \( C_{i+1} \) of \( L_{i+1} \) with the desired properties.

If follows from the two claims that \( u' \) is \( k \)-protected by \( U' \) in \( H' \) if and only if there is an external cycle \( C_k \) of \( L_k \) such that \( V(\text{Clins}(C_k)) \subseteq U' \). This can easily be checked by a polynomial time algorithm.

**Lemma 13.3.37.** For every \( k \in \mathbb{N}^+ \) there is an \( \text{IFP} \)-formula \( \text{horizon}_k(Z, z) \) such that for all \( u' \in V(G) \) and \( U' \subseteq V(G) \) it holds that

\[
G \models \text{horizon}_k[U', u'] \iff G[U'] \text{ is a } k\text{-horizon for } u'.
\]

**Proof.** This follows easily from the fact that the class of 3-connected planar graphs is \( \text{IFP} \)-definable (Theorem 9.3.5) and Lemma 13.3.36.

**Proof of Lemma 13.3.3** Let \( \varphi(x, z) \) be the \( \text{IFP} \)-formula obtained from the formula \( \text{horizon}_{p+4}(Z, z) \) of Lemma 13.3.37 replacing each subformula of the form \( Z(y) \) by the formula \( \text{def-H}_0(x, y) \) of Lemma 13.3.30. Then for every tuple \( a' = (a'_1, \ldots, a'_{54p+139}) \in V(G)^{54p+139} \) and every \( u \in V(G) \) it holds that \( G \models \varphi(a', u) \) if and only if the graph \( H'_0(\bar{a}') := G[\text{def-H}_0(G, \bar{a}', u)] \) is a \((p+4)\)-horizon for \( u \). In particular, \( G \models \varphi(\bar{a}, u_0) \) if and only if \( H_0 = H'_0(\bar{a}) \) is a \((p+4)\)-horizon for \( u_0 \).
• $G \models \text{angle'}([\pi', u_1, u_2, u_3]$ if and only if $H_0(\pi')$ is a $(p + 4)$-horizon for $u_2$ and $(u_1, u_2, u_3)$ is an angle of $H'(\pi)$, where $H'(\pi)$ is graph obtained from $G$ by contracting all connected components of $G \setminus H'_0(\pi)$ to single vertices.

• $G \models \text{aligned}'([\pi', u_1, u_2, u_3, u_4]$ if and only if $H_0(\pi')$ is a $(p + 4)$-horizon for $u_2$ and $(u_1, u_2, u_3, u_4)$ are aligned angles of $H'(\pi)$.

We let

$$\text{angle}_p(z_1, z_2, z_3) := \exists x_2 \ldots \exists x_{54p+139} \text{angle}'(z_2, x_2, x_3, \ldots, x_{54p+139}, z_1, z_2, z_3),$$

$$\text{aligned}_p(z_1, z_2, z_3, z_4) := \exists x_2 \ldots \exists x_{54p+139} \text{aligned}'(z_2, x_2, x_3, \ldots, x_{54p+139}, z_1, z_2, z_3, z_4),$$

It follows from Lemma 13.3.35 that these formulae have the desired properties. □

### 13.4 Centres and Skeletons

In this section, we prove that almost planar graphs can be decomposed into a planar “central” part, which admits a definable linear order, and “skeleton” which has bounded tree width. For the rest of this chapter, we let $p \in \mathbb{N}^+$ and

$$p'' := 5p + 15.$$

#### 13.4.1 Bounding the Tree Width of the Geometric Skeleton

**Definition 13.4.1.** Let $q \in \mathbb{N}^+$, and let $(G_0, \pi, R, \bar{\tau})$ be a local $p$-arrangement of a graph $G$ in a closed disk $D$.

1. The **geometric $q$-centre** of $(G_0, \pi, R, \bar{\tau})$ is the induced subgraph $g\text{-Cen}_q(G_0, \pi, R, \bar{\tau})$ of $G$ with vertex set $\{\pi(v) \mid v$ $q$-central in $G_0\}$.

2. The **geometric $q$-skeleton** of $(G_0, \pi, R, \bar{\tau})$ is the minor

$$g\text{-Skel}_q(G_0, \pi, R, \bar{\tau}) := G/g\text{-Cen}_q(G_0, \pi, R, \bar{\tau})$$

of $G$ obtained by contracting each connected component of the geometric $q$-centre into a single vertex.

The following lemma, which is crucial for most of what follows, shows that the tree width of $q$-skeletons of $p$-arrangements can be bounded in terms of $p$ and $q$.

**Lemma 13.4.2.** There is a constant $c_1(p) \in \mathbb{N}$ such that for all $q \in \mathbb{N}^+$ the following holds. Let $(G_0, \pi, R, \bar{\tau})$ be a $p$-arrangement of a graph $G$ in a closed disk $D$. Then

$$\text{tw} \left( g\text{-Skel}_q(G_0, \pi, R, \bar{\tau}) \right) \leq c_1(p) \cdot q.$$

The $r$-neighbourhood of a vertex $v$ in a graph $G$ is the set $N_r^G(v)$ of all vertices of distance at most $r$ from $v$. We use the following fact from [44].

**Fact 13.4.3.** There is a constant $c_2(p) \in \mathbb{N}$ such that for all $r \in \mathbb{N}$ the following holds. Let $G$ be a minor of a graph that has an injective $p$-arrangement in a disk. Then for all $v \in V(G)$ we have

$$\text{tw} \left( G[N_r(v)] \right) \leq c_2(p) \cdot r.$$
Proof of Lemma \ref{thm:claim3}. We shall define a graph $G^3$ and a vertex $v_0 \in V(G^3)$ such that the following conditions are satisfied:

(i) \( G^3 \in \mathcal{AP}_{p+2}. \)

(ii) \( g\text{-Skel}_{q}(G_0, \pi, R, r) \preceq G^3. \)

(iii) \( V(G^3) \subseteq N_{2q+3}^3(v_0). \)

Then the assertion of the lemma follows directly from Fact \ref{thm:claim2}

We will proceed in three steps in which we define graphs $G^1, G^2, G^3$ and arrangements of these graphs in $D$. In the first step, we make the arrangement injective. We choose a graph $G^1$, a subgraph $R^1 \subseteq G^1$, and an injective mapping $\pi^1 : V(G_0) \to V(G^1)$ according to Lemma \ref{thm:claim1} such that $(G_0, \pi^1, R^1, \tau)$ is an injective $(p+1)$-arrangement of $G^1$ in $D$ and $G = G^1/E^1$ for some set $E^1 \subseteq E(R^1)$. As the embedded graph $G_0$ is the same, the two arrangements $(G_0, \pi, R, \tau)$ and $(G_0, \pi^1, R^1, \tau)$ have the same geometric $q$-centre, and the geometric $q$-skeleton of $G$ is a minor of the geometric $q$-skeleton of $G^1$.

In the second step, we let $G^2 := g\text{-Skel}_{q}(G_0, \pi^1, R^1, \tau)$. Without loss of generality we assume that $\pi^1$ is the identity on $G_0$. Then $g\text{-Cent}_{q}(G_0, \pi, R, \tau) = g\text{-Cent}_{q}(G_0, \pi^1, R^1, \tau) \subseteq G_0 \setminus \tilde{\tau}$. We let $G_0^2 := G_0 / g\text{-Cent}_{q}(G_0, \pi^1, R^1, \tau)$. We may view $G_0^2$ as a graph embedded in $D$ that strongly coincides with $G_0 \cap G^2 = G_0 \setminus g\text{-Cent}_{q}(G_0, \pi^1, R^1, \tau)$. We let $\pi^2$ be the identity on $V(G_0^2)$. Then $(G_0^2, \pi^2, R^1, \tau)$ is an injective $(p+1)$-arrangement of $G^2$ in $D$.

In the third step, we first define two graphs $G^3_0$ and $R^3$. To define $R^3$, let $(P, \beta^3)$ be a path decomposition of $R^1$ of width at most $p$ such that $\tau$ is increasing in $(P, \beta^3)$. We take a fresh vertex $v_0$ (not in $V(G^2)$) and let $V(R^3) := V(R^2) \cup \{v_0\}$. We let

\[
E(R^3) := \bigcup_{t \in V(P)} K[\beta^3(t) \cup \{v_0\}].
\]

Then $R^2 \subseteq R^3$. If we define $\beta^3 : V(P) \to 2^{V(R^3)}$ by letting $\beta^3(t) := \beta^1(t) \cup \{v_0\}$ for all $t \in V(P)$, then $(P, \beta^3)$ is a path decomposition of $R^3$ of width at most $(p+1)$ such that $\tau$ is increasing in $(P, \beta^3)$. Thus $(R^3, \tau)$ is a $(p+2)$-ring.

To define $G^3_0$, for every face $f \in F(G^3_0)$ we let $v_f$ be a new vertex. We let $V(G^3_0) := V(G^2_0) \cup \{v_f \mid f \in F(G^3_0)\}$ and

\[
E(G^3_0) := E(G^2_0) \cup \{v_f w \mid f \in F(G^3_0), w \in V(G^3_0) \cap \text{bd}(f)\}.
\]

We extend the embedding of $G^3_0$ in $D$ to $G^3_0$ by embedding $v_f$ and all edges $v_f w$ incident with $v_f$ into $f$. Henceforth, we view $G^3_0$ as a graph embedded in $D$. We let $G^3 := R^3 \cup G^3_0$, and we let $\pi^3$ be the identity on $V(G^3_0)$. Then $(G^3_0, \pi^3, R^3, \tau)$ is an injective $(p+2)$-arrangement of $G^3$ in $D$, and we have $G^2 \subseteq G^3$ and thus $g\text{-Skel}_{q}(G_0, \pi, R, \tau) \preceq G^2 \preceq G^3$. Thus $G^3$ satisfies (i) and (ii). It remains to prove that $G^3$ and $v_0$ satisfy (iii), that is, that $\text{dist}^3_0(v_0, w) \leq 2q + 3$ for all $w \in V(G^3)$. Note that $\text{dist}^{G_0^3}(v_0, w) = 1$ for all $w \in V(R^3)$ and in particular for all $w \in \tilde{\tau}$. Hence it suffices to prove the following claim.

Claim 1. For every $w \in V(G^3_0)$ there is an $r \in \tilde{\tau}$ such that $\text{dist}^{G^3_0}(r, w) \leq 2q + 2$.

Proof. It follows from Lemma \ref{thm:claim5} (with $k = q - 1$) that for every

\[ w \in V(G_0 \setminus g\text{-Cent}_{q}(G_0, \pi, R, \tau)) \]

there is a $G^3_0$-normal curve $g \subseteq D$ from $w$ to a point in $\text{bd}(D)$ with $|g \cap V(G_0)| \leq q$. By moving the endpoint of $g$ along $\text{bd}(D)$ to the next vertex in $\tilde{\tau}$, we get the following:
(A) For every $w \in V(G_0 \setminus g\text{-}Cen_q(G_0, \pi, R, \overline{r}))$ there is an $r \in \overline{r}$ and a $G_0^1$-normal curve $g \subseteq D$ from $w$ to $r$ with $|g \cap V(G_0)| \leq q + 1$.

Then it follows from the definition of $G_0^q$ that:

(B) For every $w \in V(G_0 \setminus g\text{-}Cen_q(G_0, \pi, R, \overline{r}))$ there is an $r \in \overline{r}$ such that $\text{dist}^{G_0^q}(r, w) \leq 2q + 1$.

The claim follows, because every vertex of $G_0^q$ is adjacent to a vertex $V(G_0 \setminus g\text{-}Cen_q(G_0, \pi, R, \overline{r}))$, and for every face $f \in F(G_0^2)$ there is a vertex $v \in V(G_0 \setminus g\text{-}Cen_q(G_0, \pi, R, \overline{r})) \cap \text{bd}(f)$. □

**Definition 13.4.4.** Let $q \in \mathbb{N}^+$, and let $G$ be a graph.

1. A vertex $v \in V(G)$ is *geometrically $(p, q)$-central* if there is a local $p$-arrangement $(G_0, \pi, R, \overline{r})$ of $G$ in a closed disk $D$ such that $v \in V(g\text{-}Cen_q(G_0, \pi, R, \overline{r}))$.

2. The *geometric $(p, q)$-centre* of $G$ is the induced subgraph $g\text{-}Cen_{p,q}(G) := G[\{v \in V(G) \mid v \text{ geometrically $(p, q)$-central}\}]$.

3. The *geometric $(p, q)$-skeleton* of $G$ is the minor $g\text{-}\text{Skel}_{p,q}(G) := G/g\text{-}Cen_{p,q}(G)$.

Note that the geometric $(p, q)$-centre of a graph $G$ is the union of the geometric $q$-centres of all local $p$-arrangements of $G$ in disks. Thus the geometric $(p, q)$-skeleton of $G$ is a minor of the $q$-skeleton of any local $p$-arrangement of $G$.

**Corollary 13.4.5.** For every 3-connected graph $G \in AP_p$ and every $q \in \mathbb{N}^+$ it holds that $\text{tw}(g\text{-}\text{Skel}_{p,q}(G)) \leq c_1(p) \cdot q$, where $c_1(p)$ is the constant from Lemma 13.3.3.

13.4.2 The Definable Centre and Skeleton

Let $\text{angle}_p(x_1, x_2, x_3)$ be the formula of Lemma 13.3.3

**Definition 13.4.6.** Let $G$ be graph.

1. A vertex $v \in V(G)$ is *definably $p$-central* if it satisfies the following conditions.

   (i) For all $w, w' \in N(v)$, if $G \models \text{angle}_p[w, v, w']$ then $G \models \text{angle}_p[w', v, w]$.

   (ii) The graph $C_v := \left( N(v), \{ww' \mid G \models \text{angle}_p[w, v, w']\} \right)$ is an undirected cycle.

   (iii) For all $w, w' \in N(v)$ such that $G \models \text{angle}_p[w, v, w']$ there is exactly one $w'' \in N(w')$ such that $G \models \text{angled}_p[w, v, w']$ and $G \models \text{aligned}_p[w, v, w']$.

2. The *definable $p$-centre* of $G$ is the induced subgraph $\text{Cen}_p(G) := G[\{v \in V(G) \mid v \text{ definably $p$-central}\}]$. 

Preliminary Version
(3) The definable $p$-skeleton of $G$ is the minor

$$Skel_p(G) := G / Cen_p(G).$$

Lemma 13.4.7. There is an IFP-formula $cen_p(x)$ such that for every graph $G$ and vertex $v \in V(G)$ we have

$$G \models cen_p[v] \iff v \text{ is definably } p\text{-central.}$$

Proof. Straightforward.

Recall the definitions of subgraph transductions and contraction transductions from Examples 2.4.11 and 2.4.12.

Corollary 13.4.8. (1) There is an IFP subgraph transduction $\Theta_{p-Cen}$ such that for every graph $G$ it holds that $\Theta_{p-Cen}[G] \cong Cen_p(G)$.

(2) There is an IFP contraction transduction $\Theta_{p-Skel}$ such that for every graph $G$ it holds that $\Theta_{p-Skel}[G] \cong Skel_p(G)$.

Lemma 13.4.9. For every 3-connected graph $G$ it holds that $g\text{-Cen}_{p,p'}(G) \subseteq Cen_p(G)$.

Proof. Let $G$ be a 3-connected graph. It suffices to prove that for every local $p$-arrangement $(G_0, \pi, R, \tau)$ of $G$ in a closed disk $D$ it holds that

$$g\text{-Cen}_{p,p'}(G_0, \pi, R, \tau) \subseteq Cen_p(G). \quad (13.4.1)$$

Let $(G_0, \pi, R, \tau)$ be a local $p$-arrangement of $G$ in a closed disk $D$. Let $v \in V(G)$ such that $\Pi(v)$ is $p''$-central in $G_0$. By Lemma 13.3.3, for all $w, w' \in N_G(v)$ it holds that $G \models \text{angle}_p(w, v, w')$ if and only if $(\Pi(w), \Pi(v), \Pi(w'))$ is an angle of $G_0$. As the embedding of $G_0$ in the disk $D$ induces a cyclic ordering of $N_{G_0}(\Pi(v)) = \Pi(N_G(v))$ and hence on $N_G(v)$, it follows that the graph $C_v$ is a cycle. Furthermore, for all $w, w' \in N_G(v)$ and $w'' \in N_G(w')$ it holds that $G \models \text{aligned}_p(w, v, w', w'')$ if and only if $(\Pi(w), \Pi(v), \Pi(w'))$ and $(\Pi(v), \Pi(w'), \Pi(w''))$ are aligned angles of $G_0$. By Lemma 13.1.13(3), this implies that for all $w, w' \in N_G(v)$ there is exactly one $w'' \in N_G(w')$ such that $G \models \text{aligned}_p(w, v, w', w'')$. Thus $v \in V(Cen_p(G))$. \qed

Corollary 13.4.10. For every 3-connected graph $G$ it holds that $\text{Skel}_p(G) \leq g\cdot \text{Skel}_{p,p'}(G)$.

Corollary 13.4.11. There is a constant $c_3(p)$ such that for every 3-connected $p$-almost planar graph $G$ it holds that $\text{tw}(\text{Skel}_p(G)) \leq c_3(p)$.

Lemma 13.4.12. There is an IFP-formula $\text{ord-cen}_p(\overline{\tau}, y_1, y_2)$ such that for every 3-connected graph $G$ the following holds. For every connected component $A$ of $Cen_p(G)$ there is a tuple $\overline{\tau} \in V(G)^{\overline{\tau}}$ such that $\text{ord-cen}_p[G, \overline{\tau}, y_1, y_2]$ is a linear order of $V(A)$.

Proof. Follows from the definition of the centre and the Second Angle Lemma 9.2.5. Note that from a linear order of $V(A) \cap N(A)$ one can definably extract a linear order of $V(A)$, because the set $V(Cen(G))$ is definable. \qed
13.5 Decomposing Almost Planar Graphs and Their Minors

Our goal is to prove the following theorem. Remember that we fixed $p$ and set $p'' := 5p + 15$.

**Theorem 13.5.1 (Definable Structure Theorem for Almost Planar Graphs).**

The class $\mathcal{M}(\mathcal{AP}_p)$ of all minors of $p$-almost planar graphs admits $\text{IFP}$-definable ordered treelike decompositions.

The rest of this subsection is devoted to a proof of this theorem. We first prove that the class of 3-connected $p$-almost planar graphs admits $\text{IFP}$-definable ordered treelike decompositions. By analysing the minors of almost planar graphs, we then derive the corresponding result for 3-connected minors of $p$-almost planar graphs. Then the theorem follows from the 3CC Lifting Lemma (Corollary 8.3.3).

### 13.5.1 Almost Planar Graphs

We start with yet another extension lemma for definable ordered treelike decompositions.

**Lemma 13.5.2.** Let $k \in \mathbb{N}$. Furthermore, let $\Theta$ be an $\text{IFP}$-subgraph transduction and $\varphi(\vec{x},y_1,y_2)$ an $\text{IFP}$-formula. Then there exists an od-scheme $\Lambda$ defining an ordered treelike decomposition on all graphs $G$ satisfying the following conditions:

(i) $G \in D_\Theta$;

(ii) $\text{tw}(G/\Theta[G]) \leq k$;

(iii) for every connected component $A$ of $\Theta[G]$ there is a tuple $\vec{v} \in G^\vec{x}$ such that $\varphi[G,\vec{v},y_1,y_2]$ is a linear order of $V(A)$.

Before we prove the lemma, we state its most important implications. For every $\ell \in \mathbb{N}$, we let

$$SK_{p,\ell}$$

be the class of all 3-connected graphs whose definable $p$-skeleton has tree width at most $\ell$.

**Corollary 13.5.3.** For all $\ell \in \mathbb{N}$, the class $SK_{p,\ell}$ admits $\text{IFP}$-definable ordered treelike decompositions.

**Proof.** We apply Lemma 13.5.2 to $k := \ell$, the transduction $\Theta := \Theta_{p-Cen}$ of Corollary 13.4.8 and the formula $\varphi(\vec{x},y_1,y_2) := \text{ord-cen}_p$ of Lemma 13.4.12. \hfill \Box

**Corollary 13.5.4.** The class $Z_3 \cap \mathcal{AP}_p$ of 3-connected $p$-almost planar graphs admits $\text{IFP}$-definable ordered treelike decompositions.

**Proof.** We have $Z_3 \cap \mathcal{AP}_p \subseteq SK_{p,c3(p)}$ by Corollary 13.4.11. \hfill \Box

**Proof of Lemma 13.5.2** To explain the definition of $\Lambda$, it will be convenient to fix a graph $G$ that satisfies (i)–(iii). Of course the formulae we shall define will not depend on $G$. Let $H := \Theta[G]$ and $G' := G/H$. Suppose that the connected components of $H$ are $A_1, \ldots, A_m$, and let $a_1, \ldots, a_m$ be the elements of $G'$ corresponding to the respective components.

We define a graph minor transduction $\Theta'$ as follows:

- $\theta'_{\text{dom}} := \text{true};$
• \( \theta'_V(y) := \text{true}; \)
• \( \theta'_E(y_1, y_2) \) is an \( \mathbf{IFP} \)-formula such that for all \( v_1, v_2 \in V(G) \) it holds that \( G \models \theta'_E[v_1, v_2] \) if and only if \( v_1, v_2 \in V(H) \) and \( v_1, v_2 \) belong to the same connected component of \( H \) or \( v_1, v_2 \not\in V(H) \) and \( v_1 = v_2; \)
• \( \theta'_E(y_1, y_2) := E(y_1, y_2) \land \neg \theta'_E(y_1, y_2). \)

Let \( \equiv := \theta'_E[G, y_1, y_2] \). Obviously, \( \equiv \) is an equivalence relation, and we have \( V(\Theta'[G]) = V(G)/\equiv \). Observe that \( \Theta'[G] \cong G/H \), and the mapping \( f : V(G)/\equiv \) \( \rightarrow \) \( V(G') \) defined by \( f(\{v\}) := v \) for \( v \in V(G) \setminus V(H) \) and \( f(V(A_i)) := a_i \) for \( i \in [m] \) is an isomorphism from \( \Theta'[G] \) to \( G' \).

Let \( \Delta' = (\lambda'_V(\bar{x}), \lambda'_E(\bar{x}, \bar{y}), \lambda'_\sigma(\bar{x}, y), \lambda'_\alpha(\bar{x}, y)) \) be a \( d \)-scheme that defines a treelike decomposition of width at most \( k \) on all graphs of tree width at most \( k \). Such a \( d \)-scheme exists by Theorem 6.1.1. Remember that \( \text{tw}(G') \leq k \), and let \( \Delta' := (D', \sigma', \alpha') \) be the decomposition defined by \( \Delta \) on \( G' \).

We apply the Transduction Lemma (Fact 2.4.6) to \( \Theta' \) and the formulae of \( \Lambda' \). We obtain a new \( d \)-scheme \( \Lambda := (\lambda_V(\bar{x}), \lambda_E(\bar{x}, \bar{y}), \lambda_\sigma(\bar{x}, y), \lambda_\alpha(\bar{x}, y)) \), where \( \lambda_V(\bar{x}) := (\lambda'_V(\bar{x}))^\cong(\bar{y}) \) and \( \lambda_E(\bar{x}, \bar{y}), \lambda_\sigma(\bar{x}, y), \lambda_\alpha(\bar{x}, y) \) are defined similarly. Let \( \Delta := (D, \sigma, \alpha) \) be the decomposition defined by \( \Lambda \) on \( G \). Then we have

\[
V(D) = \{ \bar{v} \in G^2 \mid f(\bar{v}/\equiv) \in V(D') \},
\]
\[
E(D) = \{ (\bar{v}, \bar{v}') \in V(D)^2 \mid (f(\bar{v}/\equiv), f(\bar{v}'/\equiv)) \in E(D') \}.
\]

Moreover, for all \( \bar{v} \in V(D) \) we have

\[
\alpha(\bar{v}) = (\alpha'(f(\bar{v}/\equiv)) \setminus V(H)) \cup \bigcup_{i \in [m]} V(A_i),
\]
\[
\text{with } a_i \in \alpha'(f(\bar{v}/\equiv))
\]
\[
\sigma(\bar{v}) = (\sigma'(f(\bar{v}/\equiv)) \setminus V(H)) \cup \bigcup_{i \in [m]} V(A_i),
\]
\[
\text{with } a_i \in \sigma'(f(\bar{v}/\equiv))
\]

With the usual notation, we have similar relationships between \( \beta, \beta' \) and \( \gamma, \gamma' \). It is now straightforward to verify that \( \Delta \) is a treelike decomposition of \( G \).

As the tree width of \( \Delta' \) is at most \( k \), for every bag \( \beta(t) \) of \( \Delta \) there is an \( \ell \leq k + 1 \) such that \( \beta(t) \) is the union of \( \ell \) sets \( V(A_i) \), for \( i \in [m] \), with at most \( k + 1 - \ell \) elements of \( V(G) \setminus V(H) \).

Using the formula \( \varphi \) that defines linear orders on the sets \( V(A_i) \), it is easy to define a formula \( \varphi'(\bar{x}, y_1, y_2) \) such that for every \( t \in V(D) \) there is a tuple \( \bar{w} \in G^2 \) such that \( \varphi'[G, \bar{w}, y_1, y_2] \) is a linear order of \( \beta(t) \). Now the lemma follows from Corollary 7.1.9.

13.5.2 The Structure of Minors of Almost Planar Graphs

Unfortunately, the class of \( p \)-almost planar graphs does not seem to be closed under taking minors. In this section, we study the structure of graphs in the class \( \mathcal{M}(\mathcal{AP}_p) \) of minors of \( p \)-almost planar graphs.

**Definition 13.5.5.** An \( m \)-arrangement of a graph \( G^* \) in a closed disk \( D \) is a tuple \( (G_0, \pi, R, \tau, F) \) such that:

M. Grohe, *Definable Graph Structure Theory*
(i) \((G_0, \pi, R, \vec{r})\) is an arrangement of the graph \(G := \pi(G_0) \cup R\) in \(D\).

(ii) \(F \subseteq E(\pi(G_0))\), and all edges in \(F\) have both endvertices in \(\pi(\vec{r})\).

(iii) \(G^* = G/F\).

\((G_0, \pi, R, \vec{r}, F)\) is a \(p\)-\(m\)-arrangement if \((R, \pi(\vec{r}))\) is a \(p\)-ring, and it is a local \(p\)-\(m\)-arrangement if \((R, \pi(\vec{r}))\) is a 2\(p\)-vortex.

The class of all graphs that have a \(p\)-\(m\)-arrangement is denoted by \(\mathcal{MAP}_p\).

\textbf{Lemma 13.5.6.} \(\mathcal{M}(\mathcal{AP}_p) = \mathcal{MAP}_p\).

\textit{Proof.} The inclusion \(\mathcal{M}(\mathcal{AP}_p) \supseteq \mathcal{MAP}_p\) is trivial. To prove the converse inclusion \(\mathcal{M}(\mathcal{AP}_p) \subseteq \mathcal{MAP}_p\), let \((G_0, \pi, R, \vec{r})\) be a \(p\)-arrangement of a graph \(G\) in disk \(D\) and let \(G^* \subseteq G\). We shall construct a \(p\)-\(m\)-arrangement of \(G^*\) in \(D\).

For every subgraph \(H \subseteq G\) we let \(\pi^{-1}(H)\) be the graph with vertex set \(\pi^{-1}(V(H) \cap \pi(V(G_0)))\) and edge set \(\{vw \in E(G_0) \mid \pi(v) \pi(w) \in E(H)\}\). There exists a subgraph \(G' \subseteq G\) and a set \(E' \subseteq E(G')\) of edges of \(G'\) such that \(G^* \supseteq E' := G' / E'\). The arrangement \((G_0, \pi, R, \vec{r})\) of \(G\) induces an arrangement \((G_0', \pi', R', \vec{r}')\) of \(G'\) defined as follows:

1. \(G_0' := \pi^{-1}(G')\).
2. \(\pi'\) is the restriction of \(\pi\) to \(V(G_0')\).
3. \(R' := R \cap G'\).
4. \(\vec{r}' := \vec{r} \cap V(G_0')\) is the subtuple of \(\vec{r}\) consisting of all entries in \(V(G_0')\).

Clearly, \((G_0', \pi', R', \vec{r}')\) is an arrangement of \(G'\) in \(D\). It is easy to see that \((R', \pi(\vec{r}'))\) is a \(p\)-ring, but for later reference we define a path decomposition \((P, \beta_p')\) of \(R'\) of width at most \((p - 1)\) such that \(\pi'(\vec{r}')\) is increasing in \((P, \beta_p')\). Let \((P, \beta_P)\) be a path decomposition of \(R\) of width at most \((p - 1)\) such that \(\pi(\vec{r})\) is increasing in \((P, \beta_P)\). We define \(\beta'_P : V(P) \rightarrow 2^{V(G')}\) by \(\beta'_P(t) := \beta_P(t) \cap V(G')\).

Without loss of generality we may assume that the graph \((V(G'), E')\), which we contract when forming the minor \(G^*\), is a forest. Then if we contract a subset \(E'' \subseteq E'\), each edge of \(E' \setminus E''\) corresponds to an edge of the graph \(G'/E''\). To simplify the following discussion, we will not distinguish between the edges in \(E' \setminus E'' \subseteq E(G')\) and the corresponding edges in \(E(G'/E'')\). The order in which the edges in \(E'\) are contracted does not matter. Thus we can delay the contraction of edges \(e \in \pi'(G_0')\) with both endvertices in \(\vec{r}'\), or what has become of \(\vec{r}'\) by the previous edge contractions. More formally, we can partition \(E'\) into two sets \(E'', F\) such that conditions (A) and (B) below are satisfied.

(A) For all connected components \(T\) of the forest \((V(G'), E'')\) and \(r, r' \in \pi'(\vec{r}') \cap V(T)\) with \(r \neq r'\), the path \(rTr'\) is contained in \(R'\).

Note that we do not require all components \(T\) of \((V(G'), E'')\) to contain vertices of \(\pi'(\vec{r}')\); for components \(T\) that do not contain at least two vertices in \(\pi'(\vec{r}')\) condition (A) is vacuous.

(B) \(F \subseteq E(\pi'(G_0'))\), and for all edges \(f = v_1v_2 \in F\) there are connected components \(T_1, T_2\) of \((V(G'), E'')\) such that for \(i = 1, 2\) we have \(v_i \in V(T_i)\) and \(V(T_i) \cap \pi'(\vec{r}') \neq \emptyset\).
Intuitively, we add edges from $E'$ to $E''$ until we only have edges in $E(\pi'(G_0'))$ left whose contraction would identify two vertices in $\pi'(\mathcal{F})$, after contracting all edges in $E''$.

Let $G' := G'/E''$, and let $E'_0 := (\pi')^{-1}(E'') = \{vw \in E(G_0') \mid \pi'(v)\pi'(w) \in E''\}$. The $p$-arrangement $(G_0', \pi', R', \mathcal{F}')$ of $G'$ induces a $p$-arrangement $(G''_0, \pi'', R'', \mathcal{F}'')$ of $G''$ defined as follows:

- We let $G''_0 := G'_0/E'_0$. By Fact 9.1.18 and (A), we may view $G'_0$ as a graph embedded in $D$.

For every $v \in V(G')$, we let $a(v)$ be the vertex of $G''_0$ that corresponds to the connected component $A(v)$ of $(V(G'), E'')$ with $v \in V(A(v))$. Similarly, for every $v \in V(G'_0)$, we let $a_0(v)$ be the vertex of $G''_0$ that corresponds to the connected component $A_0(v)$ of $(V(G'_0), E''_0)$ with $v \in V(A_0(v))$. Note that $V(G''_0) = \{a(v) \mid v \in V(G')\}$ and $V(G''_0) = \{a_0(v) \mid v \in V(G'_0)\}$, and for all vertices $v, w \in V(G'_0)$, if $a_0(v) = a_0(w)$ then $a(\pi'(v)) = a(\pi'(w))$.

- We define the mapping $\pi'': V(G''_0) \rightarrow V(G'')$ by $\pi''(a_0(v)) := a(\pi'(v))$.

- We let $R''$ be the subgraph of $G''_0$ with vertex set $V(R'') := \{a(v) \mid v \in V(R')\}$ and edge set $E(R'') := \{a(v)a(w) \mid \exists v', w' \in V(R') : a(v') = a(v) \text{ and } a(w') = a(w) \text{ and } v'w' \in E(R')\} \setminus E(\pi''(G''_0))$.

- Suppose that $\mathcal{F}' = (r_1, \ldots, r_n)$. We let $\mathcal{F}'' := (a_0(r_1), \ldots, a_0(r_n))$.

**Claim 1.** $(G''_0, \pi'', R'', \mathcal{F}'')$ is a $p$-arrangement of $G''$ in $D$.

**Proof.** To prove that $(G''_0, \pi'', R'', \mathcal{F}'')$ is an arrangement, we only need to verify (AD.1) and (AD.5) follow immediately from the definitions. (AD.1) follows from the observation that for every vertex $a_0(v) \in V(G''_0) \setminus \mathcal{F}''$, the connected component $A_0(v)$ of $(V(G''_0), E''_0)$ corresponding to $a_0(v)$ has an empty intersection with $\mathcal{F}'$. Then $A(\pi'(v))$ is a connected component of $(V(G'), E'')$, because no vertex of $A$ is incident with an edge $E(G') \setminus E(\pi'(G'_0))$.

It remains to prove that $(R'', \pi''(\mathcal{F}''))$ is a $p$-ring. Let $(P, \beta'_P)$ be the path decomposition of $R'$ defined above. We define $\beta''_{P} : V(P) \rightarrow 2^{V(G'')}$ by

$$\beta''_{P}(t) := \{a(v) \mid V(A(v)) \cap \beta'_P(t) \neq \emptyset\}.$$ 

It is easy to see that $(P, \beta''_{P})$ is a path decomposition of $R''$ of width at most $(p - 1)$, and $\pi''(\mathcal{F}'')$ is increasing in $(P, \beta''_{P})$.

Observe that by (B), all edges in $F$, or more precisely the set of edges of $G'/E''$ corresponding to $F$, are in $E(\pi(G''_0))$ and have both endvertices in $\pi(\mathcal{F}'')$. Thus $(G''_0, \pi'', R'', \mathcal{F}'', F)$ is a $p$-m-arrangement of $(G/E'')/F$ in $D$. As $G^* \cong (G/E'')/F$, this completes the proof. 

**Remark 13.5.7.** The arrangement $(G''_0, \pi'', R'', \mathcal{F}'')$ we construct in the previous proof is not necessarily injective, even if the arrangement $(G_0, \pi, R, \mathcal{F})$ we start with is, because it may happen that for two vertices $r, r' \in \mathcal{F}$ we have $a_0(r) = a_0(r')$, but $a(\pi'(r)) = a(\pi'(r'))$.

This is the reason we introduced non-injective arrangements in the first place.
Lemma 13.5.8. Let \((G_0, \pi, R, \tau, F)\) be an \(m\)-arrangement of a graph \(G^*\) in a disk \(D\), and let \(F_0 := \pi^{-1}(F) = \{vw \in E(G_0) \mid \pi(v)\pi(w) \in F\}\). Then there is a tree decomposition \((T^*, \beta^*)\) of \(G^*\) and for every \(t \in V(T^*)\) a closed disk \(D_t\) such that the following conditions are satisfied.

(i) \(\bigcup_{t \in V(T^*)} D_t = D\).

(ii) \(\text{int}(D_t) \cap \text{int}(D_u) = \emptyset\) for all distinct \(t, u \in V(T^*)\).

(iii) \(\bigcup_{t \in V(T^*)} \text{bd}(D_t) = \text{bd}(D) \cup \bigcup_{f \in F_0} f\).

(iv) For all \(t \in V(T^*)\) the torso \(\tau^*(t)\) has a \(3p\)-arrangement \((G_{10}, \pi_t, R_t, r_t)\) in \(D_t\) such that:

\[
\begin{align*}
\text{a.} & \text{ for all } u \in \{t\} \cup N^*_t(t) \text{ it holds that } K[\sigma^*(u)] \subseteq R_t; \\
\text{b.} & \text{ } V(R_u) \subseteq V((R + F)/F); \\
\text{c.} & \text{ } G_{10} \subseteq G_0/F_0, \text{ and the embeddings of } G_{10} \text{ in } D_t \text{ and of } G_0 \text{ in } D \text{ strongly coincide on } G_{10} \cap (G_0 \setminus \cup F_0). \\
\end{align*}
\]

Proof. For every subgraph \(H \subseteq G\) we let \(\pi^{-1}(H)\) be the graph with vertex set \(\pi^{-1}(V(H) \cap \pi(V(G_0)))\) and edge set \(\{vw \in E(G_0) \mid \pi(v)\pi(w) \in E(H)\}\). Observe that each edge \(f_0 \in F_0\) has both endvertices in \(\tau \subseteq \text{bd}(D)\) and thus cuts the disk \(D\) into two pieces. This cut induces a separation of \(G_0\), which yields a separation of \(G\) of order \(2p\) and a separation of \(G^*\) of order \(2p - 1\). We will decompose \(G\) and \(G^*\) using these separations induced by the edges in \(F\).

Let \((P, \beta P)\) be a path decomposition of \(R\) of width \((p - 1)\) such that \(\tau\) is increasing in \((P, \beta P)\). Without loss of generality we assume that \(P\) is the natural path on \([m]\). Suppose that \(\tau = (r_1, \ldots, r_n)\). Let \(1 \leq i_1 < i_2 < \ldots < i_n \leq m\) such that for all \(j \in [n]\) we have \(r_j \in \beta P(i_j)\). Fix an orientation of \(D\) and suppose that the vertices \(r_1, \ldots, r_n\) appear on \(\text{bd}(D)\) in clockwise direction. For all \(i, j \in [n]\) with \(i \neq j\), let \(q_{ij}\) be the segment of \(\text{bd}(D)\) from \(r_i\) to \(r_j\) in clockwise direction. Thus if \(i < j\), then \(r_i, r_{i+1}, \ldots, r_j \in q_{ij}\), and if \(i > j\) then \(r_i, r_{i+1}, \ldots, r_j, r_{i+1}, r_{i+2}, \ldots, r_j \in q_{ij}\). For every edge \(f \in F_0\), let \(x(f)_0, y(f)_0 \in [n]\) such that \(x(f)_0 < y(f)_0\) and \(f = x(f)_0 y(f)_0\). We let \(d_f \subseteq D\) be the closed disk with boundary \(q_{x(f)_0}(y(f)_0) \cup f\).

We define a partial order \(\preceq\) on \(F_0\) by letting \(f \preceq g\) if \(d_f \supseteq d_g\). We extend the partial order \(\preceq\) to the set \(F_0 \cup \{\bot\}\), where \(\bot\) is just a symbol not contained in \(F_0\), by letting \(\bot \preceq f\) for all \(f \in F_0\), and we let \(d_\bot := D\). We let \(T\) be the directed graph with \(V(T) := F \cup \{\bot\}\) and

\[
E(T) := \{tu \in V(T) \mid t \prec u \text{ and there is no } v \in V(T) \text{ with } t \preceq u \prec v\}.
\]

Then \(\preceq^T = \preceq\).

Let \(f, g \in E(D)\) be edges in \(D\), the edges \(f, g\) do not cross, and thus either \(d_f \subset d_g\) or \(d_f \subset d_g\) or \(\text{int}(d_f) \cap \text{int}(d_g) = \emptyset\). If \(\text{int}(d_f) \cap \text{int}(d_g) = \emptyset\) and \(d_f \cap d_g \neq \emptyset\), then the only point in \(d_f \cap d_g\) is a common endvertex of \(f\) and \(g\) (either \(y(f) = x(g)\) or \(y(g) = x(f)\)). It follows that for every \(f \in F_0\) the set \(\{g \in F_0 \mid g \preceq f\}\) is linearly ordered by \(\preceq\). Thus \(T\) is a tree. \(\bot\) is obviously the root of this tree.

For every \(t \in V(T)\), let

\[
D_t := \text{cl}(d_t \setminus \bigcup_{u \in N^*_t(t)} d_u).
\]

Then \(D_t\) is a closed disk. Suppose that \(N^*_t(t) = \{f^1, \ldots, f^l\}\). We may assume that \(x(f^1) < y(f^1) \leq x(f^2) < y(f^2) \leq x(f^3) < \ldots \leq x(f^l) < y(f^l)\). If \(t = \bot\) then

\[
\text{bd}(D_t) = f^1 \cup q_{y(f^1)} x(f^2) \cup f^2 \cup q_{y(f^2)} x(f^3) \cup \ldots \cup f^l \cup q_{y(f^l)} x(f^1),
\]
If $t = f \in F_0$ then

$$bd(D_t) = f \cup q_{x(f)x(f^1)} \cup f^1 \cup q_{y(f)x(f^2)} \cup f^2 \cup \ldots \cup f^t \cup q_{y(f)y(f)}.$$ 

Note furthermore that $\{\text{int}(D_t) \mid t \in V(T)\}$ is precisely the set of arcwise connected components of $\text{int}(D) \setminus \bigcup_{f \in F_0} f$. This implies (i)–(iii).

For every $t \in V(T)$ we define a path $P_t$ as follows. Suppose that $N^+_t(t) = \{f^1, \ldots, f^t\}$ with $x(f^1) < y(f^1) \leq x(f^2) < y(f^2) \leq x(f^3) < \ldots \leq x(f^t) < y(f^t)$. Recall that $V(P) = [m]$ and $i_j \in [m]$ such that $r_j = \beta_P(i_j)$, for all $j \in [n]$.

- If $t = \bot$ then we let $V(P_t) = [1, i_{x(f^1)}] \cup [i_{y(f^1)}, i_{x(f^2)}] \cup [i_{y(f^2)}, i_{x(f^3)}] \cup \ldots \cup [i_{y(f^{t-1})}, i_{x(f^t)}] \cup [i_{y(f^t)}, m]$.
- If $t = f \in F_0$ we let $V(P_t) = [i_{x(f)}, i_{x(f^1)}] \cup [i_{y(f^1)}, i_{x(f^2)}] \cup [i_{y(f^2)}, i_{x(f^3)}] \cup \ldots \cup [i_{y(f^{t-1})}, i_{x(f^t)}] \cup [i_{y(f^t)}, i_{y(f)}]$.
- We let $E(P_t) := \{ij \in V(P_t)^2 \mid i < j \text{ and } \forall k \in V(P_t) : (k \leq i \text{ or } j \leq k)\}$.

We define a mapping $\beta : V(T) \to 2^{V(G)}$ by

$$\beta(t) := \pi(V(G_0) \cap D_t) \cup \bigcup_{i \in V(P_t)} \beta_P(i),$$

for all $t \in V(T)$.

**Claim 1.** $(T, \beta)$ is a tree decomposition of $G$.

**Proof.** To verify axiom (T.1) let $v \in V(G)$ and $\beta^{-1}(v) := \{t \in V(T) \mid v \in \beta(t)\}$. If $v \in \pi(V(G_0) \setminus \pi)$, then there is a unique $t \in V(T)$ such that $\Pi(v) \in \text{int}(D_t)$. In this case, we have $\beta^{-1}(v) = \{t\}$. Similarly, if

$$v \in V(R) \setminus \bigcup_{f \in F_0} (\beta_P(i_{x(f)}) \cup \beta_P(i_{y(f)}))$$

then there is a unique $t \in V(T)$ such that $v \in \beta_P(i)$ for some $i \in V(P_t)$. Again, we have $\beta^{-1}(v) = \{t\}$.

So suppose that $v \in \beta_P(i)$ for some $i \in \{i_{x(f)}, i_{y(f)} \mid f \in F_0\}$. Then $v \in \beta(t)$ for all $t \in V(T)$ such that there is an $f \in F_0$ with $i \in \{i_{x(f)}, i_{y(f)}\}$ and either $f = t$ or $f \in N_T^+(t)$. To see that this set of nodes is connected in $T$, let $f_1, \ldots, f_k$ be the set of all edges in $f \in F_0$ with $i_y(f) = i$, sorted by increasing $x(f_j)$, and let $g_1, \ldots, g_\ell$ be the set of all edges in $g \in F_0$ with $i_x(g) = i$, sorted by increasing $y(g_j)$ (see Figure 3.13). Then there is a unique node $t \in V(T)$ such that $f_1, g_\ell \in N_T^+(t)$, and we have $\beta^{-1}(v) \supseteq \{t, f_1, \ldots, f_k, g_1, \ldots, g_\ell\}$. Moreover, for all $j \in [k - 1]$ it holds that $f_{j+1} \in N_T^+(f_j)$, and for all $j \in [\ell - 1]$ it holds that $g_j \in N_T^+(g_{j+1})$. Hence $T\{t, f_1, \ldots, f_k, g_1, \ldots, g_\ell\}$ is connected. This proves (T.1).

(T.2) follows from the fact that $\{\text{int}(D_t) \mid t \in V(T)\}$ is precisely the set of arcwise connected components of $\text{int}(D) \setminus \bigcup_{f \in F_0} f$. 

M. Grohe, Definable Graph Structure Theory
13.5. Decomposing Almost Planar Graphs and Their Minors

Figure 13.13. The arrangement of edges in the proof of Claim [ ]. The disks $d_{f_1}$ and $d_{g_k}$ are filled.

Let us now turn to the graph $G^*$. For every $v \in V(G)$, let $A(v)$ be the connected component of the graph $(V(G), F)$ that contains $v$, and let $a(v)$ be the vertex of $G^*$ corresponding to $A(v)$. The tree decomposition $(T, \beta)$ of $G$ induces a tree decomposition $(T^*, \beta^*)$ of $G^*$ defined as follows: we let $T^* := T$, and for every $t \in V(T)$ we let

$$\beta^*(t) := \{ a(v) \mid v \in \beta(t) \}.$$ 

Note that $\beta^*(t) = \{ a(v) \mid v \in V(G) \text{ such that } V(A(v)) \cap \beta(t) \neq \emptyset \}$. It is easy to see that $(T^*, \beta^*)$ is a tree decomposition of $G^*$.

We shall prove that every torso of the decomposition $(T^*, \beta^*)$ has an arrangement satisfying (iv) a–c. For the rest of the proof, we fix a node $t \in V(T)$ and let $H := \tau(t)$ and $H^* := \tau^*(t)$. We need some additional notation. For every vertex $v \in V(G_0)$, let $A_0(v)$ be the connected component of the graph $(V(G_0), F_0)$ with $v \in V(A_0(v))$. Let $G_0^*$ be the minor of $G_0$ obtained from $G_0$ by contracting all components $A_0(v)$, for $v \in V(G_0)$, to vertices $a_0(v)$. We define a mapping $\pi^* : V(G_0^*) \to V(G^*)$ by $\pi^*(a_0(v)) := a(\pi(v))$. This mapping is well-defined, because $\pi(A_0(v))$ is a connected subgraph of $(V(G), F)$ for every $v \in V(G_0)$. The mapping $\pi^*$ is not necessarily injective, but if $a_0(v) \neq a_0(w)$ and $\pi^*(a_0(v)) = \pi^*(a_0(w))$ then $a_0(v) = a_0(r)$ and $a_0(v') = a_0(r')$ for some $r, r' \in \tau$, because all edges in $F_0$ have both endvertices in $\tau$.

Let

$$H_0 := G_0[\pi^{-1}(V(H) \cap \pi(V(G_0)))].$$

Note that $H_0$ is embedded in $D_t$. Let

$$H_0^* := G_0^*[(\pi^*)^{-1}(V(H^*) \cap \pi^*(V(G_0^*)))].$$

Claim 2. Let $v \in V(H_0)$. Then $A_0(v) \cap H_0$ is connected. Furthermore, if $|A_0(v) \cap H_0| \geq 2$ then $A_0(v) \cap H_0$ is a path or cycle in $G_0$ that is embedded in $bd(D_t)$.

Proof. To see that $A_0(v) \cap H_0$ is connected, let $w, w' \in V(A_0(v)) \cap V(H_0)$ with $w \neq w'$. Since $w, w'$ belong to the same connected component of the graph $(V(G_0), F_0)$, they are both endvertices of edges in $F_0$ and thus belong to $bd(D_t)$. Let $g, g'$ be the two segments of $bd(D_t)$ from $w$ to $w'$. As all edges in $F_0$ have an empty intersection with $int(D_t)$ and as $H_0$ is embedded in $D_t$, there is a path from $w$ to $w'$ in $A_0(v) \cap H_0$ if and only if there is a sequence of edges $f_1, \ldots, f_\ell \in F_0$ such that either $g = \bigcup_{i=1}^\ell f_i$ or $g' = \bigcup_{i=1}^\ell f_i$. 

Preliminary Version
Suppose for contradiction that there is no such edge sequence. Then there are points \( x \in g, x' \in g' \) such that \( x, x' \notin \bigcup_{f \in F_0} f_0 \). As \( bd(D_t) \setminus \bigcup_{f \in F_0} f_0 \subseteq bd(D) \), we have \( x, x' \in bd(D) \). There is a simple curve \( s \) with endpoints \( x, x' \) and all interior points in \( \text{int}(D_t) \). This curve separates \( w \) and \( w' \) in \( D \). As \( s \) is disjoint from \( f \) for all \( f \in F_0 \), this contradicts \( w, w' \) belonging to the same connected component \( A_0(v) \) of \( (V(G_0), F_0) \).

The second assertion of the claim follows immediately from the first, because for all \( f \in E(A(v)) \subseteq F_0 \) it holds that \( f \cap \text{int}(D_t) = \emptyset \).

Suppose that \( N^T_1(t) = \{f^1, \ldots, f^\ell\} \) with \( x(f^1) < y(f^1) \leq x(f^2) < y(f^2) \leq x(f^3) < \ldots \leq x(f^\ell) < y(f^\ell) \). Since

\[
\sigma(t) = \begin{cases} 
\emptyset & \text{if } t = \bot, \\
\beta_P(i_{x(f)}) \cup \beta_P(i_{y(f)}) & \text{if } t = f \in F_0
\end{cases}
\]

and thus \( \sigma(f^j) = \beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)}) \) for all \( j \in [\ell] \), we have

\[
H = \begin{cases} 
G[\beta(t)] \cup \bigcup_{j=1}^\ell K[\beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)})] & \text{if } t = \bot, \\
G[\beta(t)] \cup K[\beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)})] \cup \bigcup_{j=1}^\ell K[\beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)})] & \text{if } t = f \in F_0.
\end{cases}
\]

Remember the definition of the path \( P_t \). We define a subgraph \( R_t \subseteq H \) and subtuple \( \pi_t \) of \( \pi \) as follows.

- If \( t = \bot \), we let

\[
R_t := R[\bigcup_{i \in V(R_t)} \beta_P(i)] \cup \bigcup_{j=1}^\ell K[\beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)})],
\]

and if \( t = f \in F_0 \) we let

\[
R_t := R[\bigcup_{i \in V(R_t)} \beta_P(i)] \cup K[\beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)})] \cup \bigcup_{j=1}^\ell K[\beta_P(i_{x(f^j)}) \cup \beta_P(i_{y(f^j)})].
\]

- If \( t = \bot \), we let

\[
\pi_t := (r_1, \ldots, r_{x(f^1)}, r_{y(f^1)}, \ldots, r_{x(f^2)}, r_{y(f^2)}, \ldots, r_{x(f^\ell)}, r_{y(f^\ell)}),
\]

and if \( t = f \in F_0 \) we let

\[
\pi_t := (r_{x(f)}, \ldots, r_{x(f^1)}, r_{y(f^1)}, \ldots, r_{x(f^2)}, r_{y(f^2)}, \ldots, r_{x(f^\ell)}, r_{y(f^\ell)}).
\]

We transfer these definitions to \( H^* \) as follows:

- We define \( R^*_t \subseteq H^* \) by \( V(R^*_t) := \{a(v) \mid v \in V(R_t)\} \) and \( E(R^*_t) := \{a(v)a(w) \mid \exists v'w' \in E(R_t) : a(v') = a(v) \text{ and } a(w') = a(w)\} \setminus E(*)H^*_t\).

- To define \( \pi^*_t \), we consider the graphs \( A_0(r_i) \cap D_t \) for \( i \in [n] \). Let \( Q_1, \ldots, Q_k \) be an enumeration of these graphs. By Claim 2 either \( k = 1 \) and \( Q_1 \) is a cycle and \( bd(D_t) = Q_1 \), or \( Q_1, \ldots, Q_k \) are paths (possibly of length 0) embedded in \( bd(D_t) \). For each \( i \in [k] \), let \( j_i \) be the minimum \( j \) such that \( r_j \in V(Q_i) \). Without loss of generality we may assume that \( j_1 < j_2 < \ldots < j_k \). We let

\[
\pi^*_i := (a_0(r_{j_1}), \ldots, a_0(r_{j_k})).
\]
Claim 3. \((H^*_t, \pi^*, R^*_t, \tau^*_t)\) is an arrangement of \(H^*\) in \(D_t\).

Proof. We have already noted that (AD.1) holds when we defined \(\pi^*\). Axioms (AD.2) (AD.4) are obviously satisfied. It follows from Claim 2 that we can view \(H^*_0\) as a graph embedded in \(D_t\) and that we can contract the edges in \(F_0\) in such a way that \(H^*_0 \cap Bd(D_t) = \tau^*_t\) and that the vertices of \(\tau^*_t\) appear in cyclic order on \(\text{bd}(D_t)\). This proves (AD.5). \(\blacksquare\)

It remains to define a path decomposition \((P^*, \beta^*_P)\) of \(R^*_t\) of width \(3p - 1\). We assume that \(N^T_{\pm}(t) = \{f_1, \ldots, f_{\ell}\}\) with \(x(f_1) < y(f_1) \leq x(f_2) < y(f_2) \leq x(f_3) < \ldots \leq x(f_{\ell}) < y(f_{\ell})\). We let \(P^* := P_t\).

- If \(t = \perp\), for \(i \in V(P_t)\) we let
  \[
  \beta^*_P(i) := \begin{cases} \{a(v) \mid v \in \beta_P(i)\} & \text{if } i \neq i_x(f_j) \text{ for all } j \in [\ell], \\ \{a(v) \mid v \in \beta_P(i_x(f_j)) \cup \beta_P(i_y(f_j))\} & \text{if } i = i_x(f_j) \text{ for some } j \in [\ell]. \end{cases}
  \]

- If \(t = f \in F\), for \(i \in V(P_t)\) we let
  \[
  \beta^*_P(i) := \begin{cases} \{a(v) \mid v \in \beta_P(i) \cup \beta_P(i_y(f_j))\} & \text{if } i \neq i_x(f_j) \text{ for all } j \in [\ell], \\ \{a(v) \mid v \in \beta_P(i_x(f_j)) \cup \beta_P(i_y(f_j)) \cup \beta_P(i_y(f))\} & \text{if } i = i_x(f_j) \text{ for some } j \in [\ell]. \end{cases}
  \]

\(\blacksquare\)

Corollary 13.5.9. \(\mathcal{M}(\text{AP}_p) \subseteq \mathcal{T}(\text{AP}_{3p}).\)

13.5.3 Proof of Theorem 13.5.1

Lemma 13.5.10. Let \((G_0, \pi, R, \tau, F)\) be a \(p\)-\(m\)-arrangement of a graph \(G^*\) in a closed disk \(D\), and let \(G := \pi(G_0) \cup R\). Let \(v \in V(G_0)\) be \((30p + 91)\)-central in \(G_0\). Then \(\pi(v)\) is geometrically \((6p, 30p + 90)\)-central in \(G^*\).

Proof. We apply Lemma 13.5.8 and obtain a tree decomposition \((T^*, \beta^*)\) of \(G^*\).

Let \(p_1 := 6p\) and \(p_1^* := 5p_1 + 15 = 30p + 90\). Let \(C_1, \ldots, C_{p_1^* + 1} \subseteq G_0\) be cycles such that \(u\) is \((p_1^* + 1)\)-central within these cycles. Then \(C_1, \ldots, C_{p_1^*} \subseteq G \setminus \tau = G \setminus R\). Thus none of the edges in \(F\) have an endvertex in any of the cycles \(C_i\) for \(i \in [p_1^*]\), and the cycles have an empty intersection with \(R\). As \(\sigma(t) \subseteq V(R)\) for all \(t \in V(T^*)\) by Lemma 13.5.8 iv-a, there is a node \(t \in V(T^*)\) such that \(C_1, \ldots, C_{p_1^*} \subseteq \tau^*(t)\). Let \((G_{10}, \pi_t, R_t, \tau_t)\) be a \(3p\)-arrangement in the disk \(D_t\) that satisfies Lemma 13.5.8 iv-a–c. By iv-b) and iv-c) we have \(C_1, \ldots, C_{p_1^*} \subseteq G_{10}\). We also have \(v \in V(G_{10})\), because \(C_1\) separates \(v\) from \(V(R)\), and \(v\) is \(p_1^*\)-central within these cycles because the embeddings of \(G_{10}\) and \(G_0\) coincide on \(G_{10} \setminus \tau_t\). Let \(R' := G[V(R_t) \cup \bigcup_{u \in V(T^*) \setminus \{t\}} \beta^*(u)]\). Then \((G_{10}, \pi_t, R', \tau_t)\) is an arrangement of \(G\) in \(D_t\). As \((R_t, \tau_t)\) is a \(3p\)-ring, it is a \(p_1\)-vortex, and this implies that \((R', \tau_t)\) is a \(p_1\)-vortex as well. The vertex \(v\) is \(p_1^*\)-central in \(G_{10}\) and thus contained in \(g\)-\(\text{Cen}_{p_1, p_1^*}(G_{10}, \pi_t, R', \tau_t) \subseteq g\)-\(\text{Cen}_{p_1, p_1^*}(G)\). \(\blacksquare\)

Lemma 13.5.11. There is a constant \(c_4(p)\) such that for all \(G^* \in \mathcal{Z}_3 \cap \mathcal{M}(\text{AP}_p)\) it holds that
\[
\text{tw}\left(\text{Skel}_{6p}(G^*)\right) \leq c_4(p).
\]
Proof. Let \( p_1 := 6p \) and \( p_1^* := 30p + 90 \). Let \( G^* \in \mathcal{Z}_3 \cap \mathcal{M}(\mathcal{AP}_p) \). Then by Lemma 13.5.6 \( G^* \in \mathcal{MAP}_p \). Let \((G_0, \pi, R, \bar{r}, F)\) be a \( p-m \)-arrangement of \( G^* \) in a closed disk \( D \), and let \( G := \pi(G_0) \cup R \). By Lemma 13.5.10 we have \( g\text{-}\text{Cent}_{p_1^*+1}(G_0, \pi, R, \bar{r}) \subseteq g\text{-}\text{Cent}_{p_1^*}(G^*) \) and thus \( g\text{-}\text{Skel}_{p_1^*}(G^*) \subseteq g\text{-}\text{Skel}_{p_1^*+1}(G_0, \pi, R, \bar{r}) \). Here we use that all edges in \( F \) have both endvertices in \( \bar{r} \) and therefore are disjoint from \( V(g\text{-}\text{Cent}_{p_1^*+1}(G_0, \pi, R, \bar{r})) \). By Corollary 13.4.10 we have \( \text{Skel}_{p_1}(G^*) \subseteq g\text{-}\text{Skel}_{p_1^*}(G^*) \).

We let \( c_4(p) := c_1(p) \cdot (p_1^* + 1) \), where \( c_1 \) is chosen according to Lemma 13.4.2. Then

\[
\begin{align*}
tw(\text{Skel}_{p_1}(G^*)) & \leq tw(g\text{-}\text{Skel}_{p_1^*}(G^*)) \\
& \leq tw(g\text{-}\text{Skel}_{p_1^*+1}(G_0, \pi, R, \bar{r})) \\
& \leq c_4(p) \quad \text{(by Lemma 13.4.2)}.
\end{align*}
\]

Proof of Theorem 13.5.1. By the 3CC Lifting Lemma (Corollary 8.3.3), it suffices to prove that the class \( \mathcal{Z}_3 \cap \mathcal{M}(\mathcal{AP}_p) \) admits \( \exists \text{FP} \)-definable ordered treelike decompositions. This follows from Lemma 13.5.11 and Corollary 13.5.3.
In the inductive proof of the Definable Structure Theorem \([17.2.1]\), we will have to extend partial decompositions. Each extension step will be formalised as the completion of a pre-decomposition, where the cones of the node of the pre-decomposition are the parts of the graph that we have already decomposed.

In this chapter, we describe one particular completion step. Suppose we have a pre-decomposition \(\Phi = (V, \sigma, \alpha)\) of a graph \(G\). Furthermore, suppose that there is a subset \(U \subseteq V\) of the node set such that after removing the components \(\alpha(u)\) of all nodes \(u \in U\) from \(G\), we obtain an almost planar graph. We will say that \(\Phi\) has an “almost planar star completion” in \(G\); the nodes in \(U\) are the ground nodes of this completion. We shall prove that if \(\Phi\) is \(\text{IFP}\)-definable and has an almost planar star completion, then it has an \(\text{IFP}\)-definable ordered completion. This may sound like a relatively straightforward generalisation of the Definable Structure Theorem for Almost Planar Graphs \([13.5.1]\), but turns out to be fairly complicated, because the almost planar star completion of \(\Phi\) is not assumed to be \(\text{IFP}\)-definable, and in particular we do not know which nodes of \(\Phi\) form the set \(U\) of ground nodes of the star completion.

14.1 From Almost Planar to Ordered Completions

A star is a directed tree of height at most 1. We call the root of the star the centre and if the height is 1 (that is, the star is nontrivial), we call the leaves tips. A star decomposition of a graph \(G\) is a tree decomposition \((S, \beta)\) of \(G\) where \(S\) is a star.

**Definition 14.1.1.** Let \(p \in \mathbb{N}\), and let \(G\) be a graph. An \(\text{AP}_p\)-star decomposition of \(G\) is a tuple

\[(S, \beta, s, D, H_0, \pi, R, \tau)\]

satisfying the following conditions.

(i) \((S, \beta)\) is a star decomposition of \(G\), and \(s\) is the centre of \(S\).

(ii) \(D\) is a closed disk and \((H_0, \pi, R, \tau)\) a \(p\)-arrangement of the torso \(\tau(s)\) in \(D\).

(iii) For all tips \(t \in V(S) \setminus \{s\},\)

a. either \(K[\sigma(t)] \subseteq R\),
b. or there is a subgraph $C_t \subseteq H_0$ such that $K[\sigma(t)] = \pi(C_t)$ and $C_t \cong K_i$ for some $i \in [3]$, and if $C_t \cong K_3$ then $C_t$ is a facial cycle of $H_0$.

If (iii-a) holds for all $t \in V(S) \setminus \{s\}$, then $(S, \beta, s, D, H_0, \pi, R, \tau)$ is a simple $\mathcal{AP}_{p}$-star decomposition of $G$.

Let $(S, \beta, s, D, H_0, \pi, R, \tau)$ be a simple $\mathcal{AP}_{p}$-star decomposition of a graph $G$. Let

$$R' := (R \cap G) \cup \bigcup_{t \in V(S) \setminus \{s\}} G[\beta(t)].$$

It is easy to see that $(R', \tau)$ is a 2p-vortex and thus $(H_0, \pi, R', \tau)$ is a local $p$-arrangement of $G$ in $D$. Also note that the adhesion of $(S, \beta)$ is at most $p$, because for every $t \in V(S) \setminus \{s\}$ the separator $\sigma(t)$ is a clique in $R$ and thus contained in a bag of every path decomposition of $R$. Since $R$ has a path decomposition of width at most $p - 1$, it follows that $|\sigma(t)| \leq p$. If the $\mathcal{AP}_{p}$-star decomposition $(S, \beta, s, D, H_0, \pi, R, \tau)$ is not simple, then $(S, \beta)$ has adhesion at most $\max\{p, 3\}$.

**Definition 14.1.2.** Let $p \in \mathbb{N}$, and let $\Phi$ be a pre-decomposition of a graph $G$. Then an $\mathcal{AP}_{p}$-star completion of $\Phi$ is an $\mathcal{AP}_{p}$-star decomposition $(S, \beta, s, D, H_0, \pi, R, \tau)$ of $G$ such that for all tips $t$ of $S$ there is a $t' \in V(\Phi)$ such that $t \parallel (S, \beta, \Phi, t')$.

Recall the definition of the $(d, d')$-derivation of a pre-decomposition from Section 12.4.

The goal of this chapter is to prove the following completion theorem.

**Theorem 14.1.3 (Almost Planar Completion Theorem).** Let $\Psi$ be a pd-scheme and $p, d, d' \in \mathbb{N}$ with $d \geq 10 \max\{p, 3\} + 3$ and $d' \geq 4 \max\{p, 3\}$. Then there exists an od-scheme $\Lambda$ such that for every graph $G$ the following holds: if $\Psi[G]$ is a tight pre-decomposition of $G$ that has an $\mathcal{AP}_{p}$-star completion, then $\Lambda[G]$ is an ordered completion of the $(d, d')$-derivation of $\Psi[G]$.

The proof will be given in Section 14.6. The idea is as follows. We use the Q4C Completion Lemma 12.6.2 to obtain a reduction to quasi-4-connected graphs. Then we observe that $\mathcal{AP}_{p}$-star decompositions of quasi-4-connected graphs can always be turned into simple $\mathcal{AP}_{p}$-star decompositions. Hence we may assume that $\Psi[G]$ has a simple $\mathcal{AP}_{p}$-star completion. Now we want to follow the lines of the proof of Theorem 13.5.1. We contract the centre of our graph. The skeleton will not have bounded tree width, but a bounded width completion of the “pre-decomposition induced by $\Phi$ on the skeleton”. Hence instead of Theorem 6.1.1 we can use the Bounded Width Completion Lemma 12.3.2 to define such a completion. The main problem with this approach is to define the “pre-decomposition induced by $\Psi[G]$ on the skeleton” in the right way. By contracting edges, we may turn a tight pre-decomposition into one that is no longer tight. But tightness of the pre-decomposition is a crucial assumption of the Bounded Width Completion Lemma. The solution to this problem will be to only contract a part of the centre, the supercentre, and to make sure that the relevant nodes of the pre-decomposition $\Psi[G]$ are not affected by this contraction. To define the supercentre, we have to re-work parts of the structure theory for almost planar graphs.

### 14.2 Grids

For every $k \in \mathbb{N}^+$, the *elementary hexagonal grid* $\Gamma_k$ of radius $k$ is defined as shown in Figure 14.1. Since all grids considered in this book are hexagonal, in the following we omit
the qualifier “hexagonal”. A grid of radius $k$ is a graph isomorphic to a subdivision of $\Gamma_k$, and a grid of radius $k$ in a graph $G$ is a subgraph $\Gamma \subseteq G$ that is a grid of radius $k$. Since grids are graphs of maximum degree 3, there is a grid of radius $k$ in a graph $G$ if and only if $\Gamma_k$ is a minor of $G$. (Here we use the simple fact that a graph $H$ of maximum degree 3 is a minor of a graph $G$ if and only if $H$ is a topological subgraph of $G$.)

Let $\Gamma$ be a grid of radius $k \geq 1$. Then $\Gamma$ is a subdivision of a $3$-connected planar graph. Hence up to homeomorphism it has a unique embedding into the sphere, which coincides with the natural embedding shown in Figure 14.1. The facial cycles of $\Gamma$ are the facial cycles of this embedding. In the following, let us assume that $k \geq 2$. Then there is a unique facial cycle of $\Gamma$ that contains more than 6 branch vertices of $\Gamma$; we call it the perimeter of $\Gamma$ and denote it by $\text{Per}(\Gamma)$. We call all other facial cycles of $\Gamma$ the hexagons of $\Gamma$. We call the graph $\Gamma \setminus \text{Per}(\Gamma)$ the interior of $\Gamma$ and denote it by $\text{Int}(\Gamma)$. Note that the interior of $\Gamma_k$ is isomorphic to a subdivision of $\Gamma_{k-1}$. However, in general the interior of a grid of radius $k$ is not a grid of radius $k - 1$, because there may be vertices on subdivided edges from the perimeter to the interior. The central subgrid of a grid $\Gamma$ of radius $k$ is the (unique) grid of radius $k - 1$ contained in $\text{Int}(\Gamma)$ as a subgraph. For $j \in [0, k - 1]$, we inductively define the $j$th central subgrid of a grid $\Gamma$ of radius $k$ as follows. We let $\Gamma$ be its own 0th central subgrid, and for $j \in [0, k - 2]$ we let the $(j + 1)$st central subgrid be the central subgrid of the $j$th central subgrid (see Figure 14.2(a)). Note that the $j$th central subgrid of $\Gamma_k$ is a grid of radius $k - j$. For $j \in [0, k - 2]$, the $j$th central cycle of $\Gamma$ is the perimeter of the $j$th central subgrid. Furthermore, the $(k - 1)$th central cycle of $\Gamma$ is the “exterior” facial cycle of the $(k - 1)$th central subgrid $\Gamma^{k-1}$, that is, the facial cycle $C^{k-1}$ of $\Gamma^{k-1}$ such that $\partial V(\Gamma_{k-1}) \subseteq V(C^{k-1})$. We also call $C^{k-1}$ the perimeter of $\Gamma^{k-1}$ with respect to $\Gamma$, and we call $\Gamma^{k-1} \setminus C^{k-1}$ the interior of $\Gamma^{k-1}$ with respect to $\Gamma$. Finally, we call the unique branch vertex in $V(\Gamma^{k-1}) \setminus C^{k-1}$ the central vertex of the grid $\Gamma$. The subgrid of radius $j$ around some branch vertex $v$ of $\Gamma$ consists of the $j$ concentric layers of hexagons surrounding $v$ (see Figure 14.2(b)). Only vertices of the $j$th central subgrid of $\Gamma$ have a subgrid of radius $j$ around them. Observe that the $(k - j)$th central subgrid of $\Gamma$ is the subgrid of radius $j$ around the centre of $\Gamma$. We use the term subgrid only for subgrids around some branch vertex and not for other subgraphs that happen to be
Chapter 14. Almost Planar Completions

Figure 14.2. A grid of a radius 5 with (a) its 3rd central subgrid and (b) an arbitrary subgrid of radius 2

Now let $\Gamma$ be a grid of radius $k \geq 2$ in a graph $G$. The compass of $\Gamma$ in $G$ is the union of $\text{Per}(\Gamma)$ with the unique $\text{Per}(\Gamma)$-bridge that contains a vertex of $\Gamma$, or equivalently, the union of $\Gamma$ with all $\Gamma$-bridges that have at least one vertex of attachment in the interior of $\Gamma$. We denote the compass of $\Gamma$ in $G$ by $\text{Com}^G(\Gamma)$ or just $\text{Com}(\Gamma)$ if $G$ is clear from the context. A vertex $v \in V(G)$ is $j$-central in $\Gamma$, for some $j \in [k]$, if it is either a vertex of the interior of the $(j-1)$th central subgrid of $\Gamma$ or a vertex of a connected component of $G \setminus \Gamma$ that has at least one vertex of attachment in the interior of the $(j-1)$th central subgrid. An edge $e \in E(G)$ is $j$-central in $\Gamma$ if at least one of its endvertices is $j$-central in $\Gamma$. The centre of $\Gamma$ is the set of all vertices of $G$ that are $k$-central in $\Gamma$. Observe that if $G$ is a planar graph and $v \in V(G)$ is $j$-central in a grid $\Gamma \subseteq G$, then $v$ is $j$-central in any embedding of $G$ in the sphere (in the sense of Definition 13.2.1).

A grid $\Gamma \subseteq G$ is plane in $G$ if $\text{Com}(\Gamma)$ is a planar graph. It is not hard to show that if $G$ is 3-connected and $\Gamma$ is plane in $G$, then up to homeomorphism, $\text{Com}(\Gamma)$ has a unique embedding into the sphere which embeds every connected component of $\text{Com}(\Gamma) \setminus \Gamma$ into one of the hexagonal faces of $\Gamma$. Hence every connected component of $\text{Com}(\Gamma) \setminus \Gamma$ has all its vertices of attachment in one hexagon of $\Gamma$.

Our hexagonal grids are essentially the same as the walls considered by Robertson and Seymour in [110], except that they have a slightly different shape, and we force them to be 3-regular by omitting vertices of degree 2 on the boundary. Figure 14.3 shows a wall of height 6, and it illustrates how a grid can be embedded into a wall and vice versa. It is easy to see that a wall of height $k$ can be embedded into a grid of radius $k$ (this is not a tight bound). Using the well-known fact that a wall of height $k$ has tree width $k+1$ we obtain the following lower bound on the tree width of the grids:

$$\text{tw}(\Gamma_k) > k.$$  \hspace{1cm}  (14.2.1)

It follows that every graph $G$ that contains a grid of radius $k$ has tree width at least $k+1$. The next theorem, which is one of the cornerstones of graph minor theory, provides an approximate converse of this fact.

M. Grohe, Definable Graph Structure Theory
Fact 14.2.1 (Excluded Grid Theorem; Robertson and Seymour [107]). For every $q \in \mathbb{N}$ there is a $\ell_1(q) \in \mathbb{N}$ such that for all graphs $G$, if $\text{tw}(G) \geq \ell_1(q)$ then $G$ contains a grid of radius $q$ as a subgraph.

We also need the following algorithmic version of the Excluded Grid Theorem.

Fact 14.2.2 (Robertson and Seymour [110]). For all $q \in \mathbb{N}$ there is an $\ell_2(q) \in \mathbb{N}$ and a quadratic time algorithm that, given a graph $G$, produces one of the following two outputs:

$(i)$ a tree decomposition of $G$ of width at most $\ell_2(q)$;

$(ii)$ a grid of radius $q$ in $G$.

Using results from [14, 98], the running time of the algorithm can actually be improved to linear.

Related to the fact that grids have relatively large tree width — proportional to the radius, which in turn is proportional to the square root of the order of the grid — is the fact that small separators can only split off small pieces of a grid. A convenient way to phrase this is the following isoperimetric inequality.

Fact 14.2.3. There is a $d_1 \in \mathbb{N}$ such that for every elementary grid $\Gamma$ and every subset $W \subseteq V(\Gamma)$ with $|W| \leq V(\Gamma)/2$, we have $|W| \leq d_1 \cdot |N^\Gamma(W)|^2$.

For a proof and an estimate on the constant $d_1$, see [58].

14.2.1 Grids in Almost Planar Graphs

The following Lemma 14.2.4 states that large grids in almost planar graphs contain large plane subgrids. Lemma 14.2.5 is a generalisation of this result to graphs with a simple $\mathcal{A}(\mathcal{P}_p)$-star decomposition. As in Chapter 13, we use the following convention: If $(G_0, \pi, R, \tau)$ is an arrangement of a graph $G$ in a disk $\mathcal{D}$, then by $\Pi$ we denote the embedding of $G \setminus R$ into $\mathcal{D}$ that is induced by $\pi^{-1}$. 
Lemma 14.2.4. For all \( p, q \in \mathbb{N}, r \in \mathbb{N}^+ \) there is a \( q_1(p, q, r) \in \mathbb{N} \) such that the following holds: Let \( G \in \mathcal{AP}_p \), and let \((G_0, \pi, R, \tau)\) be a \( p \)-arrangement of \( G \) in a disk \( D \). Furthermore, let \( \Gamma' \) be a grid of radius \( q_1(p, q, r) \) in \( G \). Then there is a subgrid \( \Gamma \subseteq \Gamma' \setminus R \) of radius \( q \) such that \( \Pi(\text{Com}^G(\Gamma)) \) is \( r \)-central in \( G_0 \).

Proof. Without loss of generality we assume that \( \pi \) is the identity on \( V(G_0) \setminus \tau \). Let

\[
q_1 := q_1(p, q, r) := c_1(p) \cdot (q + r) + q,
\]

where \( c_1(p) \) is the constant from Lemma 13.4.2. Let \( G' := \Gamma' \cup R \) and \( G'_0 \subseteq G_0 \) be the subgraph with vertex set \( \pi^{-1}(V(\Gamma') \setminus V(R)) \cup \tau \) and edge set \( \{vw \in E(G_0) \mid v, w \in V(G'_0), \pi(v)\pi(w) \in E(\Gamma')\} \). Let \( \pi' \) be the restriction of \( \pi \) to \( V(G'_0) \). Then \((G'_0, \pi', R, \pi)\) is a \( p \)-arrangement of \( G' \) in \( D \). Let \( \Gamma'_q \) be the \( q \)th central subgrid of \( \Gamma' \). Then \( \Gamma'_q \) is a grid of radius \( q_1 - q = c_1(p) \cdot (q + r) + q \). Hence \( \text{tw}(\Gamma'_q') > c_1(p) \cdot (q + r) \). By Lemma 13.4.2, the geometric \((q + r)\)-skeleton of \((G'_0, \pi', R, \pi)\) has tree width at most \( c_1(p) \cdot (q + r) \). Thus \( \Gamma'_q \) has a nonempty intersection with the geometric \((q + r)\)-centre of \( G' \). Let \( v \) be a vertex in the intersection of \( \Gamma'_q \) with the geometric \((q + r)\)-centre, and let \( \Gamma \) be the subgrid of \( \Gamma' \) of radius \( q \) with centre \( v \). Then \( \Gamma \) is \( r \)-central in \( G'_0 \).

Moreover, as \( G_0 \supseteq G'_0 \) is a plane graph, \( \text{Com}^G(\Gamma) \) is \( r \)-central in \( G_0 \).

In the following lemma, \( d_1 \) is the constant of Fact 14.2.3.

Lemma 14.2.5. For all \( p, q, r \in \mathbb{N} \) there is a \( q_2(p, q, r) \in \mathbb{N} \) such that the following holds. Let \( G \) be a graph and \((S, \beta, s, D_H, H_0, \pi, R, \tau)\) a simple \( \mathcal{AP}_p \)-star decomposition of \( G \). Furthermore, let \( \Gamma' \) be a grid of radius \( q_2(p, q, r) \) in \( G \). Then either there is a \( t \in V(S) \setminus \{s\} \) such that all but at most \( d_1 \cdot p^2 \) branch vertices of \( \Gamma \) are in \( \gamma(t) \) or there is a subgrid \( \Gamma \subseteq \Gamma' \) of radius \( q \) such that \( \Pi(\text{Com}^G(\Gamma)) \) is \( r \)-central in \( G_0 \).

Proof. \((P, \beta_P)\) be a path decomposition of \( R \) of width at most \((p - 1)\) such that \( \tau \) is increasing in \((P, \beta_P)\). Let \( t \in V(S) \setminus \{s\} \). Then \( K[\sigma(t)] \subseteq R \) and thus there is a \( u \in V(P) \) such that \( \sigma(t) \subseteq \beta_P(u) \). For every \( u \in V(P) \), let \( T(u) \) be the set of all \( t \in V(S) \setminus \{s\} \) with \( \sigma(t) \subseteq \beta_P(u) \), and let

\[
\alpha(u) := \bigcup_{t \in T(u)} \alpha(t).
\]

Note that \( \beta_P(u) \) separates the sets \( \alpha(t) \) for \( t \in T(u) \) from one another and from the set \( V(G) \setminus (\beta_P(u) \cup \alpha(u)) \).

Now let \( \Gamma' \) be a grid in \( G \) of sufficiently large radius (to be determined in the proof). The first condition on the radius is that it is large enough so that \( |V(\Gamma')| \geq 2d_1 \cdot p^2 \). Let \( u \in V(P) \). As \( |\beta_P(u)| \leq p \), it follows from Fact 14.2.3 that either there is a \( t \in T(u) \) such that all but at most \( d_1 \cdot p^2 \) branch vertices of \( \Gamma' \) are contained in \( \gamma(t) \), or at most \( d_1 \cdot p^2 \) branch vertices of \( \Gamma' \) are contained in \( \alpha(u) \). In the former case, there is nothing to prove, so assume that for all \( u \in V(P) \) at most at most \( d_1 \cdot p^2 \) branch vertices of \( \Gamma' \) are contained in \( \alpha(u) \).

For every \( u \in V(P) \), let \( \beta'_P(u) \) be the union of \( \beta_P(u) \) with all branch vertices of \( \Gamma' \) that are contained in \( \alpha(u) \). Let

\[
R' := \bigcup_{u \in V(P)} K[\beta'_P(u)].
\]

Then \((P, \beta'_P)\) is a path decomposition of \( R' \) of width at most \( p' := d_1 \cdot p^2 + p - 1 \). Let \( G' := R' \cup \pi(H_0) \). Then \((H_0, \pi, R', \tau)\) a \( p' \)-arrangement of \( G' \) in the disk \( D \). There is a grid \( \Gamma'' \subseteq G' \) that has the same radius and the same branch vertices as \( \Gamma' \) and coincides with \( \Gamma' \) on

M. Grohe, Definable Graph Structure Theory
\( \pi(H_0) \). Provided the radius of \( \Gamma' \) and thus of \( \Gamma'' \) is sufficiently large, by Lemma 14.2.4 there is a subgrid \( \Gamma \subseteq \Gamma'' \) of radius \( q \) such that \( \Pi(\text{Com}^G(\Gamma)) \) is \( r \)-central in \( H_0 \). As \( \Gamma \subseteq \pi(H_0) \) and \( \Gamma' \) and \( \Gamma'' \) coincide on \( \pi(H_0) \), we have \( \Gamma \subseteq \Gamma' \). \( \square \)

The following lemma, which is significantly deeper than the previous ones, shows that 

\textit{every} vertex of an almost planar graph that is in the centre of a sufficiently large grid must be contained in the embedded part of every arrangement of the graph in a disk.

**Lemma 14.2.6.** For all \( p \in \mathbb{N} \) there is a \( q_3(p) \in \mathbb{N} \) such that the following holds. Let \( G \) be a connected graph in \( \mathcal{A}_p \setminus \mathcal{P} \), and let \((G_0, \pi, R, \bar{\pi})\) be a \( p \)-arrangement of \( G \) in a disk \( D \). Furthermore, let \( \Gamma \) be a plane grid in \( G \), and let \( v \in V(G) \) be \( q_3(p) \)-central in \( \Gamma \). Then \( v \not\in V(R) \).

At first sight, this lemma may sound surprising. One would think that the vortex, that is, the nonplanar region of an almost planar graph, can be everywhere. Yet the lemma says that it cannot touch the centre of a large plane grid. But actually, the intuition behind this result is simple. If the whole vortex was in the centre of a large plane grid, then the vortex and a sufficiently large part surrounding it would be planar, and that would suffice to make the whole graph planar. But we assume that the graph is nonplanar. If, on the other hand, the vortex would range from the centre of the grid to the perimeter, or close to the perimeter, then many (more than \( p \)) of the central cycles of the grid would go through the vortex, which contradicts the defining property of a \( p \)-vortex.

**Proof of Lemma 14.2.6.** Let

\[
q_1 := q_1(p, p + 2, 1), \quad q_3 := q_3(p) := 2q_1 + 3p - 4,
\]

where \( q_1 \) is chosen according to Lemma 14.2.4. Let \( G \) be a connected graph in \( \mathcal{A}_p \setminus \mathcal{P} \), and let \((G_0, \pi, R, \bar{\pi})\) be a \( p \)-arrangement of \( G \) in a disk \( D \). Without loss of generality we may assume that \( \pi \) is the identity on \( V(G_0) \setminus \bar{\pi} \). Let \( \Gamma \) be a plane grid in \( G \), and let \( v \in V(G) \) be \( q_3 \)-central in \( \Gamma \). Let \( q \geq q_3 \) be the radius of \( \Gamma \). For all \( i \in [q] \), let \( \Gamma^i \) be the \( i \)th central subgrid of \( \Gamma \), let \( C^i \) be the \( i \)th central cycle, and let \( I^i := \text{Com}^G(\Gamma^i) \). Let \( J^i := I^i \setminus C^i \), and note that \( J^i \) is connected. Let \( X_1, \ldots, X_{m_i} \) be the “external” connected components of \( G \setminus \Gamma^i \), that is, those components that have no vertex of attachment in \( \text{Int}(\Gamma^i) \), and let \( X^i := \bigcup_{j=1}^{m_i} X^i_j \).

Suppose for contradiction that \( v \in V(R) \).

**Claim 1.** There is a vertex \( r_1 \in \pi(\bar{\pi}) \) that is \( (q_3 - p + 1) \)-central in \( \Gamma \).

**Proof.** Since \( J^{q_3} \) is connected, there is a path \( P \subseteq J^{q_3} \) from \( v \) to a branch vertex \( w \) of \( \Gamma \). If the path \( P \) contains a vertex \( r \in \pi(\bar{\pi}) \), then this vertex \( r \) is \( q_3 \)-central in \( \Gamma \), and we let \( r_1 := r \). So let us assume that \( V(P) \cap \pi(\bar{\pi}) = \emptyset \). Then \( w \in V(R) \). Consider the subgrid \( \Gamma' \) of \( \Gamma \) of radius \( p - 1 \) around \( w \). As \( \text{tw}(\Gamma') > p - 1 \) and \( \text{tw}(R) \leq p - 1 \), there is a vertex \( w' \in V(\Gamma') \setminus V(R) \). Let \( P' \subseteq \Gamma' \) be a path from \( w \) to \( w' \). Then \( V(P') \cap \pi(\bar{\pi}) \neq \emptyset \), and we let \( r_1 \) be a vertex in \( V(P') \cap \pi(\bar{\pi}) \).

**Claim 2.** \( X^1 \cap R \neq \emptyset \).

**Proof.** Let \( G^1 := G/X^1 \). Without loss of generality we may assume that \( X^1_1 \) is the connected component of \( G \setminus \Gamma^1 \) that contains the perimeter of \( \Gamma \). Let \( X_1 := X^1_1 \cap I^0 \). Note that \( X^1_1 \subseteq I^0 \).
for \( i \geq 2 \), and let \( X_i := X^1_i \). It is easy to see that \( G^1 = G^0 / X^1_1 / X^1_2 / \ldots / X^1_m \). As \( \Gamma \) is a plane grid, the graph \( G^0 \) is planar, and thus \( G^1 \preceq G^0 \) is a planar as well.

Now we can argue as in the proof of Claim 1 in the proof of Lemma 13.3.5 to show that if \( X^1 \cap R = \emptyset \) and hence \( X^1 \) is planar, then \( G \) is planar, which contradicts our assumption that \( G \notin \mathcal{P} \).

\( \textbf{Claim 3.} \) There is a vertex \( r_X \in \pi(\tilde{r}) \) that is not \((2p - 2)\)-central in \( \Gamma \).

\( \textbf{Proof.} \) Let \( x \in V(X^1 \cap R) \), and let \( P \) be a path from \( x \) to a branch vertex \( y \) of \( \Gamma \) on the cycle \( C^{p-1} \) such that all internal vertices of \( P \) are in \( V(X^{p-1}) \cup V(C^{p-1}) \). If the path \( P \) contains a vertex \( r \in \pi(\tilde{r}) \), then this vertex \( r \) is not \( p - 1 \)-central in \( \Gamma \), and we let \( r_X := r \). So let us assume that \( V(P) \cap \pi(\tilde{r}) = \emptyset \). Then \( y \in V(R) \). Consider the subgrid \( \Gamma' \) of \( \Gamma \) of radius \( p - 1 \) around \( y \). As \( tw(\Gamma') > p - 1 \) and \( tw(R) \leq p - 1 \), there is a vertex \( y' \in V(\Gamma') \setminus V(R) \). Let \( P' \subseteq \Gamma' \) be a path from \( y \) to \( y' \). Then \( V(P') \cap \pi(\tilde{r}) \neq \emptyset \), and we let \( r_X \) be a vertex in \( V(P') \cap \pi(\tilde{r}) \).

In the following, we fix \( r_I, r_X \in \pi(\tilde{r}) \) such that \( r_I \) is \((q_3 - p + 1)\)-central in \( \Gamma \) and \( r_X \) is not \((2p - 2)\)-central. Consider the “cylinder”

\[
Z := \Gamma^{2p - 2} \setminus \text{Int}(\Gamma^{q_1 - p + 1})
\]

and the cycles \( C^{2p - 2}, \ldots, C^{q_3 - p + 1} \) it contains. All these cycles separate \( r_I \) from \( r_X \). In the next, crucial step of the proof we show that at least three of these cycles have an empty intersection with \( R \) and hence with the \( G \)-normal curve \( \text{bd}(D) \), which connects \( r_I \) and \( r_X \). This will lead to a contradiction.

Let \( v' \) be a vertex on the “middle cycle” \( C^{2p - 2 + q_1} \) of \( Z \), and let \( \Gamma' \) be the subgrid of \( \Gamma \) of radius \( q_1 \) around \( v' \). Then \( \Gamma' \subseteq Z \). By Lemma 14.2.4 there is a subgrid \( \Gamma'' \) of \( \Gamma \) of radius \( p + 2 \) such that \( \Gamma'' \cap R = \emptyset \). Let \( v'' \) be the centre of \( \Gamma'' \), and let \( j'' \in [2p - q_3 - 2p + 1] \) such that \( v'' \in V(C^{j''}) \). Then each of the \( 2p + 5 \) cycles \( C^{j'' - 2 - p}, C^{j'' - p - 1}, \ldots, C^{j'' - 1}, C^{j''}, C^{j'' + 1}, \ldots, C^{j'' + p + 2} \) has a nonempty intersection with \( \Gamma'' \), and each of the \( 2p + 3 \) cycles

\[
C^{j'' - p - 1}, \ldots, C^{j'' - 1}, C^{j''}, C^{j'' + 1}, \ldots, C^{j'' + p + 1}
\]

is incident with at least two adjacent hexagons of \( \Gamma'' \).

\( \textbf{Claim 4.} \) At least three of the cycles \( C^{j'' - p - 1}, \ldots, C^{j'' - 1}, C^{j''}, C^{j'' + 1}, \ldots, C^{j'' + p + 1} \) have an empty intersection with \( R \).

\( \textbf{Proof.} \) Suppose for contradiction that at least \( 2p + 1 \) of the cycles have a nonempty intersection with \( R \). Let \( C_1, \ldots, C_{2p + 1} \) be these cycles, listed in the same order as in

\[
C^{j'' - p - 1}, \ldots, C^{j'' - 1}, C^{j''}, C^{j'' + 1}, \ldots, C^{j'' + p}.
\]

We fix an orientation of the disk \( D \). For all \( i \in [2p + 1] \), let \( v_i \in V(C_i \cap \Gamma'') \). Let \( x_i \) be the first vertex on \( C_i \) in \( V(R) \) walking from \( v_i \) in clockwise direction, and let \( y_i \) be the first vertex on \( C_i \) in \( V(R) \) walking from \( v_i \) in anti-clockwise direction. Then \( x_i, y_i \in \pi(\tilde{r}) \) for all \( i \in [2p + 1] \). Let \( Q_i \) be the segment of \( C_i \) from \( x_i \) to \( y_i \) through \( v_i \). Then there is a path \( Q'_i \subseteq G_0 \) such that \( \pi(Q'_i) = Q_i \). Let \( x'_i \in \pi^{-1}(x_i) \) and \( y'_i \in \pi^{-1}(y_i) \) be the endvertices of \( Q_i \) and remember that \( Q'_i \setminus \{x'_i, y'_i\} = Q_i \setminus \{x_i, y_i\} \), because we assumed \( \pi \) to be the identity on \( V(G_0) \setminus \tilde{r} \). The paths \( Q'_1, \ldots, Q'_{2p + 1} \) are mutually disjoint and their endvertices appear on \( \text{bd}(D) \) in the following cyclic order:

\[
x'_1, x'_2, \ldots, x'_{2p + 1}, y'_2, \ldots, y'_1.
\]
This follows from the fact that the grid $\Gamma''$ has a unique embedding into the sphere, because it is a subdivision of a 3-connected graph. (See Figure 14.4 for an illustration.) Say, $x'_1 = r_{i_1}$ and $x'_{2p+1} = r_{i_2}$, where without loss of generality we assume that $1 \leq i_1 < i_2 \leq n$. As the cycles $C_1, \ldots, C_{2p+1}$ are mutually disjoint, they all contain a segment in $R$ from a vertex $\pi(r_j)$ with $j \in [i_1, i_2]$ to a vertex $\pi(r_{j'})$ with $j' \in [1, i_1] \cup [i_2, n]$. Thus there are $(2p + 1)$ mutually disjoint paths in $R$ from $\{\pi(r_{i_1}), \ldots, \pi(r_{i_2})\}$ to $\{\pi(r_1), \ldots, \pi(r_{i_2})\} \cup \{\pi(r_{i_2}), \ldots, \pi(r_n)\}$. This contradicts Lemma 13.1.2.

Let $i_1, i_2, i_2 \in [j'' - p - 1, j'' + p + 1]$ with $i_1 < i_2 < i_3$ such that $C^{i_j} \cap R = \emptyset$ for $j \in [3]$. To simplify the notation, let $C_j := C^{i_j}$. It is easy to see that there are $P_1, P_2, P_3 \subseteq \Gamma''$ such that for $j \in [3]$, the endvertices of $P_j$ are on $C_1$ and $C_3$, and $P_j$ is internally disjoint from $C_1$ and $C_3$, and $P_j \cap C_2$ is a path (see Figure 14.5). Note that $P_j \cap R = \emptyset$ because $P_j \subseteq \Gamma''$. Let

$$Y := C_1 \cup C_2 \cup C_3 \cup P_1 \cup P_2 \cup P_3.$$  

Then $Y$ is a subdivision of a 3-connected graph and thus has a unique embedding in the sphere, which puts $C_2$ between $C_1$ and $C_3$. Furthermore, $Y \cap R = \emptyset$. Let $S \supseteq \mathcal{D}$ be a sphere. We view $Y \subseteq G_0$ as a graph embedded in $S$. As $bd(D)$ is $G$-normal and $Y \cap R = \emptyset$, we have $bd(D) \cap Y = \emptyset$. Thus there is a face $f$ of $Y$ such that $bd(D) \subseteq f$. Let $P_X \subseteq G$ be a path from $r_X$ to a vertex $v_1 \in V(C_1)$ that is internally disjoint from $Y$. Such a path exists because $r_X$ is in the “exterior” of $C_1$. Let $r'_X$ be the last vertex of $P_X$ in $\pi(\tilde{r})$ and $P'_X \subseteq G_0$ such that $\pi(P'_X) = r'_X P_X v_1$. Then $P'_X$ is a path from a vertex in $bd(D)$ to $C_1$ that is internally disjoint from $Y$. Thus $P'_X \subseteq cl(f)$, which implies $bd(f) \cap C_1 \neq \emptyset$. Similarly, there is a path $P_I \subseteq G$ from $r_I$ to $C_3$ that is internally disjoint from $Y$, and thus $bd(f) \cap C_3 \neq \emptyset$. However, this is impossible because $C_1$ and $C_3$ belong to different arcwise connected components of $S \setminus C_2$.

**Lemma 14.2.7.** For all $p \in \mathbb{N}$ there is a $q_4(p) \in \mathbb{N}$ such that the following holds. Let $G$ be a nonplanar graph and $(S, \beta, s, \mathcal{D}, H_0, \pi, R, \tau)$ a simple $AP_p$-star decomposition of $G$. Let
\( \Gamma \subseteq G \) be a plane grid in \( G \), and let \( v \in V(G) \) be \( q_4(p) \)-central in \( \Gamma \). Then if \( v \in \gamma(t) \) for some \( t \in V(S) \setminus \{s\} \), all but at most \( d_1 \cdot p^2 \) branch vertices of \( \Gamma \) are in \( \gamma(t') \) for some \( t' \in V(S) \setminus \{s\} \).

**Proof.** Let \( H := \tau(s) \). Suppose for contradiction that \( v \in \gamma(t) \) for some \( t \in V(S) \setminus \{s\} \) and that for all \( t' \in V(S) \setminus \{s\} \), more than \( d_1 \cdot p^2 \) branch vertices of \( \Gamma \) are in \( V(G) \setminus \gamma(t) \). We can argue as in the proof of Lemma 14.2.6 using Lemma 14.2.5 instead of Lemma 14.2.4: Starting from \( v \), we find a vertex \( r_I \in \pi(\tilde{r}) \) that is sufficiently central in \( \Gamma \). Since \( \Gamma \) is plane grid and \( G \) is not planar, we find an \( r_X \in \tilde{r} \) that is not very central in \( \Gamma \). Then by Lemma 14.2.5 we find a subgrid \( \Gamma' \) of \( \Gamma \) radius \( (p + 2) \) that is 1-central in \( G_0 \). Observe that

\[
G \left[ V(R) \cup \bigcup_{t \in V(S) \setminus \{s\}} \gamma(t) \right]
\]

is a 2p-vortex. Using this observation, we can complete the proof as the proof of Lemma 14.2.6.

**Definition 14.2.8.** An \( \mathcal{AP}_p \)-star decomposition \((S, \beta, s, D, H_0, \pi, R, \tau)\) of a graph \( G \) is \( \ell \)-wide, for some \( \ell \in \mathbb{N} \), if the torso \( \tau(s) \) has tree width at least \( \ell \).

Let \( G \) be a graph and \( \ell \in \mathbb{N} \). Let \((S, \beta, s, D, H_0, \pi, R, \tau)\) be a sufficiently wide simple \( \mathcal{AP}_p \)-star decomposition of \( G \). Let \( H := \tau(s) \). Without loss of generality we may assume that \( \pi \) is the identity on \( V(H_0) \setminus \tilde{r} \). Since \( H \) has large tree width, by the Excluded Grid Theorem, it contains a large grid \( \Gamma' \). By Lemma 14.2.4 this grid \( \Gamma' \) has a large subgrid \( \Gamma \) whose compass is 1-central in \( H_0 \). As \( H_0 \) is planar, \( \text{Com}(\Gamma) \subseteq H_0 \) is planar as well, and thus \( \Gamma \) is plane in \( G \). Thus graphs with a sufficiently wide simple \( \mathcal{AP}_p \)-star decomposition contain large plane grids. Our next lemma states that such a plane grid can be computed in polynomial time.

To prove the lemma, we make use of a another deep result due to Robertson and Seymour.
A matching of a set $S$ is a matching of the complete graph $K[S]$, that is, a set
\[ \{x_1y_1, x_2y_2, \ldots, x_ly_l\} \subseteq \binom{S}{2} \]
such that $x_1, y_1, x_2, y_2, \ldots, x_l, y_l$ are pairwise distinct. Let $p \in [n]$, and let $G, G'$ be graphs and $\pi = (v_1, \ldots, v_p) \in V(G)^p$, $\pi' = (v_1', \ldots, v_p') \in V(G')^p$. Then we say that $(G, \pi)$ and $(G', \pi')$ have the same folio if for all matchings $\{i_1j_1, \ldots, i_lj_l\}$ of $[p]$, there is a family of mutually disjoint paths $P_1, \ldots, P_l \subseteq G$ such that the endvertices of $P_k$ are $v_{i_k}$ and $v_{j_k}$ if and only if there is a family of mutually disjoint paths $P_1', \ldots, P_l' \subseteq G'$ such that the endvertices of $P_k'$ are $v_{i_k}'$ and $v_{j_k}'$. Our notion of folio is actually a special case of a more general notion introduced in [110].

**Fact 14.2.9 (Robertson and Seymour [110]).** For all $n \in \mathbb{N}$ there is an $\ell_3(p) \in \mathbb{N}$ and a cubic time algorithm that, given an graph $G$ and a tuple $\pi \in V(G)^p$ of vertices, computes a subgraph $G' \subseteq G$ satisfying the following conditions:

(i) $\pi \subseteq V(G')$;
(ii) $(G, \pi)$ and $(G', \pi)$ have the same folio;
(iii) $\text{tw}(G') \leq \ell_3(p)$.

The fact is not explicitly stated in this form in [110], but the algorithm of Fact 14.2.9 is just a variation of Algorithm (10.5) of [110].

**Lemma 14.2.10.** For all $p, q$ there is an $\ell_4(p, q) \in \mathbb{N}$ and a polynomial time algorithm that, given a graph $G$ that has an $\ell_4(p, q)$-wide simple $\mathcal{AP}_p$-star decomposition, computes a grid $\Gamma \subseteq G$ of radius $q$ that is plane in $G$.

**Proof.** We let
\[
\ell_3 := \ell_3(p),
q_2 := q_2(p, q, 1),
q_2' := \max\{q_2, \ell_3 + d_1 \cdot p^2\},
q_3 := \ell_2(q_2'),
q_4 := q_2(p, q, 3, 1),
\ell_4 := \ell_4(p, q) := \ell_1(q_4),
\]
where $d_1, \ell_1, \ell_2, \ell_3, q_2$ are chosen according to Fact 14.2.3, the Excluded Grid Theorem (Fact 14.2.1, Fact 14.2.2, Fact 14.2.9, and Lemma 14.2.5) respectively. To explain the algorithm, we fix a graph $G$ and an $\ell_4$-wide simple $\mathcal{AP}_p$-star decomposition $(S, \beta, s, D, H_0, \pi, R, \tau, F)$ of $G$. Let $H := \tau(s)$.

**Claim 1.** There is a grid $\Gamma \subseteq G$ of radius $q_3$ such that $\text{Com}^G(\Gamma)$ is 1-central in $H_0$.

**Proof.** By the Excluded Grid Theorem, $H$ contains a grid $\Gamma'$ of radius $q_4$. Thus by Lemma 14.2.5, $\Gamma'$ contains a grid $\Gamma$ of radius $q_3$ such that $\text{Com}^G(\Gamma)$ is 1-central in $H_0$.

**Claim 2.** Let $W \subseteq V(G)$ and $\pi = (x_1, \ldots, x_p) \in V(G)^p$ with $W \cap \overline{x} = \emptyset$. Let $A \subseteq G \setminus \overline{x}$ with $N^G(A) \subseteq \overline{x}$ and $V(A) \cap W = \emptyset$. Let $B := G[V(A) \cup \overline{A}]$, and let $B' \subseteq B$
with \( \bar{x} \subseteq V(B') \) such that \((B, \bar{x})\) and \((B', \bar{x})\) have the same folio. Finally, let \( G' \) be the subgraph of \( G \) obtained by replacing \( B \) by \( B' \), that is, \( V(G') := (V(G) \setminus V(B)) \cup V(B') \) and \( E(G') := (E(G) \setminus E(B)) \cup E(B') \).

Then \( G \) contains a grid whose branchvertices are precisely the vertices in \( W \) if and only if \( G' \) contains such a grid.

**Proof.** For the forward direction, suppose that \( \Gamma \subseteq G \) is a grid such that \( W \) is the set of branchvertices of \( \Gamma \). As \( W \cap V(B) = \emptyset \), the subgraph \( \Gamma \cap B \) is a union of mutually disjoint paths \( P_1, \ldots, P_\ell \) (possibly empty, that is, \( \ell = 0 \)) with both endvertices in \( \bar{x} \). As \((B, \bar{x})\) and \((B', \bar{x})\) have the same folio, there is a family \( P'_1, \ldots, P'_\ell \subseteq B' \) of mutually disjoint paths such that \( P_i \) and \( P'_i \) have the same endvertices. Let \( \Gamma' \) be the grid obtained from \( \Gamma \) by replacing \( P_i \) by \( P'_i \) for all \( i \in [\ell] \). Then \( \Gamma' \) is a grid in \( G' \) with the same branchvertices as \( \Gamma \). This proves the forward direction. The backward direction is trivial because \( G' \subseteq G \).

Our algorithm proceeds in the following steps.

**Step 1.** “Guess” a set \( W \subseteq V(G) \) of \( O(q_3^2) \) vertices that are intended to be the branch vertices of a grid \( \Gamma \) of radius \( q_3 \) such that \( \text{Com}^G(\Gamma) \) is 1-central in \( H_0 \). (This nondeterministic guess can be simulated by a deterministic algorithm, increasing the running time by a (polynomial) factor of \( |G|^{O(q_3^2)} \).)

In the following steps, we assume that the algorithm guessed correctly, that is, \( W \) is the set of branch vertices of a grid \( \Gamma_W \subseteq G \) of radius \( q_3 \) such that \( \text{Com}^G(\Gamma_W) \) is 1-central in \( H_0 \). If there is no such grid, the algorithm may fail and not compute a plane grid of radius \( q \). Note that by Claim 1, there is at least one correct guess if the input graph \( G \) satisfies the pre-conditions of the algorithm.

**Step 2.** Let \( G' := G \).

While there is a \( \bar{x} \in V(G')^p \) with \( \bar{x} \cap W = \emptyset \) and a subgraph \( A \subseteq G' \setminus \bar{x} \) with \( N^{G'}(A) \subseteq \bar{x} \) and \( V(A) \cap W = \emptyset \) such that the graph \( B := G'[V(A) \cup \bar{x}] \) has tree width greater than \( \ell_3 \), use the algorithm of Fact [14.2.9] to compute a \( B' \subseteq B \) such that \( \bar{x} \in V(B') \) and \( (B, \bar{x}), (B', \bar{x}) \) have the same folio and \( \text{tw}(B') \leq \ell_3 \). Replace \( B \) by \( B' \) in \( G' \).

After the loop in Step 2 is executed, the resulting \( G' \) has the following two properties.

(i) \( G \) contains a grid whose branchvertices are precisely the vertices in \( W \) if and only if \( G' \) contains such a grid (this follows from Claim 2).

(ii) For all \( \bar{x} \in V(G')^p \) with \( \bar{x} \cap W = \emptyset \) and all subgraphs \( A \subseteq G' \setminus \bar{x} \) with \( N^{G'}(A) \subseteq \bar{x} \) and \( V(A) \cap W = \emptyset \) the graph \( B := G'[V(A) \cup \bar{x}] \) has tree width at most \( \ell_3 \).

Note that by our assumption that \( G \) contains a grid of radius \( q_3 \) with branch vertices in \( W \), the graph \( G' \) contains such a grid. Thus by [14.2.1], its tree width is greater than \( q_3 = \ell_2(q_2) \).

**Step 3.** Use the algorithmic version of the Excluded Grid Theorem (Fact [14.2.2]) to compute a grid \( \Gamma' \subseteq G' \) of radius \( q_2 \).

**Step 4.** Test if \( \Gamma' \) contains a subgrid of radius \( q \) that is plane in \( G \). If this is the case, output such a subgrid and halt. Otherwise, report failure.

Suppose for contradiction that the algorithm fails. Note that \( \Gamma' \) is a grid of radius \( q'_2 \geq q_2 = q_2(p, q, 1) \). Suppose first that there is a subgrid \( \Gamma \subseteq \Gamma' \) of radius \( q \) such that \( \text{Com}^G(\Gamma) \) is
1-central in $H_0$. Then $\Gamma$ is plane in $G$, because $H_0$ is a planar graph. This contradicts failure of the algorithm in Step 4. Hence by Lemma 14.2.5, there is a $t \in V(S) \setminus \{s\}$ such that all but at most $d_1 \cdot p^2$ branch vertices of $\Gamma'$ are in $\gamma(t)$. As $\Gamma$ is a grid of radius $q_2' \geq \ell_3 + p^2$ and thus has tree width greater than $\ell_3 + d_1 \cdot p^2$, it follows that $\Gamma \cap G[\gamma(t)]$ has tree width greater than $\ell_3$. Let $\bar{x} \in V(G')^p$ such that $(\sigma(t) \cap V(G')) \subseteq \bar{x}$ and $\bar{x} \cap W = \emptyset$. Such an $\bar{x}$ exists because $W \cap V(R) = \emptyset$ and $\sigma(t) \subseteq V(R)$. Let $A$ be the union of all connected components of $G' \setminus \bar{x}$ that have a nonempty intersection with $\Gamma \cap G[\gamma(t)]$, and let $B := G[V(A) \cup \bar{x}]$. Then $B$ has tree width greater than $\ell_3$. This contradicts (ii).\[\square\]

We actually need the following strengthening of Lemma 14.2.10.

**Lemma 14.2.11.** For all $p, q$ there is a polynomial time algorithm that, given a graph $G$ and a set $P \subseteq V(G)$ satisfying condition (i) below, computes a grid $\Gamma \subseteq G$ satisfying (ii)–(iv).

**Condition on input $G, P$:**

(i) $G$ has a simple $\mathcal{AP}_p$-star decomposition $(S, \beta, s, D, H_0, \pi, R, \tau)$ such that

$$\text{tw} \left( \tau(s)/(\tau(s)[P \cap \beta(s)]) \right) \geq \ell_4(p, q),$$

where $\ell_4(p, q)$ is chosen according to Lemma 14.2.10.

**Properties of output $\Gamma$:**

(ii) the radius of $\Gamma$ is $q$;

(iii) $\Gamma$ is plane in $G$;

(iv) for each connected component $A$ of $G[P]$ the intersection $A \cap \Gamma$ is connected and contains at most one branch vertex of $\Gamma$.

The pre-condition (i) corresponds to the wideness condition in Lemma 14.2.10. Intuitively, property (iv) just says that in the minor $G/G[P]$ the grid $\Gamma$ contracts to a grid of the same radius as $\Gamma$.

**Proof.** The proof is a fairly straightforward generalisation of the proof of Lemma 14.2.10. We give the details for the sake of completeness. We define $\ell_3, q_2, q_2', q_3, q_4, \ell_4 = \ell_4(p, q)$ exactly as in the proof of Lemma 14.2.10. To explain the algorithm, we fix a graph $G$ and a simple $\mathcal{AP}_p$-star decomposition $(S, \beta, s, D, H_0, \pi, R, \tau)$ of $G$ satisfying (i). Let $H := \tau(s)$. Without loss of generality we assume that $\pi$ is the identity on $H_0 \setminus \tau$.

Let $G^* := G/G[P]$ and $H^* := H/H\{P \cap V(H)\}$. By (i), we have $\text{tw}(H^*) \geq \ell_4(p, q)$. A grid $\Gamma$ in $G$ is **good** if it satisfies (iv). Similarly, a grid $\Gamma$ in $H$ is **good** if for each connected component $A$ of $H\{P \cap V(H)\}$ the intersection $A \cap \Gamma$ is connected and contains at most one branch vertex of $\Gamma$. Note that for every grid $\Gamma^* \subseteq G^*$ there is a good grid $\Gamma \subseteq G$ of the same radius as $\Gamma^*$ such that $\Gamma^* = \Gamma/\Gamma\{P \cap V(\Gamma)\}$. The analogous statement holds for good grids in $H^*$. Note further that if $A$ is a connected subgraph of $G[P]$ then $A_H := H[V(A) \cap V(H)]$ is connected as well. This follows from the definition of $H$ as a torso. Hence if $\Gamma \subseteq G \cap H$ is good width respect to $H$ it is also good with respect to $G$.

**Claim 1.** There is a good grid $\Gamma \subseteq G$ of radius $q_3$ such that $\text{Com}^G(\Gamma)$ is 1-central in $H_0$.\[\square\]
Claim 2. Let $W^* \subseteq V(G^*)$ and $\overline{x} = (x_1, \ldots, x_p) \in V(G^*)^p$ with $W^* \cap \overline{x} = \emptyset$. Let $A \subseteq G^* \setminus \overline{x}$ with $N^{G^*}(A) \subseteq \overline{x}$ and $V(A) \cap W^* = \emptyset$. Let $B := G^*[V(A) \cup \overline{x}]$, and let $B' \subseteq B$ with $\overline{x} \subseteq V(B')$ such that $(B, \overline{x})$ and $(B', \overline{x})$ have the same folio. Finally, let $G'$ be the subgraph of $G^*$ obtained by replacing $B$ by $B'$, that is, $V(G') := (V(G^*) \setminus V(B)) \cup V(B')$ and $E(G') := (E(G^*) \setminus E(B)) \cup E(B')$.

Then $G'$ contains a grid whose branch vertices are precisely the vertices in $W^*$ if and only if $G'$ contains such a grid.

Proof. Similar to the proof of Claim 2 in Lemma 14.2.10.

Our algorithm proceeds in the following steps.

Step 1. “Guess” a set $W \subseteq V(G)$ of $O(q_3^2)$ vertices such that every connected component of $G[P]$ contains at most 1 vertex in $W$. Let $W^* \subseteq V(G^*)$ be the set of vertices of $G^*$ corresponding to $W$.

The vertices in $W$ are intended to be the branch vertices of a good grid $\Gamma \subseteq G$ of radius $q_3$ such that $Com^G(\Gamma)$ is 1-central in $H_0$. In the following steps, we assume that the algorithm guessed correctly, that is, $W$ is the set of branch vertices of a good grid $\Gamma_W \subseteq G$ of radius $q_3$ such that $Com^G(\Gamma_W)$ is 1-central in $H_0$, and we let $\Gamma^*_W \subseteq G^*$ be the corresponding grid with branch vertices $W^*$ in $G^*$.

Step 2. Let $G^{**} := G^*$.

While there is a $\overline{x} \in V(G^{**})^p$ with $\overline{x} \cap W^* = \emptyset$ and a subgraph $A \subseteq G^{**} \setminus \overline{x}$ with $N^{G^{**}}(A) \subseteq \overline{x}$ and $V(A) \cap W^* = \emptyset$ such that the graph $B := G^{**}[V(A) \cup \overline{x}]$ has tree width greater than $\ell_3$, use the algorithm of Fact 14.2.9 to compute a $B' \subseteq B$ such that $\overline{x} \in V(B'\pi)$ and $(B, \overline{x}), (B', \overline{x})$ have the same folio and $tw(B') \leq \ell_3(p)$. Replace $B$ by $B'$ in $G^{**}$.

After the loop in Step 2 is executed, the resulting $G^{**}$ has the following two properties:

(i) $G^*$ contains a grid whose branch vertices are precisely the vertices in $W^*$ if and only if $G^{**}$ contains such a grid;

(ii) for all $\overline{x} \in V(G^{**})^p$ with $\overline{x} \cap W^* = \emptyset$ and all subgraphs $A \subseteq G^{**} \setminus \overline{x}$ with $N^{G^{**}}(A) \subseteq \overline{x}$ and $V(A) \cap W^* = \emptyset$ the graph $B := G^{**}[V(A) \cup \overline{x}]$ has tree width at most $\ell_3$.

Note that by our assumption that $\Gamma^*_W$ is a grid in $G^*$ with branch vertices $W^*$, the graph $G^{**}$ contains a grid of radius $q_3$ with branch vertices in $W^*$. Thus by (14.2.1), its tree width is greater than $q_3 = \ell_2(q_2')$.

Step 3. Use the algorithmic version of the Excluded Grid Theorem (Fact 14.2.2) to compute a grid $\Gamma^{***} \subseteq G^{**}$ of radius $q_3'$ in $G^{**}$.

The grid $\Gamma^{***} \subseteq G^{**} \subseteq G^*$ corresponds to a good grid $\Gamma' \subseteq G$ such that $(\Gamma')^* = \Gamma'$.

Step 3a. Compute a a good grid $\Gamma' \subseteq G$ such that $(\Gamma')^* = \Gamma^{**}$.

M. Grohe, Definable Graph Structure Theory
Step 4. Test if \( \Gamma' \) contains a subgrid of radius \( q \) that is plane in \( G \). If this is the case, output such a subgrid and halt. Otherwise, report failure.

Suppose for contradiction that the algorithm fails. Note that \( \Gamma' \) is a grid of radius \( q'_2 \geq q_2 = q_2(p,q,1) \). Suppose first that there is a subgrid \( \Gamma \subseteq \Gamma' \) of radius \( q \) such that \( \text{Com}^\gamma(\Gamma) \) is 1-central in \( H_0 \). Then \( \Gamma \) is plane in \( G \), because \( H_0 \) is a planar graph. This contradicts failure of the algorithm in Step 4. Hence by Lemma 14.2.5, there is a \( t \in V(S) \setminus \{s\} \) such that all but \( p^2 \) branch vertices of \( \Gamma' \) are in \( \gamma(t) \). Let \( S^* \subseteq V(G^*) \) be the set of vertices of \( G^* \) corresponding to the vertices in \( \sigma(t) \). Let \( \pi \in V(G^{**})^p \) such that \( (S^* \cap V(G^{**})) \subseteq \pi \) and \( \pi \cap W^* = \emptyset \). Let \( \Gamma^* \subseteq G^* \) be the grid corresponding to \( \Gamma \). Such a grid exists, because \( \Gamma \subseteq \Gamma' \) is good. Then \( \Gamma^* \) is a grid of radius \( q'_2 \geq \ell_3 + p^2 \) and thus has tree width greater than \( \ell_3 + p^2 \). As all but \( p^2 \) vertices of \( \Gamma \) are in \( G[\gamma(t)] \), all but \( p^2 \) vertices of \( \Gamma^* \) are in the subgraph \( C^* \subseteq G^* \) corresponding to \( G[\gamma(t)] \). Hence \( \text{tw}(C^* \cap \Gamma^*) > \ell_3 \). As \( \pi \) separates \( C^* \cap \Gamma^* \) from \( W^* \), this contradicts (ii).

A corollary of Lemma 14.2.10 and the Immerman-Vardi Theorem 3.1.5 states that in ordered graphs that have sufficiently wide simple \( \text{AP}_p \)-star decompositions, large plane grids are definable. We only state the corresponding corollary of Lemma 14.2.11. Remember that an ordered graph is an \( \{E, \leq\} \)-structure \( G = (V(G), E(G), \leq^G) \) where \( (V(G), E(G)) \) is a graph and \( \leq^G \) is a linear order of \( V(G) \). In the following two corollaries, when we say that an ordered graph \( G \) has a \( \text{AP}_p \)-star decomposition or contains a grid \( \Gamma \), we always refer to the underlying graph \( (V(G), E(G)) \). Let \( X \) be a unary relation variable.

Corollary 14.2.12. For all \( p, q \in \mathbb{N} \) there is an \( \text{IFP} \{\{E, \leq\}, \{E\}\} \)-transduction \( \Theta(X) \) such that for every ordered graph \( G \) and every subset \( P \subseteq V(G) \) satisfying condition (i) of Lemma 14.2.11, \( \Gamma := \Theta[G, P] \) is a grid satisfying (ii)–(iv).

Lemma 14.2.13. For all \( p, q \in \mathbb{N} \) there is an \( \text{IFP} \{\{E, \leq\}, \{E\}\} \)-formula \( \text{sc}_{p,q}(X,x) \) such that for every ordered graph \( G \) and every set \( P \subseteq V(G) \):

(i) for all \( v \in V(G) \), if \( G \models \text{sc}_{p,q}[P,v] \) there is a grid \( \Gamma \subseteq G \) that is plane in \( G \), and \( v \) is \( q \)-central in \( \Gamma \);

(ii) if \( G \) has a simple \( \text{AP}_p \)-star decomposition \( (S, \beta, s, D, H_0, \pi, R, \tau) \) such that

\[
\text{tw} \left( \tau(s)/\left(\tau(s)[P \cap \beta(s)]\right) \right) \geq \ell_4(p, q + 2),
\]

where \( \ell_4(p, q + 2) \) is chosen according to Lemma 14.2.10, then there is a \( v \in V(G) \setminus P \) such that \( G \models \text{sc}_{p,q}[P,v] \).

Furthermore,

(iii) there is an \( \text{IFP} \{\{E, \leq\}, \{E\}\} \)-transduction \( \Theta(X,x) \) such that for every ordered graph \( G \), every subset \( P \subseteq V(G) \), and every vertex \( v \in V(G) \) such that \( G \models \text{sc}_{p,q}[P,v] \) it holds that \( \Theta[G, P, v] \) is a grid that is plane in \( G \) and \( v \) is \( q \)-central in this grid.

Proof. The trick is to apply Corollary 14.2.12 with \( p, q + 2 \) to define a grid of radius \( q + 2 \) and then let \( \text{sc}_{p,q}(X,x) \) define all \( q \)-central vertices of this grid. There is at least one vertex in \( V(G) \setminus P \) that is \( q \)-central in such a grid of radius \( q + 2 \), because the vertices in \( V(G)\setminus P \) correspond to the connected components of \( G[P] \), form an independent set in \( G/G[P] \). □
14.3 Supercentre and Superskeleton

Let \( p, q \in \mathbb{N} \), and let \( G \) be a 3-connected graph. Let \( A \) be a connected component of \( \text{Cen}_p(G) \), the definable \( p \)-centre of \( G \). Remember the formula \( \text{ord-cent}_p(\tau, y_1, y_2) \) of Lemma 13.4.12 that defines a linear order on \( V(A) \) with parameters. For every tuple \( \tau \in G^2 \) such that \( \leq_{\text{ord-cent}_p}(G, \tau, y_1, y_2) \) is a linear order of \( V(A) \), we define a sequence \( S_i(A, \tau) \), for \( i \in \mathbb{N} \), of subsets of \( V(A) \) inductively as follows. We let

\[
S_0(A, \tau) := \emptyset.
\]
\[
S_{i+1}(A, \tau) := S_i(A, \tau) \cup \text{sc}_{p,q}([A, \leq_{\text{ord-cent}_p}(A, \tau)], S_i(A, \tau), x],
\]

where \( \text{sc}_{p,q}(X, x) \) is the formula from Lemma 14.2.13.

**Definition 14.3.1.** Let \( p, q \in \mathbb{N} \), and let \( G \) be a 3-connected graph.

1. Let \( S := \bigcup_{A, \tau, i} S_i(A, \tau) \), where \( S_i(A, \tau) \) is defined as in (14.3.1) and (14.3.2) and the union ranges over all connected components \( A \) of \( \text{Cen}_p(G) \), all tuples \( \tau \in G^2 \) such that \( \leq_{\text{ord-cent}_p}(G, \tau, y_1, y_2) \) is a linear order of \( A \), and all \( i \in \mathbb{N} \).

The \((p, q)\)-supercentre of \( G \) is the subgraph \( \text{SCen}_{p,q}(G) := G[S] \).

2. The \((p, q)\)-superskeleton of \( G \) is the minor \( \text{SSkel}_{p,q}(G) := G/\text{SCen}_{p,q}(G) \).

**Lemma 14.3.2.** For all \( p, q \in \mathbb{N} \) there is an \( \text{IFP} \)-formula \( \text{scen}_{p,q}(x) \) such that for every 3-connected graph \( G \),

\[
\text{scen}_{p,q}[G, x] = V(\text{SCen}_{p,q}(G)).
\]

**Proof.** Follows immediately from the definitions.

**Lemma 14.3.3.** For all \( p, q \in \mathbb{N} \) there is an \( \text{IFP} \)-graph transduction \( \Theta(x) \) such that for every 3-connected graph \( G \) and every vertex \( v \in V(\text{SCen}_{p,q}(G)) \) it holds that \( \Gamma_v := \Theta[G, v] \) is a grid that is plane in \( G \) and \( v \) is \( q \)-central in \( \Gamma_v \).

We call the grid \( \Gamma_v \) the **definable witness** for \( v \in V(\text{SCen}_{p,q}(G)) \).

**Proof.** Follows from Lemma 14.2.13(iii).

Let \( G \) be a graph and \((S, \beta, s, D, H_0, \pi, R, \tau)\) a \( \mathcal{AP}_p \)-star decomposition of \( G \). Let \( H := \tau(s) \). Let \( A_1, \ldots, A_m \) be the connected components of \( \text{SCen}_{p,q}(G) \). For all \( i \in [m] \), let \( A_i' := H[V(A_i) \cap V(H)] \), and note that \( A_i' \) is connected. Let \( H^* := H/A_1'/\cdots/A_m' \). We call \( H^* \) the restriction of the \((p, q)\)-superskeleton to \( H \).

**Lemma 14.3.4.** For all \( p, q \in \mathbb{N} \) there is an \( \ell_5(p, q) \in \mathbb{N} \) such that the following holds. Let \( G \) be a 3-connected graph and \((S, \beta, s, D, H_0, \pi, R, \tau)\) a simple \( \mathcal{AP}_p \)-star decomposition of \( G \). Let \( H := \tau(s) \), and let \( H^* \) be the restriction of the \((p, q)\)-superskeleton to \( H \). Then

\[
\text{tw}(H^*) \leq \ell_5(p, q).
\]

**Proof.** Let

\[
\ell_4 := \ell_4(p, q + 1),
\]
\[
p^* := 5 p + 15,
\]

M. Grohe, *Definable Graph Structure Theory*
\[ q_1 := q_1(p, \ell_4, p^*), \]
\[ \ell_5 := \ell_5(p, q) := \ell_1(q_1), \]

where the functions \( \ell_4, q_1, \ell_1 \) are chosen according to Lemma \[ \text{[14.2.10]} \] and the Excluded Grid Theorem (Fact \[ \text{[14.2.1]} \]). Let \((H_0, \pi, R, \tau)\) be a \( p \)-arrangement of \( H \) in a disk \( D \) such that for all \( t \in V(S) \setminus \{s\} \) we have \( K[\sigma(t)] \subseteq R \). Let \( A_1, \ldots, A_m \) be the connected components of \( SCen_{p,q}(G) \), and for all \( i \in [m] \), let \( A'_i := H[V(A_i) \cap V(H)] \).

Let \( G' := Cen_{p}(G) \). For all \( t \in V(S) \), let \( \beta'(t) := \beta(t) \cap V(G') \). Then \((S, \beta')\) is a star decomposition of \( G' \). Let \( H' := \tau'(s) \). The \( p \)-arrangement \((H_0, \pi, R, \tau)\) of \( H \) induces a \( p \)-arrangement \((H'_0, \pi', R', \tau')\) of \( H' \) in the obvious way. Hence \((S, \beta', D, H'_0, \pi', R', \tau')\) is a simple \( AP_{p}\)-star decomposition of \( G' \). Remember that \( SCen_{p,q}(G) \subseteq G' \) by definition, and let \( G'' := G'/SCen_{p,q}(G) \) and \( H'' := H'/A'_1/\cdots/A'_m \). It follows from Lemma \[ \text{[14.2.13]} \] and the definition of the \((p, q)\)-supercentre that \( \text{tw}(H'') < \ell_4 \).

Suppose for contradiction that \( \text{tw}(H'') > \ell_5 \). Then by the Excluded Grid Theorem, there is a grid \( \Gamma^*_1 \subseteq H'' \) of radius \( q_1 \). As \( H'' = H'/A'_1/\cdots/A'_m \), there is a grid \( \Gamma_1 \subseteq H'' \) such that for all \( i \in [m] \) the intersection \( A'_i \cap \Gamma_1 \) is connected and contains at most one branchvertex of \( \Gamma_1 \). By Lemma \[ \text{[14.2.4]} \] there is a subgrid \( \Gamma_2 \subseteq \Gamma_1 \) of radius \( \ell_4 \) that is \( p^*\)-central in \( H_0 \). By Lemma \[ \text{[13.4.9]} \] we have \( \Gamma_2 \subseteq H' \). Let
\[ \Gamma_2^* := \Gamma_2/(A'_1 \cap \Gamma_2)/\cdots/(A'_m \cap \Gamma_2) \]

and note that \( \Gamma_2^* \subseteq H'' \) is a grid of the same radius \( \ell_4 \). Thus \( \text{tw}(H'') > \ell_4 \), which is a contradiction. \( \square \)

### 14.4 The Completion Theorem for Quasi-4-Connected Graphs

In this section, we prove the Almost Planar Completion Theorem for quasi-4-connected graphs (Lemma \[ \text{[14.4.2]} \]). Quasi-4-connected graphs are much easier to deal with than general graphs, because their \( AP_{p}\)-star decompositions can easily be transformed into simple \( AP_{p}\)-star decompositions. The following lemma and its proof make this precise.

**Lemma 14.4.1.** Let \( G \) be a quasi-4-connected graph and \( \Phi \) a pre-decomposition of \( G \) that has an \( AP_{p}\)-star completion. Then \( \Phi \) has a simple \( AP_{p}\)-star completion.

**Proof.** Let \((S, \beta, s, D, H_0, \pi, R, \tau)\) be an \( AP_{p}\)-star completion of \( \Phi \) in \( G \) with the minimum number of tips \( t \) of \( S \) violating the “simplicity condition” Definition \[ \text{[14.1.1]} \] (iii-a). I claim that \((S, \beta, s, D, H_0, \pi, R, \tau)\) is a simple \( AP_{p}\)-star completion of \( \Phi \).

Suppose for contradiction that \( t \) is a tip of \( S \) that violates Definition \[ \text{[14.1.1]} \] (iii-a). Then \( t \) satisfies (iii-b). We have \( |\sigma(t)| = 3 \), because \( G \) is 3-connected. Furthermore, \( |\alpha(t)| \leq 1 \), because \( G \) is quasi-4-connected. Actually, \( |\alpha(t)| = 1 \), because otherwise, we can just delete the tip \( t \) from the decomposition, and it remains an \( AP_{p}\)-star completion of \( \Phi \), which contradicts the minimality condition on \((S, \beta)\). Let \( v_t \) be the unique vertex in \( \alpha(t) \). We delete the tip \( t \) from \( S \) and add \( v_t \) to the bag of the centre \( s \). Let \((S', \beta')\) be the resulting decomposition of \( G \). Let \( H'_t := \tau'(s) \). We let \( H'_0 \) be the graph obtained from \( H_0 \) by adding \( v_t \) and edges from \( v_t \) to the vertices \( v \) with \( \pi(v) \in \sigma(t) \). Recall that these vertices lie on a facial cycle \( C \) of \( H_0 \) with \( \pi(V(C)) = \sigma(t) \). We extend the embedding of \( H_0 \) in the disk \( D \) to an embedding of \( H'_t \) by embedding \( v \) in the interior of the face bounded by the cycle \( C \). We define a mapping \( \pi' : V(G'_0) \to V(G) \) by letting \( \pi'(v_t) := v_t \) and \( \pi'(v) := \pi(v) \) for all \( v \in V(H_0) \). Then \((H'_0, \pi', R, \tau)\) is a \( p \)-arrangement of \( H' \) in \( D \), and \((S', \beta', D, H'_0, \pi', R, \tau)\) is an \( AP_{p}\)-star completion of \( \Phi \).
that has fewer tips violating Definition 14.1.1(iii-a) than \((S, \beta, s, D, H_0, \pi, R, \tau)\). This is a contradiction.

\[\square\]

**Lemma 14.4.2 (Almost Planar Completion for Quasi-4-Connected Graphs).** Let \(\Psi\) be a pd-scheme and \(p \in \mathbb{N}\). Then there exists an od-scheme \(\Lambda\) such that for every quasi-4-connected graph \(G\) the following holds: If \(\Psi[G]\) is a tight pre-decomposition of \(G\) that has an \(\mathcal{AP}_p\)-star completion, then \(\Lambda[G]\) is an ordered completion of \(\Psi[G]\).

**Proof.** To explain the proof, we fix a quasi-4-connected graph \(G\); as usually, the od-scheme \(\Lambda\) we shall define will not depend on this specific graph. Suppose that \(\Phi := \Psi[G]\) is a tight pre-decomposition of \(G\) and \((S, \beta, s, D, H_0, \pi, R, \tau)\) is an \(\mathcal{AP}_p\)-star completion of \(\Phi\). By Lemma 14.4.1 we may assume without loss of generality that \(\Phi\) is a d-scheme \(\Lambda\) and \((p,q)\)-superskeleton to \(\Lambda\) is an ordered completion of \(\Psi[G]\).

To simplify the notation, we assume without loss of generality that \(V(S) \\{s\} \subseteq V(\Phi)\) and that for every \(t \in V(S) \\{s\}\) it holds that \(\sigma(t) = \sigma^\Phi(t)\) and \(\alpha(t) = \alpha^\Phi(t)\). Let \(H := \tau(s)\).

Let
\[
q := \max\{q_1(p), 3(d_1 \cdot p^2 + p + 1)\}
\]
\[
\ell := \ell_5(p, q),
\]
where \(q_1\) and \(\ell_5\) are from Lemmas 14.2.7 and 14.3.4 respectively. Let \(Z := S\text{Cen}_{p,q}(G)\) and \(G^* := S\text{Skel}_{p,q}(G) = G/Z\). Let \(A_1, \ldots, A_m\) be the connected components of \(Z\), and for every \(i \in [m]\), let \(a_i\) be the vertex of \(G^*\) corresponding to \(A_i\). Let \(H^*\) be the restriction of \(G^*\) to \(H\) (as defined before Lemma 14.3.4).

To explain the proof, we make the following simplifying assumption first.

**Assumption 1.** For all \(t \in V(S) \\{s\}\) it holds that \(\sigma(t) \cap V(Z) = \emptyset\).

We define a decomposition \(\Phi^* = (V(\Phi^*), \sigma^*, \alpha^*)\) of \(G^*\) “induced” by \(\Phi\) as follows:

\[
V(\Phi^*) := \{t \in V(\Phi) \mid \sigma^\Phi(t) \cap V(Z) = \emptyset\},
\]
\[
\sigma^*(t) := \sigma(t),
\]
\[
\alpha^*(t) := (\alpha^\Phi(t) \cap V(Z)) \cup \{a_i \mid V(A_i) \subseteq \alpha^\Phi(t)\},
\]

Note for all \(i \in [m]\) and all \(t \in V(\Phi^*)\) we have
\[
a_i \in \alpha^*(t) \iff V(A_i) \cap \alpha^\Phi(t) \neq \emptyset,
\]
because \(A_i\) is connected and \(\alpha^\Phi(t)\) separates \(\alpha^\Phi(t)\) from \(V(G) \setminus \alpha^\Phi(t)\).

**Claim 1.** \(\Phi^*\) is a tight pre-decomposition of \(G^*\).

**Proof.** Follows immediately from the tightness of \(\Phi\).

**Claim 2.** \(\Phi^*\) has a completion of width at most \(\ell\).

**Proof.** By Lemma 14.3.4 \(H^*\) has tree width at most \(\ell\). Let \((T_0, \beta_0)\) be a tree decomposition of \(H^*\) of width at most \(\ell\). Let \(t \in V(S) \\{s\}\). By Assumption 1, \(t \in V(\Phi^*)\) an \(\sigma(t) \subseteq V(H^*)\). Then \(\sigma(t)\) is a clique in \(H^*\), and hence there is a node \(t_0 \in V(T_0)\) such that \(\sigma(t) \subseteq \beta_0(t_0)\). We attach \(t\) as a new child to \(t_0\). Doing this for all \(t \in V(S) \\{s\}\), we obtain the desired width-\(\ell\) completion of \(\Phi^*\).

Hence by (a relativised version of) the Bounded Width Completion Lemma 12.3.2 there is a d-scheme \(\Lambda^*\) that defines a width-\(\ell\) completion of \(\Phi^*\) (which is a pre-decomposition

M. Grohe, Definable Graph Structure Theory
of $G^*$) within $G$. Now a straightforward modification of Lemma 13.5.2 and Corollary 13.5.3 completes the proof of the lemma under Assumption 1.

Let us drop Assumption 1. This makes the proof considerably harder. The obvious idea of extending $\Phi^*$ to nodes $t \in V(\Phi)$ with $\sigma(t) \cap V(Z) \neq \emptyset$ by letting $\sigma^*(t) := (\sigma^\Phi(t) \setminus V(Z)) \cup \{a_i \mid V(A_i) \cap \sigma^\Phi(t) \neq \emptyset\}$ does not work because the resulting pre-decomposition is not necessarily tight, and tightness is a crucial assumption of the Bounded Width Extension Lemma. Instead, we modify the pre-decomposition $\Phi$ and work with a pre-decomposition $\Phi'$ that satisfies Assumption 1. Then by the proof above we obtain an ordered completion of $\Phi'$, and we will turn this into an ordered completion of $\Phi$ in the last step of the proof.

Let $t \in V(\Phi)$. Let $C(t)$ be the set of all connected components $A$ of $G \setminus \gamma^\Phi(t)$ such that $N(A) \cap V(Z) \neq \emptyset$ and $G[V(A) \cup N(A)]$ is a planar graph. We define mappings $\gamma', \sigma', \alpha' : V(\Phi) \to 2^{V(G)}$ by

\[ \gamma'(t) := \gamma^\Phi(t) \cup \bigcup_{A \in C(t)} V(A), \]
\[ \sigma'(t) := \partial^G(\gamma'(t)), \]
\[ \alpha'(t) := \gamma'(t) \setminus \sigma'(t), \]

for all $t \in V(\Phi)$. We define the pre-decomposition $\Phi'$ by

\[ V(\Phi') := \{ t \in V(\Phi) \mid |\sigma^\Phi(t)| \leq p, \sigma'(t) \cap V(Z) = \emptyset \} \]

and $\sigma^\Phi'(t) := \sigma'(t)$, $\alpha^\Phi'(t) := \alpha'(t)$ for all $t \in V(\Phi')$.

Claim 3. $\Phi'$ is a tight pre-decomposition of $G$. Furthermore, for all $t \in V(\Phi')$ we have

\[ \sigma'(t) \subseteq \sigma^\Phi(t) \quad \text{and} \quad \gamma'(t) \supseteq \gamma^\Phi(t). \]

Proof. The tightness of the decomposition and $\gamma'(t) \supseteq \gamma^\Phi(t)$ are immediate from the definitions and the tightness of $\Phi$. To see that $\sigma'(t) = \partial^G(\gamma'(t)) \subseteq \sigma^\Phi(t)$, remember that $\sigma^\Phi(t) = \partial^G(\gamma^\Phi(t))$. We only added connected components of $G \setminus \gamma^\Phi(t)$ to $\gamma^\Phi(t)$, we have $\partial^G(\gamma'(t)) \subseteq \partial^G(\gamma^\Phi(t))$.

Claim 4. There is a pd-scheme $\Psi'$ such that $\Phi' = \Psi'[G]$.

Proof. Straightforward. (But note that we use Theorem 9.3.5, the lFP-definability of the class of planar graphs.)

The following claim is crucial. It shows that all relevant nodes of $\Phi$ are kept in $\Phi'$.

Claim 5. $\ V(S) \setminus \{ s \} \subseteq V(\Phi')$.

Proof. Let $t \in V(S) \setminus \{ s \}$. Then $|\sigma(t)| \leq p$, because the adhesion of $(S, \beta)$ is at most $p$. We have to prove that $\sigma'(t) \cap V(Z) = \emptyset$. If $\sigma(t) \cap V(Z) = \emptyset$, this follows from Claim 3. So let $z \in \sigma(t) \cap V(Z)$. Let $A$ be a connected component of $G \setminus \gamma(t)$ with $z \in N(A)$. Such a component exists because $\Phi$ is tight. Let $\Gamma_z$ be the definable witness for $z \in V(Z)$ (see Lemma 14.3.3). Let $C_0, \ldots, C_{d_1 \cdot p^2 + p}$ be the first $(d_1 \cdot p^2 + p + 1)$ central cycles of $\Gamma_z$. As $q \geq (d_1 \cdot p^2 + p + 1)$ and $z$ is $q$-central in $\Gamma_z$, these cycles exist, and $z$ is in the “interior” of these cycles. By Lemma 14.2.7, at most $d_1 \cdot p^2$ branch vertices of $\Gamma_z$ belong to $V(G) \setminus \gamma(t)$,
and at most \( p \) vertices belong to \( \sigma(t) \). Thus at least one of the cycles \( C^i \) is contained in \( G[\alpha^\Phi(t)] \) and therefore has an empty intersection with \( V(A) \cup N(A) \subseteq G \setminus \alpha(t) \). It follows that \( G[V(A) \cup N(A)] \subseteq Com^G(G) \). Since \( Com^G(G) \) is planar, this proves that \( A \in C(t) \).

Hence for all connected components of \( G \setminus \gamma(t) \) with \( z \in N(A) \) it holds that \( V(A) \subseteq \gamma(t) \). It follows that \( z \notin \partial(\gamma'(t)) = \sigma'(t) \) and thus \( V(Z) \cap \sigma'(t) = \emptyset \).

**Claim 6.** Let \( t_1, t_2 \in V(S) \setminus \{s\} \). Then either \( \gamma'(t_1) \subseteq \gamma'(t_2) \) or \( \gamma'(t_2) \supseteq \gamma'(t_1) \) or \( \gamma'(t_1) \cap \gamma'(t_2) = \sigma'(t_1) \cap \sigma'(t_2) \) or there is a \( t \in V(S) \setminus \{s,t_1,t_2\} \) such that \( \gamma'(t_1) \cup \gamma'(t_2) \subseteq \gamma'(t) \).

**Proof.** Of course we may assume that \( t_1 \neq t_2 \). We make a case distinction.

**Case 1:** There is a component \( A \in C(t_1) \) such that \( V(A) \cap \gamma(t_2) \neq \emptyset \).

We shall prove that \( \gamma'(t_2) \subseteq \gamma'(t_1) \). Let \( v \in V(A) \cap \gamma(t_2) \). Since \( \Phi \) is tight, for every \( w \in \gamma(t_2) \setminus \gamma(t_1) = \alpha(t_2) \cup (\sigma(t_2) \setminus \sigma(t_1)) \) there is a path \( P \) from \( v \) to \( w \) with all internal vertices in \( \alpha(t_2) \). Then \( P \cap \gamma(t_1) = \emptyset \) and thus \( P \subseteq A \). It follows that

\[
\gamma(t_2) \setminus \gamma(t_1) \subseteq V(A)
\]

(14.4.1)

and thus \( \gamma(t_2) \subseteq \gamma'(t_1) \).

Now let \( A' \in C(t_2) \). Let \( A'_1, \ldots, A'_m \) be the connected components of \( A' \setminus \gamma(t_1) \). Let \( i \in [m] \). We shall prove that either \( A'_i \in C(t_1) \) or \( A'_i \subseteq A \). This will imply \( V(A') \subseteq \gamma'(t_1) \) and thus \( \gamma'(t_2) \subseteq \gamma'(t_1) \). Note that \( V(A'_i) \cup N(A'_i) \subseteq V(A') \cup N(A') \). Thus \( G[V(A'_i) \cup N(A'_i)] \) is a planar graph. Further note that \( \partial(A'_i) \subseteq \sigma(t_1) \cup \sigma(t_2) \). If \( \partial(A'_i) \subseteq \sigma(t_1) \), then \( A'_i \in C(t_1) \). Otherwise, \( \emptyset \neq N(A'_i) \cap (\sigma(t_2) \setminus \sigma(t_1)) \subseteq N(A'_i) \cap (\gamma(t_2) \setminus \gamma(t_1)) \subseteq N(A'_i) \cap V(A) \),

where the last inclusion holds by (14.4.1). This implies \( A'_i \subseteq A \).

**Case 2:** There is a component \( A \in C(t_2) \) such that \( V(A) \cap \gamma(t_1) \neq \emptyset \).

By symmetry, in this case we have \( \gamma'(t_1) \subseteq \gamma'(t_2) \).

**Case 3:** Neither Case 1 nor Case 2.

**Case 3a:** \( V(Z) \cap \gamma(t_1) \cap \gamma(t_2) \neq \emptyset \).

Let \( z \in V(Z) \cap \gamma(t_1) \cap \gamma(t_2) \neq \emptyset \). Then \( z \in \sigma(t_1) \cap \sigma(t_2) \), because \( (S, \beta) \) is a tree decomposition. Furthermore, for \( i = 1, 2 \) there is a \( w_i \in N(z) \cap \alpha(t_i) \), because \( \Phi \) is tight.

Let \( \Gamma_z \) the definable witness for \( z \). By Lemma 14.2.7 there is a \( t \in V(S) \setminus \{s\} \) such that all but at most \( d_1 \cdot p^2 + p \) branch vertices of \( \Gamma_z \) belong to \( \gamma(t) \). Hence all but at most \( d_1 \cdot p^2 + p \) branch vertices of \( \Gamma_z \) belong to \( \alpha(t) \). Let \( C_0, \ldots, C_q = (d_1 \cdot p^2 + p + 1) \) be the first \( (d_1 \cdot p^2 + p + 1) \) central cycles of \( \Gamma_z \). As \( q \geq (d_1 \cdot p^2 + p + 1) \) and \( z \) is \( q \)-central in \( \Gamma_z \), these cycles exist, and \( z \) is in the “interior” of these cycles. Furthermore, at least one of the cycles \( C^i \) is contained in \( \alpha(t) \). Let \( C \) be such a cycle. Without loss of generality we may assume that \( t \neq t_1 \). Let \( A' \) be the connected component of \( G \setminus C \) that contains \( z \). Then \( A' \subseteq Com(\Gamma_z) \) and thus \( G[V(A') \cup N(A')] \) is planar. Let \( A \) be the connected component of \( G \setminus \gamma(t) \) that contains \( w_1 \in \gamma(t_1) \cap N(z) \). Then \( A \subseteq A' \) and thus \( G[V(A) \cup N(A)] \) is planar. Hence \( A \in C(t) \). Moreover, \( \gamma(t_1) \setminus \gamma(t) \subseteq V(A) \) and thus \( \gamma(t_1) \subseteq \gamma'(t) \), which implies \( \gamma'(t_1) \subseteq \gamma'(t) \).

If \( t = t_2 \), this implies the claim. If \( t \neq t_2 \), then \( \gamma'(t_2) \subseteq \gamma'(t) \) by a similar argument and thus \( \gamma'(t_1) \cup \gamma'(t_2) \subseteq \gamma'(t) \).
4.4. The Completion Theorem for Quasi-4-Connected Graphs

351

Case 3b: \( V(Z) \cap \gamma(t_1) \cap \gamma(t_2) = \emptyset \).
We shall prove that \( \gamma'(t_1) \cap \gamma'(t_2) = \sigma'(t_1) \cap \sigma'(t_2) \). Note first that for all \( A_1 \in C(t_1), A_2 \in C(t_2) \) we have \( A_1 \cap A_2 = \emptyset \). To see this, suppose for contradiction that \( x \in V(A_1) \cap V(A_2) \). Let \( z \in N(A_1) \cap V(Z) \subseteq \gamma(t_1) \cap V(Z) \). Then \( z \notin \gamma(t_2) \). Let \( P \) be a path from \( z \) to \( x \) with \( P \setminus \{ z \} \subseteq A_1 \). Then \( V(P) \cap \gamma(t_2) = \emptyset \) and thus \( P \subseteq A_2 \). Thus \( V(A_2) \cap \gamma(t_1) \supseteq \{ z \} \neq \emptyset \), and we are in Case 2, which is a contradiction.

This proves \( A_1 \cap A_2 = \emptyset \).

It follows that

\[
\gamma'(t_1) \cap \gamma'(t_2) = \gamma(t_1) \cap \gamma(t_2) = \sigma(t_1) \cap \sigma(t_2) \supseteq \sigma'(t_1) \cap \sigma'(t_2).
\]

To see the converse inclusion, let \( v \in \sigma(t_1) \cap \sigma(t_2) \). As \( \Phi \) is tight, there is a \( w_i \in N(v) \cap \alpha(t_1) \), and as \( \alpha(t_1) \cap \gamma(t_2) = \emptyset \), we have \( v \in \partial(\gamma(t_2)) = \sigma'(t_2) \).

Similarly, we have \( v \in \partial(\gamma(t_1)) = \sigma'(t_1) \) and thus \( v \in \sigma'(t_1) \cap \sigma'(t_2) \).

We define a star decomposition \((S', \beta')\) of \( G \) as follows. Suppose that \( V(S) \setminus \{ s \} = \{ t_1, \ldots, t_m \} \). We let

\[
S' := S \setminus \{ t_i \mid \text{there is a } j \in [m] \text{ such that } \gamma'(t_i) \subseteq \gamma'(t_j) \text{ or } j < i \text{ and } \gamma'(t_i) = \gamma'(t_j) \}.
\]

We let

\[
\beta'(s) := \beta(s) \setminus \bigcup_{t \in V(S') \setminus \{ s \}} \alpha'(t)
\]

and for every \( t \in S' \setminus \{ s \} \) we let

\[
\beta'(t) := \gamma'(t).
\]

Observe \( H' := \tau'(s) \subseteq H \); this follows from Claim 3. The \( p \)-arrangement \((H_0, \pi, R, \tau)\) of \( H \) in \( D \) induces a \( p \)-arrangement \((H'_0, \pi', R', \tau')\) of \( H' \) in \( D \) in the obvious way.

Claim 7. \((S', \beta', D, H'_0, \pi', R', \tau')\) is an \( AP_{\text{p-star}} \) completion of \( \Phi' \).

Proof. It follows from Claim 6 that \((S', \beta')\) is a tree decomposition. It is immediate from the definitions that \((S', \beta')\) is a completion of \( \Phi' \).

The pre-decomposition \( \Phi' \) satisfies Assumption 1, that is, for all \( t \in V(\Phi') \) it holds that \( \sigma'(t) \cap V(Z) = \emptyset \).

Hence by the first part of the proof we can now construct an od-scheme \( \Lambda' \) that defines an ordered completion \( \Delta' := \Lambda'[G] \) of \( \Phi' \). Without loss of generality we may identify the ground nodes of the completion \( \Delta' \) with the nodes of \( \Phi' \) they correspond to.

That is, for every ground node \( t \) we assume that \( t \in V(\Phi') \) and that \( \sigma^{\Delta'}(t) = \sigma'(t) \) and \( \alpha^{\Delta'}(t) = \alpha'(t) \). In the following, we write \( \Delta' := (D', \sigma', \alpha') \).

To define the desired ordered completion \( \Delta \) of \( \Phi \), we modify \( \Delta' \) as follows. For nodes \( t \in V(\Phi') \) with \( C(t) = \emptyset \) we have \( \sigma^\Phi(t) = \sigma^\Phi(t) = \alpha^\Phi(t) \), and \( \alpha^\Phi(t) = \alpha^\Phi(t) \), so if \( t \in V(\Delta') \) is a ground node with \( C(t) = \emptyset \), it is parallel to a node of \( \Phi \) and hence may serve as a ground node of \( \Delta \). Consider a ground node \( t \) with \( C(t) \neq \emptyset \). We add a new child \( u_0 \) to \( t \). We let \( \sigma^\Delta(u_0) := \sigma^\Phi(t) \) and \( \alpha^\Delta(u_0) := \alpha^\Phi(t) \). Hence \( u_0 \) will be a ground node of \( \Delta \). Suppose that \( C(t) = \{ A_1, \ldots, A^n \} \). For \( i \in [n] \), we take an ordered treelike decomposition of the planar graph \( G[V(A_1) \cup N(A_1)] \) with the property that all roots of the decomposition contain \( N(A_i) \) in their bag. It is possible to construct such a decomposition because \( N(A_i) \subseteq \sigma(t) \) and thus \( |N(A_i)| \leq p \). Then we make all roots of this decomposition children of \( t \). It remains to define an order on the node \( t \). Its bag in the new decomposition is \( \sigma(t) \), and as \( |\sigma(t)| \leq p \), we can
simply add \( p! - 1 \) new copies of \( t \) with the same children and parents as \( t \), so that we have one copy for each possible order.

\[ \ \ \ \ ]

\textit{Remark 14.4.3.} In our later application of Lemma \([14.4.2] \) we need a parametrised version where we have a parametrised pd-scheme \( \Psi(X) \) and obtain a parametrised od-scheme \( \Lambda(X) \).

I leave the precise formulation of this version of the lemma and its proof, which is a straightforward generalisation of the proof of the unparametrised version, to the reader.

\[ \ \ \ \ ]

\textit{Remark 14.4.4.} The whole purpose of introducing grids and the graph theory accompanying them, and of defining the supercentre and superskeleton of a graph was to obtain the \textit{tight} pre-decomposition \( \Phi' \) in the previous proof. Let me repeat that tightness is a necessary condition of the Bounded Width Completion Lemma.

\[ \ \ \ \ ]

\section{14.5 MAP\(_p\)-Star Completions}

To complete the proof of the Almost Planar Completion Theorem, we would like to apply Lemma \([14.4.2] \) to the quasi-4-connected components of a graph and then lift the decompositions obtained for the components to a decomposition of the whole graph with the Q4C Completion Lemma \([12.6.2] \). The problem is that Lemma \([14.4.2] \) is not directly applicable to the quasi-4-connected components of a graph because they are minors of the graph, and almost-planarity is not preserved under taking minors. In this section, we generalise our results on the structure of minors of almost planar graphs (see Section \([13.5.2] \)) to the completion setting.

\textbf{Definition 14.5.1.} Let \( p \in \mathbb{N} \), and let \( G \) be a graph.

(1) A \textit{MAP\(_p\)-star decomposition} of \( G \) is a tuple

\[ (S, \beta, s, D, H_0, \pi, R, \tau, F) \]

satisfying the following conditions.

(i) \( (S, \beta) \) is a star decomposition of \( G \), and \( s \) is the centre of \( S \).

(ii) \( D \) is a closed disk and \( (H_0, \pi, R, \tau, F) \) a \( p \)-m-arrangement of the torso \( \tau(s) \) in \( D \).

(iii) For all tips \( t \in V(S) \setminus \{s\} \),

a. either \( K[\sigma(t)] \subseteq (R + F)/F \),

b. or there is a subgraph \( C_t \subseteq H_0 \) such that \( K[\sigma(t)] = \pi(C_t)/F \) and \( C_t \cong K_i \) for some \( i \in [3] \), and if \( C_t \cong K_3 \) then \( C_t \) is a facial cycle of \( H_0 \).

If (iii-a) holds for all \( t \in V(S) \setminus \{s\} \), then \( (S, \beta, s, D, H_0, \pi, R, \tau, F) \) is a \textit{simple MAP\(_p\)-star decomposition} of \( G \).

(2) Let \( \Phi \) be a pre-decomposition of \( G \). A \textit{MAP\(_p\)-star completion} of \( \Phi \) in \( G \) is a \textit{MAP\(_p\)-star decomposition} \( (S, \beta, s, D, H_0, \pi, R, \tau, F) \) of \( G \) such that for all tips \( t \) of \( S \) there is a \( t' \in V(\Phi) \) with \( t \parallel (S, \beta, \Phi) t' \).

\[ \ \ \ \ ]

\textbf{Lemma 14.5.2.} Let \( p \in \mathbb{N} \) such that \( p \geq 3 \). Let \( G \) be a graph and \( \Phi \) a tight pre-decomposition of \( G \) that has a MAP\(_p\)-star completion. Let \( J^* \) be a quasi-4-connected component of \( G \) and \( \Phi^* \) the pre-decomposition induced by \( \Phi \) on \( J^* \). Then \( (\Phi^*)^{p,p} \) has a simple MAP\(_p\)-star completion in \( J^* \).
Proof. Let \((S, \beta, s, D, H_0, \pi, R, \tau, F)\) be a \MAP\_\(p\)-star completion of \(\Phi\) in \(G\), and let \(H := \pi(H_0) \cup R\). Then \(\tau(s) := H/F\). Let \(U\) be the set of all tips of \(S\). Without loss of generality we may assume that \(V(\Phi) = U\) and that \(t \parallel (S, \beta, \Phi) \, t\) for all \(t \in U\). Hence we may write \(\Phi = (U, \sigma, \alpha)\) (just noting that here \(\sigma\) and \(\alpha\) are restricted to \(U = V(S) \setminus \{s\}\)). We partition \(U\) into two sets \(U_a\) and \(U_b\) by letting \(U_a\) be the set of all tips \(t \in U\) satisfying condition (iii-a) of Definition \([14.5.1]\) and \(U_b := U \setminus U_a\). Then all tips \(t \in U_b\) satisfy condition (iii-b). Let \(J\) and \(M\) be the torso and matching associated with \(J^*\). We use the notation introduced in Proviso \([10.3.3]\).

Let \((Y_v)_{v \in V(J)}\) be a faithful image of \(J\) in \(G\). Such an image exists by the Q4C Decomposition Lemma \([10.2.4]\). For every \(v \in V(J)\), let \(I_v \subseteq G\) be a tree with vertex set \(V(I_v) = Y_v\), and for every edge \(e = vw \in E(J)\), let \(f_e = v'w' \in E(G)\) with \(v' \in Y_v\) and \(w' \in Y_w\).

Step 1. Let

\[G^1 := G[V(J)] \cup \bigcup_{v \in V(J)} I_v + \{f_e \mid e \in E(J) \setminus E(G)\}.\]

Then \(J \subseteq G^1 \subseteq G\), and \((Y_v)_{v \in V(J)}\) is a faithful image of \(J\) in \(G^1\). The \MAP\_\(p\)-star decomposition \((S, \beta, s, D, H_0, \pi, R, \tau, F)\) of \(G\) induces a \MAP\_\(p\)-star decomposition

\[(S, \beta^1, s, D, H_0^1, \pi^1, R^1, \tau^1, F^1)\]

of \(G^1\) defined as follows. For every \(t \in V(S)\), we let \(\beta^1(t) := \beta(t) \cap V(G^1)\). For every vertex \(v \in \beta(s)\), we let \(Z_v\) be the vertex set of the connected component of \((V(H), F)\) that is contracted to \(v\). Then \((Z_v)_{v \in \beta(s)}\) is an image of \(\tau(s)\) in \(H\). We let \(H^1\) be the induced subgraph of \(H\) with vertex set \(\bigcup_{v \in \beta(s)} Z_v\), and we let \(F^1\) be the set of all edges in \(F\) with both endvertices in \(V(H^1)\). Then \(\tau^1(s) = H^1/F^1\). We let \(H_0^1\) be the subgraph of \(H_0\) with vertex set \(\pi^{-1}(\pi(H_0) \cap V(H^1))\) and edges \(vw\) for all \(vw \in E(H_0)\) such that \(\pi(v)\pi(w) \in E(H^1)\); we let \(R^1 := R \cap H^1\); and we let \(\tau^1\) be the subtuple \(\tau \cap V(H^1)\).

Step 2. In this step, we contract the parts of \(V(G^1) \setminus V(J)\) contained in components \(\alpha^1(t)\) for tips \(t \in U\). For each \(t \in U\), we repeatedly contract edges in \(\bigcup_{v \in V(J)} E(I_v)\) that have at least one endvertex in \(\alpha^1(t) \setminus V(J)\) until no vertices are left in \(\alpha^1(t) \setminus V(J)\). In doing this, we always identify each contracted edge with one of its endvertices, and we take the endvertex in \(V(J) \cup \sigma^1(t)\) if the edge has an endvertex in this set. We let \(G^2\) be the resulting graph. We have \(V(J) \cup \beta(s) \subseteq V(G^2) \subseteq V(G^1)\), and for each tip \(t \in U\) we have \(V(G^2) \cap \alpha^1(t) = V(J) \cap \alpha^1(t)\). Moreover, \(J\) still has a faithful image in \(G^2\).

For each \(t \in U\), we let \(\alpha^2(t) := \alpha^1(t) \cap V(J)\) and \(\sigma^2(t) := \sigma^1(t)\). We let \(\alpha^2(s) := V(G^2)\) and \(\sigma^2(s) := \emptyset\). Now we can define \(\tau^2(t) := \beta^2(t)\) for every \(t \in S\) in the usual way. We obtain a star decomposition \((S, \beta^2)\) of \(G^2\). Note that \(\beta^2(s) = \beta^1(s)\) and \(\tau^2(s) = \tau^1(s)\). Thus we obtain a \MAP\_\(p\)-star decomposition \((S, \beta, s, D, H_0^1, \pi^1, R^1, \tau^2, F^2)\) of \(G^2\) by simply letting \(H_0^2 := H_0^1\), \(\pi^2 := \pi^1\), \(R^2 := R^1\), \(\tau^2 := \tau^1\), and \(F^2 := F^1\).

Step 3. In this step, we contract all edges in the matching \(M\) that have at least one endvertex in \(\alpha^2(t)\) for some \(t \in U\). Let \(G^3\) be the resulting graph. We obtain a \MAP\_\(p\)-star decomposition \((S, \beta^3, s, D, H_0^3, \pi^3, R^3, \tau^3, F^3)\) of \(G^3\) in the obvious way. Let \(H^3 := \pi^2(H_0^3) \cup R^3\).

Step 4. Now \(J^*\) is a minor of \(G^3\) that can be obtained by only contracting edges with both endvertices in \(\beta(s)\). Let \(M'\) be the set of these edges. Let \(H^4 := H^3/F^3/M'\). By Lemma \([13.5.6]\) there is a \(p\)-m-arrangement \((H_0^4, \pi^4, R^4, \tau^4, F^4)\) of \(H^4\) in the disk \(D\), and from this we obtain a \MAP\_\(p\)-star decomposition \((S, \beta^4, D, H_0^4, \pi^4, R^4, \tau^4, F^4)\) of \(J^*\).

Preliminary Version
Step 5. In this step, we turn our MAP$p$-star decomposition $(S, \beta^4, D, H_0^4, \pi^4, R^4, \tau^4, F^4)$ into a simple MAP$p$-star decomposition. Consider a tip $t \in U_b$ of type b. If $\alpha^4(t) = \emptyset$, then we can just drop $t$ from the decomposition. So let us assume that $\alpha^4(t) \neq \emptyset$. We have $|\sigma^4(t)| \leq |\sigma(t)| \leq 3$, and as $\sigma^4(t)$ separates $\alpha^4(t)$ from $V(J^*) \setminus \gamma^4(t)$ and $J^*$ is quasi-4-connected, either $|V(J^*) \setminus \gamma^4(t)| \leq 1$ or $|\alpha^4(t)| = 1$. We deal with the case $|V(J^*) \setminus \gamma^4(t)| \leq 1$ (later in Step 7), and until then we assume that $|V(J^*) \setminus \gamma^4(t')| > 1$ for all tips $t' \in U_b$. Thus $|\alpha^4(t)| = 1$. Let $x_t$ be the unique vertex in $\alpha^4(t)$. As $J^*$ is 3-connected, we have $N^{J^*}(x_t) = \sigma^4(t)$ and $|\sigma^4(t)| = 3$. Thus by Definition 14.5.1(iii-b), there is a triangle $C_t \subseteq H_0^4$ such that $\pi^4(V(C_t)) = \sigma^4(t)$ and that $C_t$ bounds a disk $D_t \subseteq D$. Let $X := \{x_t \mid t \in U_b \text{ with } \alpha^4(t) \neq \emptyset\}$.

We define a simple MAP$p$-star decomposition $(S^5, \beta^5, D, H_0^5, \pi^5, R^5, \tau^5, F^5)$ of $J^*$ as follows. We let $S^5 := S \{\{s\} \cup U_a\}$. For each $t \in U_a$ we let $\beta^5(t) := \beta^4(t)$, and we let $\beta^5(s) := \beta^4(s) \cup X$. Assuming without loss of generality that $V(H_0^4) \cap X = \emptyset$, we let $H_0^5$ be the graph obtained from $H_0^4$ by adding all vertices $x_t \in X$ and edges from $x_t$ to the three vertices of $C_t$. We extend the embedding of $H_0^4$ in $D$ to $H_0^5$ by embedding $x_t$ in the interior of the disk $D_t$. We let $\pi^5$ be the extension of $\pi^4$ with $\pi^5(x_t) := x_t$ for all $x_t \in X$. Finally, we let $R^5 := R^4, \tau^5 := \tau^4$, and $F^5 := F^4$.

Step 6. In this step, we modify our star decomposition in such a way that it is a completion of $(\Phi^*)^{p,p}$. By going through the previous steps of the proof, it can easily be checked that for all $t \in U_a$ we have $\gamma^5(t) \cap V(J^*) = \gamma(t) \cap V(J)^* = \gamma^\Phi(t)$ and that $\gamma^5(t) \setminus V(J^*) \subseteq \sigma^5(t)$, which implies $|\gamma^5(t) \setminus V(J^*)| \leq p$. Hence $|\gamma^5(t) \setminus \gamma^\Phi(t)| \leq p$. Let $B_t := J^*[\gamma^5(t)]$ an $S_t := \partial J^*[\gamma^5(t)]$. Then $S_t \subseteq \sigma^5(t)$. Let $U^6 := \{(t, A) \mid t \in U_a, A \text{ connected component of } B_t \setminus S_t\}$, and let $S^6$ be the star with centre $s$ and tips $(t, A) \in U^6$. We define a star decomposition $(S^6, \beta^6)$ as follows. We let $\sigma^6(s) := \emptyset$ and $\alpha^6(s) := V(J^*)$. For every $(t, A) \in U^6$, let $\alpha^6(t, A) := V(A)$ and let $\sigma^6(t, A) := N^{J^*}(A)$. Observe that $\sigma^5(s) \subseteq \tau^5(s)$. Thus the $p$-m-arrangement $(H_0^5, \pi^5, R^5, \tau^5, F^5)$ of $\tau^5(s)$ induces a $p$-m-arrangement $(H_0^6, \pi^6, R^6, \tau^6, F^6)$ of $\tau^6(s)$, and we obtain a MAP$p$-star decomposition $(S^6, \beta^6, D, H_0^6, \pi^6, R^6, \tau^6, F^6)$.

I claim that it is a MAP$p$-star completion of $(\Phi^*)^{p,p}$ in $J^*$. To see this, we need to prove that every tip in $U^6$ is parallel to a node of $(\Phi^*)^{p,p}$. So let $(t, A) \in U^6$. By definition, $J^*[\alpha^6(t, A)] = A$ is connected and we have $\sigma^6(t, A) = N^{J^*}(\alpha^6(t, A))$. We also have $\sigma^6(t, A) = \partial J^*[\gamma^5(t)]$, because $\sigma^6(t, A) \subseteq S_t = \partial J^*[\gamma^5(t)]$ and $\gamma^6(t, A) \subseteq V(B_t) = \gamma^5(t)$. We have $|\sigma^6(t, A)| \leq p$, because $\sigma^6(t, A) \subseteq S_t \subseteq \sigma^5(t)$. Furthermore, we have $|\gamma^6(t, A) \setminus \gamma^*(t)| \leq |\gamma^5(t) \setminus V(J^*)| \leq p$. Thus $(t, \sigma^6(t, A), A)$ is a node of the derivation $(\Phi^*)^{p,p}$ with $(t, \sigma^6(t, A), A) \parallel (\Phi^*)^{p,p}(S^6, \beta^6)(t, A)$.

Step 7. To complete the proof, it remains to deal with the case that $|V(J^* \setminus \gamma^4(t)| \leq 1$ for some $t \in U_b$. We keep this node fixed for the rest of the argument. As $t \in U_b$, we have $|\sigma^4(t)| \leq 3 \leq p$. Similarly to the previous step, we can check that $|\gamma^4(t) \setminus \gamma^\Phi(t)| \leq 3$. Let $B := J^*[\gamma^4(t)]$ an $S := \partial J^*[\gamma^4(t)]$. Then $S \subseteq \sigma^4(t)$. Let $U^7 := \{A \mid A \text{ connected component of } B \setminus S\}$, and let $S^7$ be the star with centre $s$ and tips $A \subseteq U^7$. We define a star decomposition $(S^7, \beta^7)$ as follows. We let $\sigma^7(s) := \emptyset$ and $\alpha^7(s) := V(J^*)$. For every $A \subseteq U^7$, we let $\alpha^7(A) := V(A)$.

M. Grohe, Definable Graph Structure Theory
and \( \sigma^7(A) := N^J(A) \). Then we have \( \beta^7(s) \subseteq S \cup (V(J^*) \setminus \gamma(t)) \) and thus \( |\beta^7(s)| \leq 4 \). Thus \( \tau^7(s) \) is planar, and we can trivially find a \( p \)-m-arrangement \((H_0^7, \pi^7, R^7, F^7)\) of \( \tau^7(s) \) in \( D \). We obtain a simple \( \mathcal{MAP}_{p^\ast} \)-star decomposition \((S^7, \beta^7, D, H_0^7, \pi^7, R^7, \tau^7, F^7)\). Arguing as in the previous step, we see that it is a \( \mathcal{MAP}_{p^\ast} \)-star completion of \((\Phi^*)^{p^\ast}D \) in \( J^* \).

The next lemma is an adaptation of Corollary 13.5.9.

**Lemma 14.5.3.** Let \( p \in \mathbb{N} \) such that \( p \geq 3 \). Let \( G \) be a graph and \( \Phi \) a pre-decomposition of \( G \) that has a \( \mathcal{MAP}_{p^\ast} \)-star completion. Let \( J^* \) be a quasi-4-connected component of \( G \) and \( \Phi^* \) the pre-decomposition induced by \( \Phi \) on \( J^* \).

Then \( J^* \) has a tree decomposition \((T^*, \beta^*)\) such that for all \( t \in V(T^*) \) one of the following two conditions is satisfied.

(i) \( t \) is a leaf of \( T^* \), and there is a \( t' \in V((\Phi^*)^{p^\ast}D) \) such that \( t \parallel (T^*, \beta^*),(\Phi^*)^{p^\ast}D \).

(ii) \( \tau^*(t) \) has a \( 3p \)-arrangement \((H_0, \pi, R, \tau)\) in a disk \( D \) such that for all \( u \in \{t\} \cup N^T_\ast(t) \) it holds that \( K[\sigma(x)(u)] \subseteq R \).

**Proof.** By Lemma 14.5.2 the pre-decomposition \((\Phi^*)^{p^\ast}D \) has a simple \( \mathcal{MAP}_{p^\ast} \)-star completion \((S, \beta, s, D, H_0, \pi, R, \tau, F)\). Let \( H := \pi(H_0) \cup R \) and \( H^* := H/F = \tau(s)\).

By Lemma 13.5.8 we find a tree decomposition \((T^*, \beta^\ast)\) of \( H \) such that for all \( t \in V(T^*) \) the torso \( \tau^*(t) \) has a \( 3p \)-arrangement \((H_0, \pi_t, R_t, \tau_t)\) in a disk \( D_t \subseteq D \) satisfying the conditions of Lemma 13.5.8(iv). In particular, from Lemma 13.5.8(iv-b) and (iv-c) it follows that for all \( v \in V(R) \) there is a \( t \in V(T^*) \) such that \( a(v) \in V(R_t) \). Let \( x \in V(S) \setminus \{s\} \). Then \( \sigma(x) \) is a node \( t(x) \in V(T^*) \) such that \( K[\sigma(x)] \subseteq R_t(x) \).

We define a tree decomposition \((T^*, \beta^*)\) of \( J^* \) as follows.

- \( V(T^*) := V(T^*) \cup (V(S) \setminus \{s\}) \), where without loss of generality we assume that \( V(T^*) \cap (V(S) \setminus \{s\}) = \emptyset \).
- \( E(T^*) := E(T^*) \cup \{t(x) \mid x \in V(S) \setminus \{s\}\} \).
- For all \( t \in V(T^*) \),
  \[ \beta^*(t^*) := \begin{cases} 
  \beta^\ast(t) & \text{if } t \in V(T^*), \\
  \beta(t) & \text{if } t \in V(S) \setminus \{s\}. 
  \end{cases} \]

It is immediate from the definitions and Lemma 13.5.8 that \((T^*, \beta^*)\) satisfies (i) and (ii).

**14.6 Proof of the Almost Planar Completion Theorem**

**Lemma 14.6.1.** Let \( \Psi \) be a pd-scheme and \( p \in \mathbb{N} \). Let \( d_{\ast} := d'_{\ast} := \max\{p,3\} \). Then there exists a parametrised od-scheme \( \Lambda(\pi) \) such that the following holds. Let \( G \) be a graph such that \( \Phi := \Psi[G] \) is a tight pre-decomposition of \( G \) that has an \( \mathcal{MAP}_{p^\ast} \)-star completion. Let \( J^*_\pi \) be a quasi-4-connected component of \( G \) with index \( \pi \in V(\Lambda_{\ast_{tc}}[G]) \), and let \( \Phi^*_\pi \) be the pre-decomposition induced by \( \Phi \) on \( J^*_\pi \). Then the scheme \( \Lambda(\pi) \) defines an ordered completion of \((\Phi^*_\pi)^{d_{\ast},d'_{\ast}}\) on \( J^*_\pi \) within \((G, \pi)\).

**Proof.** Without loss of generality we may assume that \( p \geq 3 \). We fix a graph \( G \) such that \( \Phi := \Psi[G] \) is a tight pre-decomposition of \( G \) that has a \( \mathcal{MAP}_{p^\ast} \)-star completion. Without loss of generality we may assume that the adhesion of \( \Phi \) is at most \( p \). Let \( J^* \) be a quasi-4-connected component of \( G \) and \( \Phi^* \) the pre-decomposition of induced by \( \Phi \) on \( J^* \). Then the
adhesion of $\Phi^*$ is at most $p$ as well. By Lemma 14.5.3 we find a tree decomposition $(T^*, \beta^*)$ of $J^*$ such that for all $t \in V(T^*)$ one of the following two conditions is satisfied:

(i) $t$ is a leaf of $T^*$ and there is a $t' \in V((\Phi^*)^{p,p})$ such that $t \parallel (T^*, \beta^*), (\Phi^*)^{p,p} t'$;

(ii) $H_t := \tau^*(t)$ has a 3p-arrangement $(H_{t_0}, \pi_t, R_t, \tau_t)$ in a disk $D_t$ such that for all $u \in \{t\} \cup N^*_+(t) \cup H_t$ it holds that $K[\sigma^*(u)] \subseteq R_t$.

We call the nodes satisfying (i) the ground nodes of the decomposition. By Lemma 12.1.3 and since $(\Phi^*)^{p,p}$ is tight, we may assume without loss of generality that $(T^*, \beta^*)$ is tight. It follows from (ii) that the adhesion of $(T^*, \beta^*)$ is at most 3p.

Let $t \in V(T^*)$ and $J_t := J^*[\gamma^*(t)] \cup K[\sigma^*(t)]$. Observe that $J_t$ is quasi-4-connected, because if $S$ is a separator of $J^*$ and $A$ a connected component of $J^* \setminus S$, then $\hat{A} J^*$ is quasi-4-connected. Let $\Phi^t$ be the pre-decomposition of $J_t$ defined by $V(\Phi^t) := N^*_+(t)$ and $\sigma^t(u) := \sigma^*(u), \alpha^t(u) := \alpha^*(u)$ for all $u \in V(\Phi^t)$. Then $\Phi^t$ is a tight pre-decomposition of $J_t$.

Claim 1. For all $t \in V(T^*)$ the pre-decomposition $\Phi^t$ has an $AP_{3p}$-star completion in $J_t$.

Proof. Follows from (ii).

For every tuple $\bar{\nu} := (v_1, \ldots, v_{3p+1}) \in V(J^*)^{3p+1}$, let

$$S_{\bar{\nu}} := \begin{cases} \{v_1, \ldots, v_{3p}\} & \text{if } v_{3p+1} \notin \{v_1, \ldots, v_{3p}\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $A_{\bar{\nu}}$ be the connected component of $J^* \setminus S_{\bar{\nu}}$ that contains $v_{3p+1}$, and let $J_{\bar{\nu}} := \hat{A}_{J_t}$. Note that $J_{\bar{\nu}}$ is quasi-4-connected. Let $P$ be the set of all $\bar{\nu} \in V(J^*)^{3p+1}$ such that $S_{\bar{\nu}} = N^*_+(A_{\bar{\nu}}) = \partial^*_{J_t}(V(J_{\bar{\nu}}))$. For every subset $Q \subseteq P$, we define a pre-decomposition $\Phi_Q$ of $J^*$ by letting $V(\Phi_Q) := Q$ and $\sigma^Q_{\bar{\nu}} := S_{\bar{\nu}}$, $\alpha^Q_{\bar{\nu}} := V(A_{\bar{\nu}})$ for all $\bar{\nu} \in Q$. For every $\bar{\nu} \in P$, we let $\Phi_{Q,\bar{\nu}}$ be the restriction of $\Phi_Q$ to $V(J_{\bar{\nu}})$, that is, we let $V(\Phi_{Q,\bar{\nu}}) := \{ \bar{\nu} \in Q \mid V(J_{\bar{\nu}}) \subseteq V(J_{\bar{\nu}}) \}$ and $\sigma^Q_{\bar{\nu}} := S_{\bar{\nu}}, \alpha^Q_{\bar{\nu}} := V(A_{\bar{\nu}})$ for all $\bar{\nu} \in V(\Phi_{Q,\bar{\nu}})$. It follows from the definition of $P$ that $\Phi_Q$ and $\Phi_{Q,\bar{\nu}}$ are tight pre-decompositions of $J^*$ and $J_t$, respectively. Note that there is a pd-scheme $\Psi(X, \bar{\nu})$, where $X$ is a $(3p + 1)$-ary relation variable, that defines $\Phi_{Q,\bar{\nu}}$ within $(J^*, Q, \bar{\nu})$. Let $\Lambda(X, \bar{\nu})$ be the od-scheme scheme obtained by applying Lemma 14.4.2 (in a parametrised version, cf. Remark 14.4.3) to $\Psi(X, \bar{\nu})$. Let $\Delta_{Q,\bar{\nu}} := \Lambda(J_t^*, Q, \bar{\nu})$. If $\Phi_{Q,\bar{\nu}}$ has an $AP_{3p}$-star completion in $J_t$, then $\Delta_{Q,\bar{\nu}}$ is an ordered completion of $\Phi_{Q,\bar{\nu}}$.

We inductively define sequences $(P_i)_{i \in \mathbb{N}}$ of subsets of $P$ as follows: We let $P_0$ be the set of all $\bar{\nu} \in P$ such that for some $t \in (\Phi^*)^{p,p}$ it holds that $S_{\bar{\nu}} = \sigma(\Phi^*)^{p,p}(t)$ and $V(A_{\bar{\nu}}) = \alpha(\Phi^*)^{p,p}(t)$. Then $(\Phi^*)^{p,p}$ and $\Phi_{P_0}$ are equivalent pre-decompositions of $J^*$. For all $i \in \mathbb{N}$, we let $P_{i+1}$ be the set of all $\bar{\nu} \in P$ such that either $\bar{\nu} \in P_i$ or $\Delta_{P_i,\bar{\nu}}$ is an ordered completion of $\Phi_{P_i,\bar{\nu}}$. We let $P_\infty := \bigcup_{i \in \mathbb{N}} P_i$.

Claim 2. For all $t \in V(T^*)$ there is a $\bar{\nu} \in P_\infty$ such that $\sigma^*(t) = S_{\bar{\nu}}$ and $\alpha^*(t) = V(A_{\bar{\nu}})$.

Proof. We say that $t \in V(T^*)$ and $\bar{\nu} \in P$ are parallel if $\sigma^*(t) = S_{\bar{\nu}}$ and $\alpha^*(t) = V(A_{\bar{\nu}})$. The proof is by induction on $T^*$. Let $t \in V(T^*)$. If $t$ is a ground node, then there is a $\bar{\nu} \in P_0$ that is parallel to $t$. Otherwise, suppose that for some $i \in \mathbb{N}$ all $u \in N^*_+(t)$ are parallel to some tuple $\bar{\nu} \in P_i$. As the adhesion of $(T^*, \beta^*)$ is at most 3p and $(T^*, \beta^*)$ is tight, there is a $\bar{\nu} \in P$ that is parallel to $P$. It follows from Claim 1 that $\Phi_{P_i,\bar{\nu}}$ has an $AP_{p}$-star completion in $J_t$. Thus $\Delta_{P_i,\bar{\nu}}$ is an ordered completion of $\Phi_{P_i,\bar{\nu}}$, and $\bar{\nu} \in P_{i+1}$.

M. Grohe, Definable Graph Structure Theory
Claim 2 implies that there is a \( \overline{v}_r \in P_\infty \), which is parallel to the root of \( T^* \), such that 
\( J_{\overline{v}_r} = J^* \).

By induction on \( i \in \mathbb{N} \), for every \( \overline{v} \in P_i \) we can define an ordered completion \( \Delta_{\overline{v}} := (D_{\overline{v}}, \sigma_{\overline{v}}, \alpha_{\overline{v}}, \leq_{\overline{v}}) \) of the restriction \( \Phi_{P_i, \overline{v}} \) of \( \Phi_{P_i} = \Phi^* \) to \( J_{\overline{v}} \) by combining the o-decompositions \( \Delta_{P_i, \overline{v}} \) appropriately. Then \( \Delta_{\overline{v}_r} \) is an ordered completion of \( \Phi^* \) in \( J^* \).

It remains to observe that the o-decomposition \( \Delta_{\overline{v}_r} \) can be defined in \( \text{IFP} \). It is important to note that in our definition of the decompositions \( \Delta_{\overline{v}} \) we make no assumption on the definability of the tree decomposition \( (T^*, \beta^*) \); we only assume that such a decomposition exists. As all steps of the construction, in particular the inductive definitions of the sets \( P_i \) and the o-decompositions \( \Delta_{\overline{v}} \), are definable in \( \text{IFP} \), there is an od-scheme \( \Lambda' (\overline{v}) \) that defines \( \Delta_{\overline{v}_r} \) within \( (G, \overline{v}_r) \).

Now the Almost Planar Completion Theorem follows.

**Proof of the Almost Planar Completion Theorem** \[14.1.3\] Without loss of generality we may assume that for every graph \( G \) the pre-decomposition \( \Phi := \Psi[G] \) has adhesion at most \( p \), because in an \( AP_p \)-star decomposition only nodes of \( t \in V(\Phi) \) with \( |\sigma^\Phi(t)| \leq \max\{p, 3\} \) can appear. Now the theorem follows immediately from the Q4C Completion Lemma \[12.6.2\] and Lemma \[14.6.1\].

For later reference, we note that Lemma \[14.6.1\] also implies the following strengthening of the Almost Planar Completion Theorem.

**Lemma 14.6.2.** Let \( \Psi \) be a pd-scheme and \( p, d, d' \in \mathbb{N} \) with \( d \geq 10 \max\{p, 3\} + 3 \) and \( d \geq 4 \max\{p, 3\} \). Then there exists an od-scheme \( \Lambda \) such that for every graph \( G \) the following holds: if \( \Psi[G] \) is a tight pre-decomposition that has a MAP\(_p\)-star completion, then \( \Lambda[G] \) is an ordered completion of the \( (d, d') \)-derivation of \( \Psi[G] \).

**Proof.** The proof is almost identical with the proof of the Almost Planar Completion Theorem. \( \square \)
Chapter 15

Almost Embeddable Graphs

An obvious generalisation of arrangements of graphs in disks is arrangements of graphs in arbitrary surfaces with boundary. This leads to the notion of almost embeddable graphs. Our goal is to generalise the Definable Structure Theorem [13.5.1] and the Completion Theorem [14.1.3] from almost planar to almost embeddable graphs. We will reach this goal in the next chapter.

The proofs of the Definable Structure Theorem [16.3.1] and the Completion Theorem [16.4.2] for almost embeddable graphs will be by induction on the genus and the number of cuffs of the surface. (Remember that the cuffs of a surface with boundary are the arcwise connected components of the boundary.) The base steps are almost planar graphs and graphs embeddable in a surface; we already know how to deal with them. The main results of this chapter will provide the tools necessary for the inductive step. The idea is the following. Suppose we are given a graph $G$ that is almost embeddable in a surface $S$ (with boundary). Then we want to delete from $G$ either a cycle $C$ that is noncontractible in $S$ or a path $P$ whose endvertices are in two different cuffs. The resulting graph $G'$ will be almost embeddable in a surface that has lower Euler genus or fewer cuffs than $S$. By the induction hypothesis, we obtain a definable ordered treelike decomposition of $G'$ (or an ordered completion of some induced pre-decomposition in the proof of the Completion Theorem). Then we want to apply the Ordered Extension Lemma [7.3.2] to extend this decomposition to $G$, exploiting the fact that the cycle $C$ or path $P$ that we deleted admits a definable order.

The problem with this approach is that we cannot necessarily define a cycle $C$ or path $P$ to be deleted in $\text{IFP}$. In this chapter, we shall introduce fairly complicated objects called belts that are $\text{IFP}$-definable and sufficiently close approximations of the desired noncontractible cycles or paths to use them in the inductive argument.

15.1 Arrangements in a Surface

Definition 15.1.1. Let $S$ be a surface, and let $c^1, \ldots, c^q$ be the cuffs of $S$.

1. An arrangement of a graph $G$ in $S$ is a tuple

   $$(G_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q),$$

   where $G_0$ is a graph normally embedded in $S$ and $\pi : V(G_0) \to V(G)$ and $R^i \subseteq G$ and $\tau^i = (r^i_1, \ldots, r^i_{n_i}) \in V(G_0)^{n_i}$ for all $i \in [q]$ such that the following conditions are
Furthermore, for all \( p, q, r \in \mathbb{N} \), let \( \mathcal{N}(G_0) = \pi(G_0) \cup R^1 \cup \ldots \cup R^n \).

Chapter 15. Almost Embeddable Graphs

In the following, let \( G \subseteq S \) be a graph. We let \( \mathcal{N}(G_0, \pi, R^1, \pi^1, \ldots, R^n, \pi^n) \) be an arrangement of \( G \) in \( S \).

1. \( (G_0, \pi, R^1, \pi^1, \ldots, R^n, \pi^n) \) is a \( p \)-arrangement of \( G \) in \( S \) if for all \( i \in [q] \) the pair \( (R^i, \pi(\pi^i)) \) is a \( p \)-ring.
2. \( (G_0, \pi, R^1, \pi^1, \ldots, R^n, \pi^n) \) is a local \( p \)-arrangement of \( G \) in \( S \) if for all \( i \in [q] \) the pair \( (R^i, \pi(\pi^i)) \) is a \( 2p \)-vortex.
3. \( G \) is \( p \)-almost embeddable in \( S \) if it has a \( p \)-arrangement in \( S \). The class of all graphs that are \( p \)-almost embeddable in \( S \) is denoted by \( \mathcal{A}_p(S) \).

Note that for every surface \( S \) it holds that \( \mathcal{A}_p(S) = \mathcal{E}_S \). If \( S \) is a surface without boundary, then \( \mathcal{A}_p(S) = \mathcal{A}_0(S) = \mathcal{E}_S \) for all \( p \in \mathbb{N} \). Let \( p, q, r \in \mathbb{N} \), and let \( G \) be a graph. We let

\[ \mathcal{A}(p, q, r) \]

be the class of all graphs \( G \) that are \( p \)-almost embeddable in a surface of Euler genus \( r \) with \( q \) cuffs. Note that for all \( p, q, r \in \mathbb{N} \) it holds that \( \mathcal{A}(p, 1, 0) = \mathcal{A}(1, p, 0) = \mathcal{A}(0, q, r) = \mathcal{E}(r) \).

Furthermore, for all \( p', q', r' \in \mathbb{N} \) such that \( p' \leq p, q' \leq q, r' \leq r \) it holds that \( \mathcal{A}(p', q', r') \subseteq \mathcal{A}(p, q, r) \).

15.1.1 Homotopic Curves

Recall that \( I \) denotes the closed unit interval on the real line, equipped with the usual topology, and \( S^1 \) denotes the 1-sphere. Let \( X, Y \) be topological spaces. Two continuous functions \( f, g : X \to Y \) are homotopic if there exists a continuous function \( H : X \times I \to Y \), where \( X \times I \) denotes the product space, such that \( f(x) = H(x, 0) \) and \( g(x) = H(x, 1) \) for all \( x \in X \).

Now let \( S \) be a surface, possibly with boundary. Recall that \( g \subseteq S \) is a simple closed curve if and only if there is an injective continuous function \( g : S^1 \to S \) such that \( g = g(S^1) \). (Such a function is automatically a homeomorphism from \( S^1 \) to \( g \).) A simple closed curve \( g \subseteq S \) is null-homotopic if and only if there is an injective continuous function \( g : S^1 \to S \) such that \( g = g(S^1) \) and \( g \) is homotopic to a constant function. It can be shown (see [85]) that a simple closed curve in a surface is null-homotopic if and only if it is contractible, that
is, the boundary of a closed disk in $S$. Two simple closed curves $g_0, g_1 \subseteq S$ are homotopic if there are homotopic injective continuous functions $g_0, g_1 : S^1 \to S$ such that $g_i = g_i(S^1)$.

It is slightly more subtle to define homotopic simple curves. A subspace $g \subseteq S$ is a simple curve if and only if there is an injective continuous function $g : I \to S$ such that $g = g(I)$ (such a function is automatically a homeomorphism from $I$ to $g$). Let $g_0, g_1 \subseteq S$ be simple curves with the same endpoints. Then $g_0$ and $g_1$ are homotopic if there are injective continuous functions $g_0, g_1 : I \to S$ such that $g_i = g_i(I)$ for $i = 1, 2$ and a continuous function $H : I \times I \to S$ such that $g_0(x) = H(x, 0)$ and $g_1(x) = H(x, 1)$ and $H(0, x) = g_0(0) = g_1(0)$ and $H(1, x) = g_0(1) = g_1(1)$ for all $x \in I$. So $H$ is a homotopy between $g_0$ and $g_1$ that keeps the endpoints fixed. If we dropped this requirement, then any two simple curves would be homotopic, because we could contract the first curve to a single point and expand it again to the second point.

In the following, we will only argue with homotopy on an intuitive level.

### 15.1.2 Simplifying Curves in Surfaces with Boundary

Recall the discussion of noncontractible curves in surfaces without boundary in Section 9.1.1. The situation is more complicated for surfaces with boundary, because there may also be noncontractible simple closed curves that separate the cuffs of the surface, but not necessarily reduce the genus. Furthermore, if the intersection of a noncontractible curve with the boundary of the surface is nonempty, then the curve may split the surface into more than two arcwise connected components, and moreover the boundary of these arcwise connected components may become quite irregular, consisting of segments of the noncontractible curve and segments of the boundary of the surface. We will not give a complete classification of the noncontractible curves in a surface with boundary, but instead describe several types of such curves that will be useful later.

Recall that $eg(S)$ denotes the Euler genus and $cf(S)$ the number of cuffs of a surface $S$. Intuitively, we think of a surface $S'$ as being “simpler” than $S$ if the pair $(eg(S'), cf(S'))$ is lexicographically smaller than $(eg(S), cf(S))$. However, we need a slightly stricter notion to make sure that for every surface $S$, up to homeomorphisms there are only finitely many surfaces simpler than $S$. We define a partial order $\leq^*$ on $\mathbb{N}^2$ by letting $(m', n') \leq^* (m, n)$ if

- either $m' \leq m$ and $n' \leq n$,
- or $(m', n') \leq^* (m - 1, n + 1)$,
- or $(m', n') \leq^* (m, n - 1),

for all $m, n, m', n' \in \mathbb{N}$. Furthermore, we let $(m', n') <^* (m, n)$ if $(m', n') \leq^* (m, n)$ and $(m', n') \neq (m, n)$. Then we say that a surface $S'$ is simpler than a surface $S$ (we write $S' <^* S$) if

$$(eg(S'), cf(S')) <^* (eg(S), cf(S))$$

We will now introduce four types of “simplifying curves” whose deletion turns the surface into one or two simpler surfaces (see Figure 15.1). Let $S$ be a surface, and let $\overline{S} \supseteq S$ be a surface without boundary such that $S$ is obtained from $\overline{S}$ by deleting the interior of finitely many mutually disjoint closed disks.

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1Of course “null-homotopic” is the notion that captures the intuitive meaning of “contractible” much better than being the boundary of a disk. We chose the latter as a definition in Chapter 9 because it is simpler.
Chapter 15. Almost Embeddable Graphs

(a) a proper noncontractible curve
(b) a noncontractible loop
(c) a link
(d) a cuff-separating curve

Figure 15.1. Simplifying curves

Proper Noncontractible Curves

A proper noncontractible curve of $S$ is a simple closed curve $g \subseteq S$ that is noncontractible in $\overline{S}$ and has an empty intersection with $\text{bd}(S)$ (see Figure 15.1(a)).

Let $g$ be a proper noncontractible curve in $S$. It follows from Fact 9.1.6 that if $S \setminus g$ is arcwise connected, then there is a a surface $S''$ such that $\text{eg}(S'') < \text{eg}(S)$ and $\text{cf}(S'') = \text{cf}(S)$ and $S \setminus g$ is obtained from $S''$ by deleting one or two disjoint closed disks. Furthermore, if $S \setminus g$ not arcwise connected then it has two arcwise connected components $A_1, A_2$, and there are surfaces $S_1, S_2$ such that for $i = 1, 2$ it holds that $\text{eg}(S_i) + \text{eg}(S_2) = \text{eg}(S)$ and $\text{cf}(S_1) + \text{cf}(S_2) = \text{cf}(S)$ and $A_i$ is obtained from $S_i$ by deleting a closed disk. Furthermore, $\text{eg}(S_i) < \text{eg}(S)$ for $i = 1, 2$, because we assumed $g$ to be noncontractible in $S$.

There is a slightly different view on essentially the same construction that allows us to be more precise about how the surface $S$ relates to the new surfaces $S'$ or $S_1, S_2$. This will be useful later. We choose a small closed neighbourhood $U \subseteq S$ of $g$ that has an empty intersection with $\text{bd}(S)$ and is either homeomorphic to the cylinder $S_{0,2}$ (if $g$ is orientable) or a Möbius strip $N_{1,1}$ (if $g$ is nonorientable). Then the spaces $S \setminus g$ and $S \setminus U$ are homeomorphic. However, we are more interested in $S'' := S \setminus \text{int}(U)$. If $S \setminus g$ is arcwise connected, then $S''$ is arcwise connected as well and hence a surface. All cuffs of $S$ are cuffs of $S''$, and $S''$ has one or two additional cuffs. More precisely, we have $\text{bd}(S'') = \text{bd}(S) \cup \text{bd}(U)$. If $U \simeq S_{0,2}$, then $S''$ has two more cuffs than $S$. In this case, we let $S' \supseteq S''$ be a surface obtained from $S''$ by gluing disks to the two new cuffs. That is, there are disks $D, D' \subseteq S'$ such that

M. Grohe, Definable Graph Structure Theory
An open loop

Noncontractible Loops

A loop \( g \) with both endpoints in the same cuff \( c \) of \( S \) and all internal points in the interior of \( S \), and a closed loop is a simple closed curve in \( S \) that has exactly one point in \( \text{bd}(S) \). (Figure 15.1b) shows an open loop. If \( g \) is an open or closed loop, we call the (either one or two) points in \( g \cap \text{bd}(S) \) the attachment points of \( g \), and we say that \( g \) is attached to the (unique) cuff of \( S \) that contains its attachment points.

First, consider an open loop \( g \subseteq S \) with attachment points \( x_1, x_2 \) in the cuff \( c \). We choose a small closed neighbourhood \( U \subseteq S \) of \( g \) such that \( U \) is homeomorphic to a closed disk and that \( U \cap \text{bd}(S) \) consists of two disjoint segments \( b_1, b_2 \subseteq c \) such that for \( i = 1, 2 \), the attachment point \( x_i \) of \( g \) is an internal point of \( b_i \) (see Figure 15.2). Let \( y_i, z_i \) be the endpoints of \( b_i \) such that the points \( y_1, x_1, z_1, y_2, x_2, z_2 \) are in cyclic order on \( c \). Let \( a_1 \) be the segment of \( c \) from \( z_2 \) to \( y_1 \) that does not contain \( x_1, x_2 \), and let \( a_2 \) be the segment of \( c \) from \( z_1 \) to

Figure 15.2. A loop \( g \) and its neighbourhood \( U \)

\[ \text{bd}(D) \cup \text{bd}(D') = \text{bd}(U) \] and \( S'' = S' \setminus (\text{int}(D) \cup \text{int}(D')) \). If \( U \cong N_{1,1} \), then \( S'' \) has one more cuff than \( S \). In this case, we let \( S' \supseteq S'' \) be a surface obtained from \( S'' \) by gluing a disk to the new cuff. In both cases we have \( \text{eg}(S') < \text{eg}(S) \) and \( \text{cf}(S') = \text{cf}(S) \) and thus \( S' < S \).

If \( S \setminus g \) is not arcwise connected and thus has two arcwise connected components, then \( S'' \) has two arcwise connected components, say, \( S'_1 \) and \( S'_2 \), as well. Furthermore, \( U \cong S_{0,2} \), and for \( i = 1, 2 \) the surface \( S'_i \) contains one of the two cuffs of \( U \) and possibly some of the cuffs of \( S \). We let \( S_i \) be a surface obtained from \( S'_i \) by gluing a disk on the new cuff. Then \( \text{eg}(S_i) < \text{eg}(S) \) and \( \text{cf}(S_i) \leq \text{cf}(S) \). Thus \( S_i < S \).

In a typical application, we will have a graph \( G \) embedded in \( S \), and \( g \) will be a \( G \)-normal curve. Let \( G' = G \setminus g \). We can choose the neighbourhood \( U \) of \( g \) in such a way that it has a nonempty intersection with \( G' \). Suppose first that \( S \setminus g \) is connected. Then \( G' \subseteq S'' \subseteq S' \), and thus we may view \( G' \) as a graph embedded in the simpler surface \( S' \). If \( S \setminus g \) is not connected, we let \( G_1 := G \cap S'_1 \) and \( G_2 := G \cap S'_2 \). Then \( G' = G_1 \cup G_2 \), because \( G' \subseteq S \setminus U \). Furthermore, for \( i = 1, 2 \) we may view \( G_i \) as a graph embedded in the simpler surface \( S_i \).

Noncontractible Loops

An open loop is a simple curve in \( S \) with both endpoints in the same cuff \( c \) of \( S \) and all internal points in the interior of \( S \), and a closed loop is a simple closed curve in \( S \) that has exactly one point in \( \text{bd}(S) \). (Figure 15.1b) shows an open loop. If \( g \) is an open or closed loop, we call the (either one or two) points in \( g \cap \text{bd}(S) \) the attachment points of \( g \), and we say that \( g \) is attached to the (unique) cuff of \( S \) that contains its attachment points.

First, consider an open loop \( g \subseteq S \) with attachment points \( x_1, x_2 \) in the cuff \( c \). We choose a small closed neighbourhood \( U \subseteq S \) of \( g \) such that \( U \) is homeomorphic to a closed disk and that \( U \cap \text{bd}(S) \) consists of two disjoint segments \( b_1, b_2 \subseteq c \) such that for \( i = 1, 2 \), the attachment point \( x_i \) of \( g \) is an internal point of \( b_i \) (see Figure 15.2). Let \( y_i, z_i \) be the endpoints of \( b_i \) such that the points \( y_1, x_1, z_1, y_2, x_2, z_2 \) are in cyclic order on \( c \). Let \( a_1 \) be the segment of \( c \) from \( z_2 \) to \( y_1 \) that does not contain \( x_1, x_2 \), and let \( a_2 \) be the segment of \( c \) from \( z_1 \) to
\[ y_2 \text{ that does not contain } x_1, x_2. \] Recall that \( U \) is a closed disk. Hence its boundary in \( \overline{S} \) is a simple closed curve that consists of the two segments \( b_1, b_2 \) and two disjoint simple curves \( h_1, h_2 \) that are internally disjoint from \( b_1 \) and \( b_2 \). Each of the curves \( h_1, h_2 \) has one endpoint in \( \{y_1, z_1\} \) and one endpoint in \( \{y_2, z_2\} \). Say, \( y_1 \) is an endpoint of \( h_1 \) and \( z_1 \) is an endpoint of \( h_2 \).

If \( S \setminus g \) is arcwise connected, then \( S' := S \setminus int(U) \) is connected as well, and we have \( eg(S') < eg(S) \). All cuffs of \( S \) except \( c \) are cuffs of \( S' \) as well, and \( S' \) has one or two additional cuffs. More precisely, we have

\[ bd(S') = (bd(S) \setminus c) \cup (a_1 \cup h_1 \cup a_2 \cup h_2). \]

If \( y_2 \) is an endpoint of \( h_1 \) and \( z_2 \) is an endpoint of \( h_2 \) then \( a_1 \cup h_1 \cup a_2 \cup h_2 \) is a simple closed curve and hence \( S' \) has the same number of cuffs as \( S \); this happens if \( g \) is a nonorientable curve. If \( z_2 \) is an endpoint of \( h_1 \) and \( y_2 \) is an endpoint of \( h_2 \), then \( a_1 \cup h_1 \cup a_2 \cup h_2 \) is the disjoint union of the two simple closed curves \( a_1 \cup h_1 \) and \( a_2 \cup h_2 \) (see Figure 15.2) and \( S' \) has the one more cuff than \( S \); this happens if \( g \) is an orientable curve. In both cases, \( S' \subseteq S \) is a surface with \( eg(S') < eg(S') \) and \( cf(S') \leq cf(S) + 1 \) and thus \( S' \prec \prec S \).

If \( S', g \) is not arcwise connected, then \( S' = S \setminus int(U) \) is not connected either. Let \( S_1, S_2 \) be the two arcwise connected components of \( S' \). Then \( eg(S_1) + eg(S_2) = eg(S) \). Moreover, \( z_2 \) is an endpoint of \( h_1 \) and \( y_2 \) is an endpoint of \( h_2 \) and thus \( a_1 \cup h_1 \cup a_2 \cup h_2 \) are two disjoint simple closed curves. One of these curves, say \( a_1 \cup h_1 \), is a cuff of \( S_1 \) and the other is a cuff of \( S_2 \). All other cuffs of \( S_1 \) and \( S_2 \) are cuffs of \( S \). Hence \( cf(S_1) + cf(S_2) = cf(S) + 1 \) and \( cf(S_i) \leq cf(S) \) for \( i = 1, 2 \). Note, we have \( S_i \prec S \), but not necessarily \( S_i \prec S \) for \( i = 1, 2 \).

Next, consider a closed loop \( g \subseteq S \) with attachment point \( x \) in the cuff \( c \). We choose a small closed neighbourhood \( U \subseteq S \) of \( g \) such that \( U \) is homeomorphic to a either a cylinder \( S_{0,2} \) or a Möbius strip \( N_{1,1} \) and that \( U \cap bd(S) \) is a segment \( b \subseteq c \). Let \( y, z \) be the endpoints of \( b \) and \( a \) the segment of \( c \) from \( y \) to \( z \) that does not contain \( x \). If \( U \) is a cylinder, then its boundary in \( \overline{S} \) consists of two simple closed curves \( h_1 \cup b \) and \( h_2 \), where \( h_1 \) is a simple curve with endpoints \( y, z \) and all internal points in \( int(S) \), and \( h_2 \) is a simple closed curve in \( int(S) \). If \( U \) is a Möbius strip, then its boundary in \( \overline{S} \) consists of a simple closed curve \( h \cup b \), where \( h \) is a simple curve with endpoints \( y, z \) and all internal points in \( int(S) \).

Let \( S' := S \setminus int(U) \). If \( S \setminus g \) and thus \( S' \) is arcwise connected, then \( eg(S') < eg(S) \) and \( cf(S') \leq cf(S) + 1 \) and thus \( S' \prec \prec S \). If \( S \setminus g \) and thus \( S' \) is not arcwise connected, then let \( S_1, S_2 \) be the arcwise connected components of \( S' \). In this case, we have \( eg(S_1) + eg(S_2) = eg(S) \) and \( cf(S_1) + cf(S_2) = cf(S) + 1 \) and \( cf(S_i) \leq cf(S) \) for \( i = 1, 2 \). Again, this implies \( S_i \prec S \), but not necessarily \( S_i \prec S \) for \( i = 1, 2 \).

We say that \( g \) is a noncontractible (open or closed) loop if either \( S \setminus g \) is arcwise connected and thus \( S' \) is simpler than \( S \) (this is the case in Figure 15.1(b)) or if both \( S_1 \) and \( S_2 \) are simpler than \( S \).

**Links**

A link is a simple curve in \( S \) with endpoints in two distinct cuffs of \( S \) and all internal points in the interior of \( S \) (see Figure 15.1(c)). The attachment points of a link are its endpoints, and we say that a link is attached to the two cuffs that contain its attachment points. Let \( g \) be a link of \( S \) with attachment points \( x_1 \in c_1, x_2 \in c_2 \). We choose a small closed neighbourhood \( U \subseteq S \) of \( g \) such that \( U \) is homeomorphic to a closed disk and that \( U \cap bd(S) \) consists of two
simplifying curve in a surface $S'$ such that $S'$ is simpler than $S$ and $S \setminus \text{int}(U) \subseteq S'$ for an arbitrarily small closed neighbourhood $U$ of $g$;

(ii) or there are two surfaces $S_1, S_2$ such that $S_1, S_2$ are both simpler than $S$ and $S \setminus \text{int}(U) \subseteq S_1 \cup S_2$ for an arbitrarily small closed neighbourhood $U$ of $g$.

Note that if the curve $g$ is cuff-separating, we are in case (ii), and we even have $S \setminus g \subseteq S_1 \cup S_2$. We call the surface $S'$ in (i) or the surfaces $S_1, S_2$ in (ii) the surface(s) obtained by simplifying $S$ with $g$.

Lemma 15.1.3. Let $S \neq S_{0,0}, S_{0,1}$ be a surface and $G$ a graph that is normally 2-cell embedded in $S$. Then there is a subgraph $Q \subseteq G$, either a path or a cycle, such that $Q$ is either a proper noncontractible curve or a noncontractible loop or a link.

Proof. Let $\overline{S}$ be a surface without boundary such that $S$ is obtained from $\overline{S}$ by deleting the interior of finitely many mutually disjoint closed disk. We let $\overline{G}$ be a graph obtained from $G$ by adding a cycle for every cuff and embedding this cycle on the cuff. We view $\overline{G}$ as a graph embedded in the surface $\overline{S}$. Note that this embedding is 2-cell.
As graphs 2-cell embedded in a surface without boundary are connected, \( \overline{G} \) is connected. Thus if \( S \) has more than one cuff, we can choose a shortest path \( Q \) that connects two cuffs. Then \( Q \) is a link in \( S \).

So let us assume that \( S \) has at most one cuff. Then \( eg(S) > 0 \) by our assumption that \( S \not\cong S_{0,0}, S_{0,1} \). Thus it follows from Fact 9.1.12 that \( \overline{G} \) contains a noncontractible cycle \( Q \). If \( Q \subseteq int(S) \), then \( Q \) is a proper noncontractible curve. So suppose that \( Q \cap bd(S) \neq \emptyset \). Let \( C \subseteq \overline{G} \) be the cycle with \( C = bd(S) \), and let \( D \subseteq S \) be the closed disk such that \( S = \overline{S} \setminus int(D) \). Then \( C = bd(D) \). Let \( Q_1, \ldots, Q_k \) be the \( C \)-bridges in the graph \( Q \cup C \). It may happen that \( k = 1 \) and \( Q_1 = Q \). In this case, \( Q \) is a noncontractible closed loop. Otherwise, for each \( i \in \{1, k\} \) the graph \( Q_i \) is a path, say with endvertices \( x_i, y_i \in V(C) \). I claim that there is an \( i \in \{1, k\} \) such that \( Q_i \) is a noncontractible open loop.

Suppose for contradiction that for each \( i \in \{1, k\} \) the loop \( Q_i \) is contractible. Then there is a disk \( D_i \) such that \( bd(D_i) \) consists of \( Q_i \) and a segment of \( C \) from \( x_i \) to \( y_i \). Now a straightforward induction based on Fact 9.1.9 shows that for every \( i \in \{0, k\} \) there is a closed disk \( D^i \subseteq S \) such that \( D \cup \bigcup_{j=1}^k D_j \subseteq D^i \). As the noncontractible curve \( Q \) is contained in \( D \cup \bigcup_{j=1}^k D_j \), this leads to a contradiction.

\[ \square \]

Remark 15.1.4. It may be worth pointing out that the proof of Lemma 15.1.3 shows that every simple closed curve \( g \subseteq S \) that is noncontractible in \( S \) either is proper noncontractible in \( S \) or contains a noncontractible loop or a link in \( S \). Thus every noncontractible simple closed curve \( g \) in \( S \) either is or contains a simplifying curve.

Definition 15.1.5. The \textit{representativity} of a graph \( G \) embedded in a surface \( S \) is the number

\[ \rho(G) := \min \{|g \cap V(G)| \mid g \subseteq S \text{ is a } G \text{-normal simplifying curve} \}. \]

Note that if \( Q \subseteq G \) is a path or cycle such that \( Q \) is a simplifying curve in \( S \), then there is a \( G \)-normal simplifying curve \( g \subseteq S \) such that \( g \cap V(G) = V(Q) \). Thus the number of vertices of such a path or cycle \( Q \) is an upper bound for the representativity.

15.1.3 Simplifying Almost Embeddable Graphs

We now describe how an almost embedded graph is simplified when we remove a simplifying curve from the surface. Let \((G_0, \pi, R^1, \bar{r}^1, \ldots, R^q, \bar{r}^q)\) be a local \( p \)-arrangement of a graph \( G \) in a surface \( S \). Let \( c^1, \ldots, c^q \) be the cuffs of \( S \). Let \( g \subseteq S \) be a simplifying curve of \( S \) that is \( G_0 \)-normal. Let \( G'_0 := G_0 \setminus g \). We let \( S' \) or \( S_1, S_2 \) be the surfaces obtained by simplifying \( S \) with \( g \). If \( g \) is separating, let \( G_{01} := G'_0 \cap S_1 \) and \( G_{02} := G'_0 \cap S_2 \). For every \( i \in [q] \), let

\[ (R^i)' := R^i \setminus \{v \in \pi(\bar{r}^i) \mid \pi^{-1}(v) \subseteq g\}, \]

and let \((\bar{r}^i)'\) be the tuple obtained from \( \bar{r}^i \) by deleting all vertices in \( g \). Let

\[ G' := \pi(G'_0) \cup \bigcup_{i=1}^q (R^i)' . \]

We call \( G' \) the \textit{\( g \)-reduction} of \( G \) with respect to the arrangement \((G_0, \pi, R^1, \bar{r}^1, \ldots, R^q, \bar{r}^q)\). If \( g \) is separating, let \( G_1 \) be the union of all connected components of \( G' \) that have a nonempty intersection with \( \pi(G_{01}) \), and let \( G_2 \) be the union of the remaining connected components of \( G' \). Then \( G' = G_1 \cup G_2 \). In the following, we will describe \( 2p \)-arrangements of \( G' \) in \( S' \) or of \( G_1, G_2 \) in \( S_1, S_2 \), respectively; if \( g \) is a noncontractible loop we will only be able to obtain such arrangements after removing a bounded number of vertices from the graph(s).
Simplifying with a proper noncontractible curve

Suppose that \( g \subseteq \text{int}(S) \) is a noncontractible simple closed curve. Then all cuffs of \( S \) are also cuffs of \( S' \) (if \( g \) is not separating) or one of the surfaces \( S_1, S_2 \) (if \( g \) is separating). If \( g \) is not separating, then we may view \( G'_0 \) as a graph embedded in the simpler surface \( S' \). We let \( \pi' \) be the restriction of \( \pi \) to \( V(G'_0) \). Then

\[
(G'_0, \pi', (R^1)', (\tau^1)', \ldots, (R^n)', (\tau^n)')
\]

is a \( p \)-arrangement of \( G' \) in \( S' \). If \( S \setminus g \) is not connected, then for \( i = 1, 2 \) we let \( \pi_i \) be restriction of \( \pi \) to \( V(G_{0i}) \). Let \( j_1, \ldots, j_k \) be the indices of the cuffs of \( S \) contained in \( S_i \). Then

\[
(G_{0i}, \pi_i, (R^{j_1})', (\tau^{j_1})', \ldots, (R^{j_k})', (\tau^{j_k})')
\]

is a \( p \)-arrangement of \( G_i \) in \( S_i \).

Simplifying with a cuff-separating curve

We proceed exactly as for separating proper noncontractible curves.

Simplifying with a Noncontractible Loop

Suppose that \( g \) is a noncontractible loop. Without loss of generality we assume that the attachment points of \( g \) are in \( c^1 \). Then the cuffs \( c^2, \ldots, c^l \) are also cuffs of \( S' \) (if \( g \) is not separating) or of one of the surfaces \( S_1, S_2 \) (if \( g \) is separating), but the cuff \( c^1 \) is not. Hence when we define the arrangement of \( G' \) over \( G_1, G_2 \), we have to modify the vortex \( (R^1, \tau^1) \). Without loss of generality we assume that \( g \) is an open loop. If it is a closed loop, we can turn it into an open loop whose attachment points are the attachment point of the closed loop and a second attachment point very close to it. We define \( U \) and \( a_1, a_2, b_1, b_2, h_1, h_2 \) as in Section 15.1.2 for \( c := c^1 \). Let \( c' := a_1 \cup h_1 \cup a_2 \cup h_2 \). Suppose that \( \tau^1 = (r_1, \ldots, r_n) \). Without loss of generality we may assume that the indices are chosen in such a way that

\[
\tau^1 \cap a_1 \cup b_1 = \{r_1, \ldots, r_k\},
\tau^1 \cap a_2 \cup b_2 = \{r_{k+1}, \ldots, r_n\},
\]

for some \( k \in [0, n] \). Here we use the fact that \( U \cap V(G) \subseteq g \) and thus \( b_i \cap a_j \cap V(G) = \emptyset \) for \( i, j = 1, 2 \). Let \( S \subseteq V(R^1) \) such that \( |S| \leq 2p \) and \( S \) separates \( \{\pi(r_1), \ldots, \pi(r_k)\} \) from \( \{\pi(r_{k+1}), \ldots, \pi(r_n)\} \) in \( R^1 \). Let \( S_0 := \pi^{-1}(S \cap \pi(V(G_0))) = \{v \in V(G_0) : \pi(v) \in S\} \). Let \( i_1, \ldots, i_{\ell} \in [k] \) such that \( i_1 < i_2 < \ldots < i_{\ell} \) and

\[
(\tau^1 \cap a_1 \cup b_1) \setminus (S_0 \cup g) = \{r_{i_1}, \ldots, r_{i_\ell}\}.
\]

Similarly, let \( j_1, \ldots, j_m \in [k+1, n] \) such that \( j_1 < j_2 < \ldots < j_m \) and

\[
(\tau^1 \cap a_2 \cup b_2) \setminus (S_0 \cup g) = \{r_{j_1}, \ldots, r_{j_m}\}.
\]

Note that \( \tau^1 \setminus g \subseteq a_1 \cup a_2 \), because \( b_1 \cup b_2 \subseteq U \) and \( U \cap V(G) = g \cap V(G) \). Hence

\[
\{r_{i_1}, \ldots, r_{i_\ell}, r_{j_1}, \ldots, r_{j_m}\} \subseteq a_1 \cup a_2.
\]

Recall the definition of \( (R^1)' \) from (15.1.1) and let

\[
R' := (R^1)' \setminus S.
\]
If \( c' \) is arcwise connected and hence a simple closed curve, let
\[
\overline{r}': = (r_{i_1}, \ldots, r_{i_k}, r_{j_m}, r_{j_{m-1}}, \ldots, r_{j_1}).
\]

Note that the vertices of \( \overline{r}' \) appear in cyclic order on \( c' \) and that \((R', \pi(\overline{r}'))\) is a 2p-vortex. If \( c' \) is not arcwise connected, then its components are the two simple closed curves \( c_1 := a_1 \cup h_1 \) and \( c_2 := a_2 \cup h_2 \). In this case, let \( \overline{r}'_1 := (r_{i_1}, \ldots, r_{i_k}) \) and \( \overline{r}'_2 := (r_{j_m}, r_{j_{m-1}}, \ldots, r_{j_1}) \). Note that for \( i = 1, 2 \), the vertices of \( \overline{r}'_i \) appear in cyclic order on \( c_i \). Let \( R'_1 \) be the union of all connected components of \( R' \) that contain at least one vertex in \( \{r_{i_1}, \ldots, r_{i_k}\} \), and let \( R'_2 \) be the union of the remaining components, that is, \( R'_2 := R' \setminus R'_1 \). Observe that \((R'_1, \pi(\overline{r}'_1))\) and \((R'_2, \pi(\overline{r}'_2))\) are 2p-vortices.

We are now ready to define the desired arrangements for \( G' \) or \( G_1, G_2 \). Let \( \pi', \pi_1, \pi_2 \) be the restrictions of \( \pi \) to \( V(G'_0 \setminus S_0) \), \( V(G_{01} \setminus S_0) \), \( V(G_{02} \setminus S_0) \), respectively.

- **If** \( g \) **is not separating and** \( c' \) **is arcwise connected, then**
\[
(G'_0 \setminus S_0, \pi', R', \overline{r}', R_2, \overline{r}_2, \ldots, R^0, \overline{r}^0)
\]
**is a** \( p \)-**arrangement of** \( G' \setminus S \) **in** \( S' \).

- **If** \( g \) **is not separating and** \( c' \) **is not arcwise connected, then**
\[
(G'_0 \setminus S_0, \pi', R'_1, \overline{r}'_1, R'_2, \overline{r}'_2, R^2, \overline{r}_2, \ldots, R^0, \overline{r}^0)
\]
**is a** \( p \)-**arrangement of** \( G' \setminus S \) **in** \( S' \).

- **If** \( g \) **is separating, then for** \( i = 1, 2 \)
\[
(G_{0i} \setminus S_0, \pi_i, R'_i, \overline{r}'_i, R^2, \overline{r}_2, \ldots, R^0, \overline{r}^0)
\]
**is a** \( p \)-**arrangement of** \( G_i \setminus S \) **in** \( S_i \).

**Simplifying with a Link**

Suppose that \( g \) is a link. Without loss of generality we assume that the endpoints of \( g \) are in \( c^1, c^2 \). Then the cuffs \( c^3, \ldots, c^d \) are also cuffs of \( S' \). For \( i = 1, 2 \), we define the endpoint \( x_i \in c^i \) of \( g \), the segments \( a_i, b_i \) of \( c^i \), and the segment \( h_i \) of \( bd(U) \) as in the Section \[15.1.2\] for \( c_1 := c^i \). Suppose that \( \pi^i = (r^i_1, \ldots, r^i_n) \) for \( i = 1, 2 \). Let \( k_i \in [n_1 + 1] \) such that either \( x_i = r^i_{k_i} \) or \( x_i \) appears between \( r^i_{k_i} \) and \( r^i_{k_i+1} \) on \( c^i \). To avoid tedious case distinctions, let us assume that \( x_1 = r^1_{k_1} \) and that \( x_2 \neq r^2_{k_2} \). Then \( \overline{r}^1 \cap b_1 = \{x_1\} \) and \( \overline{r}^2 \cap b_2 = \emptyset \).

For \( i = 1, 2 \), let \( S^i \subseteq V(R^i) \) such that \( |S^i| \leq 2p \) and \( S^i \) separates \( \{\pi(r^i_1), \ldots, \pi(r^i_n)\} \) from \( \{\pi(r^i_{k_1+1}), \ldots, \pi(r^i_n)\} \) in \( R^i \). To slightly simplify the notation, let us assume that \( S' \cap \pi(\overline{r}^2) = \emptyset \); if this is not the case we have to work with suitable subcups as we did when we discussed the simplification with noncontractible loops. Let \( S := S^1 \cup S^2 \) and \( S_0 := \pi^{-1}(S \cap \pi(V(G_0))) \).

Let \( c' := a_1 \cup h_1 \cup a_2 \cup h_2 \). Then
\[
(\overline{r}^1 \cup \overline{r}^2) \cap c' = (\overline{r}^1 \cup \overline{r}^2) \setminus (g \cup S_0) = \{r^1_1, \ldots, r^1_{k_1-1}, r^1_{k_1+1}, \ldots, r^1_{n_1}, r^2_1, \ldots, r^2_{n_2}\}.
\]

Furthermore, the vertices appear on \( c' \) in one of the following two cyclic orders:
\[
(r^1_1, r^1_2, \ldots, r^1_{k_1-1}, r^2_{k_2}, r^2_{k_2+1}, \ldots, r^2_{n_2}, r^1_1, r^1_2, \ldots, r^1_{k_1-1}, r^1_{k_1+1}, r^1_{k_1+2}, \ldots, r^1_{n_1}, r^2_{k_2}, r^2_{k_2+1}, \ldots, r^2_{n_2}), \quad (15.1.3)
\]
Lemma 15.1.6. Let $G \in \mathcal{AE}_{p,q,r}$, and let $(G_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q)$ be a $p$-arrangement of $G$ in a surface $S$ of Euler genus $r$ with $q$ cuffs. Let $g$ be a $G_0$-normal simplifying curve in $S$, and let $G'$ be the $g$-reduction of $G$ with respect to the arrangement $(G_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q)$.

Then there are $r', q' \in \mathbb{N}$ such that $(r', q') < (r, q)$ and a set $S \subseteq V(G)$ with $|S| \leq 4p$ such that every connected component of $G' \setminus S$ is contained in $\mathcal{AE}_{p', q', r'}$.

15.1.4 Protecting the Vortices

Throughout this subsection, we make the following assumptions:

Assumption 15.1.7. (1) $S \cong S_{0,1}$ is a surface with nonempty boundary, $r := \text{eg}(S)$, and $c^1, \ldots, c^q$ are the cuffs of $S$.

(2) $G$ is a 3-connected graph.

(3) $(G_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q)$ is a local $p$-arrangement of $G$ in $S$, and for all $j \in [q]$ we have $\tau^j = (r^j_1, \ldots, r^j_{n_j})$. Furthermore, we let $r^j_0 := r^j_{n_j}$. We assume that $\pi$ is the identity on $G_0 \setminus \bigcup_{j=1}^q \tau^j$.

(4) The representativity of $G_0$ in $S$ is at least 3.

(5) For all $j \in [q]$ we have $n_j \geq 3$.

(6) $S$ is a surface without boundary and $D^1, \ldots, D^q \subseteq S$ are closed disks such that $S = S \setminus \bigcup_{j=1}^q \text{int}(D^j)$ and $\text{bd}(D^j) = c^j$ for all $j \in [q]$.

Lemma 15.1.8. For all faces $f \in F(G_0)$, the closure $\text{cl}(f)$ is a closed disk. Furthermore, either $\text{Bd}(f)$ is a cycle or there is an $i \in [q]$ such that $\text{Bd}(f)$ is a path or a union of internally disjoint paths with all endvertices in $\tau^i$ and all internal vertices in $G_0 \setminus \bigcup_{j=1}^q \tau^j$.

Proof. For every $j \in [n_i]$, let $b^j_i$ be the segment of the cuff $c^i$ with endpoints $r^i_{j-1}$ and $r^i_j$ and no internal points in $\tau^i$. We construct a new graph $G \supseteq G_0$ embedded in the surface $S$ as follows. For every $i \in [q]$ we add a new vertices $v^i, s^i_1, \ldots, s^i_{n_i}$ and let $s^i_0 := s^i_{n_i}$. For all $j \in [n_i]$ we add edges $v^i r^i_{j-1} s^i_j$ and $r^i_j s^i_{j-1}$ and $s^i_j$. We embed the vertex $s^i_j$ in the interior of the segment $b^i_j$ of $\text{bd}(D^i)$. We embed the edges $r^i_{j-1} s^i_j$ and $s^i_{j-1} r^i_j$ in the segment $b^i_j$. We embed $v^i$ and the edges $v^i s^i_j$ and $r^i s^i_j$ in the disk $D^i$ in such a way that $v^i$ and the interior of the edges are contained in $\text{int}(D^i)$.

Observe that $F(G_0) \subseteq F(G)$. From Assumptions 15.1.7(2) and (5) it follows that $G$ is 3-connected. From Assumption 15.1.7(4) it follows that the representativity of $G$ is at least 3.
Hence $G$ is polyhedrally embedded in $\overline{S}$. Thus the closure of all faces of $G$ and hence of all faces of $G_0$ are closed disks (see the discussion preceding Fact 9.1.17).

Furthermore, the boundaries of all faces of $\overline{G}$ are cycles in $G$. This implies the second assertion of the lemma.

**Definition 15.1.9.** Let $i \in [q]$. A closed disk $\overline{D} \subseteq \overline{S}$ protects the vortex $(R^i, \pi(\overline{r}^i))$ if the following conditions are satisfied:

(i) $D^i \subseteq \overline{D}$.

(ii) There is a cycle $C \subseteq G_0$ such that $\text{bd}(\overline{D}) = C$.

(iii) For every $x \in \text{bd}(\overline{D})$, either $x \in \text{bd}(D^i)$ or there is a simple curve $h_x \subseteq \overline{D} \setminus \text{int}(D^i)$ from $x$ to a point in $\text{bd}(D^i)$ such that $h_x \cap G_0 \subseteq h_x \cap \text{bd}(\overline{D}) = \{x\}$.

(iv) The subgraph $\overline{R}^i := G \left[ V(R^i) \cup \{\pi(v) \mid v \in V(G_0) \cap \overline{D}\} \right]$ of $G$ is connected.

The following lemma is based on similar ideas as Lemma 13.2.4.

**Lemma 15.1.10.** There are mutually disjoint closed disks $\overline{D}^1, \ldots, \overline{D}^q \subseteq \overline{S}$ such that for all $i \in [q]$ the disk $\overline{D}^i$ protects the vortex $(R^i, \pi(\overline{r}^i))$.

**Proof.** Let $i \in [q]$. For every $j \in [n_i]$, let $b_j^i$ be the segment of the cuff $e^i$ with endpoints $r_{j-1}^i$ and $r_j^i$ and no internal points in $\overline{r}^i$. Observe that for every $j \in [n_i]$ there is a unique face $f_j^i \in F(G_0)$ with $b_j^i \subseteq \text{bd}(f_j^i)$. Let $f_j^i$ be this face and $B_j^i := \text{Bd}(f_j^i)$. By Lemma 15.1.8, $\text{cl}(f_j^i)$ is a closed disk and each connected component of $B_j^i$ is a path with both endvertices in $\overline{r}^i$. But note that $B_j^i$ is not necessarily connected because it may happen that there are distinct $j, j' \in [n_i]$ such that $f_j^i = f_j^i$.

**Claim 1.** For every $i \in [q]$ there is a disk $\overline{D}^i \subseteq \overline{S}$ such that $D^i \cup \bigcup_{j=1}^{n_i} \text{cl}(f_j^i) \subseteq \overline{D}^i$ and $\text{bd}(\overline{D}^i) \subseteq \bigcup_{j=1}^{n_i} B_j^i$.

**Proof.** Let $i \in [q]$. Let $R_0 := D^i$ and for all $j \in [n_i]$, let $R_j := R_{j-1} \cup \text{cl}(f_j^i)$. By induction on $j \in [0, n_i]$ we shall prove that there is a disk $D_j \subseteq \overline{S}$ such that $R_j \subseteq D_j$ and $\text{bd}(D_j) \subseteq \bigcup_{k=1}^{n_i} B_k^i \cup \bigcup_{k=j+1}^{n_i} b_k^i$. This will imply the claim.

The base step $j = 0$ is trivial; we simply let $D_0 := D^i$. For the inductive step, suppose that we have defined the disk $D_j$ for some $j < n_i$. Suppose for contradiction that there is no closed disk $D \subseteq \overline{S}$ such that $R_{j+1} \subseteq D$. Then the union of the two closed disks $D_j$ and $\text{cl}(f_{j+1}^i)$ is not contained in a disk in $\overline{S}$ and hence it contains a simple closed curve that is noncontractible in $\overline{S}$. Actually, it is not hard to see that there is a segment $g$ of $\text{bd}(f_{j+1}^i)$ that is internally disjoint from $D_j$ and a segment $h$ of $\text{bd}(D_j)$ that is internally disjoint from $\text{cl}(f_j^i)$ such that $g \cup h$ is a noncontractible simple closed curve in $\overline{S}$. Furthermore, there must be a path $P \subseteq B_j^i$ such that $g = P$. Let $v, w$ be the two endvertices of $P$. Let $g'$ be a simple curve from $v$ to $w$ with all internal points in $f_{j+1}^i$. There are faces $f_v, f_w \in \{f_1^i, \ldots, f_j^i\}$ such that $v \in \text{bd}(f_v)$ and $w \in \text{bd}(f_w)$.

M. Grohe, *Definable Graph Structure Theory*
If \( f_v = f_w \), let \( h' \) be a simple curve from \( v \) to \( w \) with all internal points in \( f_v \). Then \( g' \cup h' \subseteq S \) is a noncontractible simple closed curve in \( S \). As \( V(G_0) \cap (g' \cup h') = \{v, w\} \), this contradicts the representativity of \( G_0 \) being at least 3. Hence \( f_v \neq f_w \).

If \( v \in bd(D^i) \), let \( h_v := \{v\} \); otherwise, let \( h_v \) be a simple curve from \( v \) to a vertex in \( bd(D^i) \setminus \overline{\mathcal{R}} \) with all internal vertices in \( f_v \). Such a curve exists because there is a \( j' \in [n_i] \) such that \( b_{j'} \subseteq bd(f_v) \). Define \( h_w \) analogously. Then \( h_v \cup g' \cup h_w \) is a noncontractible loop in \( S \) with \( \overline{V(G_0)} \cap (h_v \cup g' \cup h_w) = \{v, w\} \). Again this leads to a contradiction.

Thus there is a disk \( D \subseteq S \setminus H \) such that \( R_{j+1} \subseteq D \). To see that we can find such a disk \( D_{j+1} \subseteq D \) with \( bd(D_{j+1}) \subseteq \bigcup_{k=1}^{j+1} B^i_k \cup \bigcup_{k=j+2}^{n_i} b^i_k \), we choose a sphere \( S_0 \supseteq D \) and view \( H := bd(D^i) \cup \bigcup_{k=1}^{j+1} B^i_k \) as the point set corresponding to a 2-connected graph \( H \) embedded in \( S_0 \). We let \( f \) be the face of this embedding that contains \( S_0 \setminus D \) and let \( D_{j+1} := S_0 \setminus f \).

As \( H \) is 2-connected, \( D_{j+1} \) is a closed disk with

\[
bd(D_{j+1}) = bd(f) \subseteq H = \bigcup_{k=1}^{j+1} B^i_k \cup bd(D^i) = \bigcup_{k=1}^{j+1} B^i_k \cup \bigcup_{k=j+1}^{n_i} b^i_k.
\]

For \( k \leq j + 1 \), the segment \( b^i_k \) is contained in the boundary of the faces \( int(D^i) \) and \( f^i_k \) of \( H \). Hence it is not contained in the boundary of \( f \). It follows that \( bd(D_{j+1}) \subseteq \bigcup_{k=1}^{j+1} B^i_k \cup \bigcup_{k=j+2}^{n_i} b^i_k \).

Choose disks \( \overline{D^1}, \ldots, \overline{D^q} \) according to Claim 1. It is easy to verify that for every \( i \in [q] \) the disk \( \overline{D^i} \) protects the vertex \( (R^i, \pi(\overline{\mathcal{R}})) \). Furthermore, the disks are disjoint, because otherwise the representativity of \( G_0 \) would be at most 1.

\section{Shortest Path Systems}

Recall that for a path \( Q \) and vertices \( v, w \in V(Q) \), by \( vQw \) we denote the segment of \( Q \) from \( v \) to \( w \).

**Definition 15.2.1.** Let \( G \) be a graph and \( u, u' \in V(G) \). A shortest path system (sps) from \( u \) to \( u' \) is a family \( \mathcal{Q} \) of shortest paths in \( G \) from \( u \) to \( u' \) such that every shortest path from \( u \) to \( u' \) in the subgraph \( \bigcup_{Q \in \mathcal{Q}} Q \) is contained in \( \mathcal{Q} \).

In the following, let \( \mathcal{Q} \) be an sps from \( u \) to \( u' \) in a graph \( G \). We let \( V(\mathcal{Q}) := \bigcup_{Q \in \mathcal{Q}} V(Q) \) and \( E(\mathcal{Q}) := \bigcup_{Q \in \mathcal{Q}} E(Q) \) and \( G(\mathcal{Q}) := (V(\mathcal{Q}), E(\mathcal{Q})) = \bigcup_{Q \in \mathcal{Q}} Q \). We call \( \mathcal{Q} \) trivial if \( |V(\mathcal{Q})| \leq 2 \), that is, if \( G(\mathcal{Q}) \) consists of a single vertex or a single edge. We call \( u \) the source and \( u' \) the sink of \( \mathcal{Q} \), and we call both \( u \) and \( u' \) the endpoints of \( \mathcal{Q} \). We define a binary relation \( \leq \mathcal{Q} \) on \( V(\mathcal{Q}) \) by letting \( v \leq \mathcal{Q} w \) if and only if \( v = w \) or there is a path \( Q \in \mathcal{Q} \) such that \( v \) precedes \( w \) on \( Q \). Note that \( \leq \mathcal{Q} \) not only depends on the family \( \mathcal{Q} \) of undirected paths, but implicitly also on the source and sink. As the source and sink will always be clear from the context, this will cause no confusion.

**Lemma 15.2.2.** Let \( \mathcal{Q} \) be an sps from \( u \) to \( u' \). Then \( \leq \mathcal{Q} \) is a partial order.

**Proof.** Trivially, \( \leq \mathcal{Q} \) is reflexive. It is antisymmetric, because if \( Q_1, Q_2 \) are shortest paths from \( u \) to \( u' \) and \( v, v' \in V(Q_1) \cap V(Q_2) \) are distinct vertices such that \( v \) precedes \( v' \) on \( Q_1 \) and \( v' \) precedes \( v \) on \( Q_2 \), then either \( uQ_1vQ_2u' \) or \( uQ_2v'Q_1u' \) is a path from \( u \) to \( u' \) that is shorter than \( Q_1 \) and \( Q_2 \).
To see that \( \leq^Q \) is transitive, let \( v, v', v'' \in V(Q) \) such that \( v \leq^Q v' \) and \( v' \leq^Q v'' \). Without loss of generality we may assume that \( v, v', v'' \) are pairwise distinct. Let \( Q \in Q \) such that \( v \) precedes \( v' \) on \( Q \) and \( Q' \in Q \) such that \( v' \) precedes \( v'' \) on \( Q' \). Then the segments \( uQv' \) and \( uQ'v' \) as well as the segments \( v'Qu' \) and \( v'Q'u' \) have the same length. Hence \( Q'' := uQv'Q'u' \) is a shortest path from \( u \) to \( u' \), and as \( Q'' \subseteq Q \cup Q' \subseteq G(Q) \), this implies \( Q'' \in Q \). As \( v \) precedes \( v'' \) on \( Q'' \), it follows that \( v \leq^Q v'' \).

We continue to let \( Q \) be an sps from \( u \) to \( u' \) in \( G \). We define a directed graph \( D^Q \) by letting \( V(D^Q) := V(Q) \) and \( E(D^Q) := \{(v, v') \mid vv' \in E(Q) \text{ and } v \leq^Q v'\} \). Then \( D^Q \) is a directed acyclic graph, and we have \( \leq^Q = \leq^{D^Q} \). The height \( \text{ht}^Q(v) \) and depth \( \text{dep}^Q(v) \) of a node \( v \) are defined as in the directed acyclic graph \( D^Q \), that is, \( \text{ht}^Q(v) \) is the distance from the source \( u \) to \( v \) and \( \text{dep}^Q(v) \) is the distance from \( v \) to the sink \( u' \). The length of \( Q \) is \( \text{len}(Q) := \text{dep}^Q(u) \), that is, the length of all paths in \( Q \). Observe that if \( v, w \in V(Q) \) such that \( vw \in E(Q) \) then \( \text{ht}^Q(v) = \text{ht}^Q(w) + 1 \) or \( \text{ht}^Q(v) = \text{ht}^Q(w) - 1 \).

Elements \( v, w \in V(Q) \) are compatible (with respect to \( Q \)) if either \( v \leq^Q w \) or \( w \leq^Q v \); otherwise they are incompatible. For \( v, w \in V(Q) \) with \( v \leq^Q w \) we let \( Q[v, w] := \{vQw \mid Q \in Q, v, w \in V(Q)\} \). We call the shortest path systems \( Q[v, w] \) for \( v, w \in V(Q) \) with \( \leq^Q \) the segments of \( Q \). A segment \( Q[v, w] \) is proper if \( \{v, w\} \neq \{u, u'\} \).

Note that \( Q \) does not necessarily contain all shortest paths from \( u \) to \( u' \) in \( G \). For all \( u, u' \in V(G) \) such there is a path from \( u \) to \( u' \) in \( G \), the canonical sps from \( u \) to \( u' \) in \( G \) is the set \( Q^G(u, u') \) of all shortest paths from \( u \) to \( u' \) in \( G \). The sps \( Q \) is canonical (in \( G \)) if it is the canonical sps from \( u \) to \( u' \). Note that if \( Q \) is a canonical sps and \( v, w \in V(Q) \) such that \( v \leq^Q w \) then \( Q[v, w] \) is the canonical sps from \( v \) to \( w \).

The vertices in \( \bigcap_{Q \subseteq Q} V(Q) \) are the articulation vertices of \( Q \). The set of all articulation vertices of \( Q \) is denoted by \( \text{art}(Q) \). All articulation vertices except the two endvertices \( u, u' \) are proper articulation vertices.

**Lemma 15.2.3.** Let \( Q \) be an sps. Then \( Q \) is nontrivial and has no proper articulation vertices if and only if the graph \( G(Q) \) is 2-connected.

**Proof.** The backward direction is trivial. For the forward direction, suppose that \( Q \) is nontrivial and has no proper articulation vertices. Then \( |G(Q)| \geq 3 \) by nontriviality. Suppose for contradiction that \( \{s\} \) is a separator of \( H := G(Q) \). Let \( u, u' \) be the endvertices of \( Q \). Clearly, we have \( s \notin \{u, u'\} \). If \( u, u' \) belong to different connected components of \( H \setminus s \) then all shortest paths from \( u \) to \( u' \) in \( H \) contain \( s \). Hence \( s \) is a proper articulation vertex of \( Q \), which is a contradiction. Hence \( u, u' \) belong to the same connected component \( A \) of \( H \setminus s \). But then all shortest paths from \( u \) to \( u' \) in \( H \) are contained in \( H[V(A) \cup \{s\}] \). This means that \( V(H) = V(A) \cup \{s\} \), which contradicts \( \{s\} \) being a separator of \( H \).

**Lemma 15.2.4.** Let \( Q \) be a nontrivial sps that has no proper articulation vertices. Then there are internally disjoint paths \( Q, Q' \in Q \).

Note that this lemma does not follow immediately from Lemma 15.2.3. The 2-connectedness of \( G(Q) \) only implies that there are internally disjoint paths from the source to the sink of \( Q \) in \( G(Q) \), but not that these paths are in \( Q \).

**Proof of Lemma 15.2.4.** The proof is by induction on the length of \( Q \). As \( Q \) is nontrivial, we have \( \text{len}(Q) \geq 2 \). For the base step \( \text{len}(Q) = 2 \), note that there must be at least two
vertices of height 1, because \( Q \) has no proper articulation vertices. This immediately implies the assertion.

For the inductive step, suppose that \( \text{len}(Q) = \ell + 1 \) for some \( \ell \geq 2 \). Let \( u, u' \) be the source and sink of \( Q \), and let \( v' \) be a vertex of height \( \ell \). Then \( v'u' \in E(Q) \). Consider the segment \( Q' := Q[u, v'] \), and let \( v_0 = u, v_1, \ldots, v_n = v' \) be the articulation vertices of \( Q' \) ordered by increasing height (possibly \( n = 1 \) and thus \( v_1 = v' \)). Let \( v := v_1 \). As \( Q \) has no proper articulation vertices, there is a path \( Q \subseteq Q \) such that \( v \notin V(Q) \). Then \( Q \) has an empty intersection with all paths in the segment \( Q'[v, v'] \), because if \( w \in V(Q) \cap V(Q') \) for some \( Q' \in Q'[v, v'] \) then \( uQwQ'v' \) is a path in \( Q' \) that does not contain the articulation vertex \( v \), which is impossible. Choose an arbitrary path \( Q' \in Q'[v, v'] \). Let \( Q'' := Q[u, v] = Q'[u, v] \).

By the choice of \( Q' \), the path \( Q'' \) has no proper articulation vertices. If \( V(Q) \cap V(Q') = \{u\} \), we choose an arbitrary path \( Q'' \in Q' \). Then the paths \( Q \) and \( uQ''vQ'v'u \) are internally disjoint paths in \( Q \). Otherwise, that is, if \( V(Q) \cap V(Q') \supset \{u\} \), the path \( Q'' \) is nontrivial, and thus by the induction hypothesis, it contains internally disjoint paths \( Q_1, Q_2 \). Let \( x \) be the last vertex on \( Q \) that is contained in \( V(Q_1 \cup Q_2) \). Say, \( x \in V(Q_1) \). Then \( uQ_1xQu \) and \( uQ_2vQ'v'u \) are internally disjoint paths in \( Q \).

Let \( Q \) be an sps from \( u \) to \( u' \) in \( G \). Let \( u_0, \ldots, u_m \) be the articulation vertices of \( Q \) sorted by increasing height. Then \( u_0 = u, u_m = u' \). For each \( i \in [m] \), let \( Q_i := Q[u_{i-1}, u_i] = \{u_{i-1}Qu_i \mid Q \in Q\} \). Then \( Q_i \) is an sps from \( u_{i-1} \) to \( u_i \) without proper articulation vertices. We call \( Q_i \) the \( i \)th block of \( Q \).

Observe that the graphs \( G(Q_i) \), for \( i \in [m] \) are the 2-connected components of the graph \( G(Q) \).

Recall Lemma 13.2.7. As an immediate consequence of that lemma and Lemma 15.2.3 we get the following corollary.

**Corollary 15.2.5.** Let \( G \) be a 3-connected graph, and let \( Q \) be a nontrivial sps in \( G \) without proper articulation vertices. Let \( A_1, \ldots, A_m \) be connected components of \( G \setminus V(Q) \) (not necessarily all of them), and let \( G^* := G/A_1/A_2/\cdots/A_m \).

Then \( G^* \) is 3-connected.

We can slightly strengthen the previous corollary by combining it with the next lemma.

**Lemma 15.2.6.** Let \( G \) be a 3-connected graph, and let \( H \subseteq G \) be a 2-connected subgraph. Let \( a_1, \ldots, a_m \in V(G) \setminus V(H) \) such that for each \( i \in [m] \), the graph \( G[a_i] \) is a connected component of \( G \setminus H \). Let \( G^* \) be the graph obtained from \( G \) by identifying the vertices \( a_1, \ldots, a_m \), that is, \( G^* \) is obtained from \( G \setminus \{a_1, \ldots, a_m\} \) by adding a new vertex \( a \) and edges from \( a \) to all \( v \in V(G) \setminus \{a_1, \ldots, a_m\} \) such that there is an \( i \in [m] \) with \( a_i \in V(G) \).

Then \( G^* \) is 3-connected.

**Proof.** Let \( W := \{a_1, \ldots, a_m\} \) and \( G' := G \setminus W \). Since \( H \) is 2-connected and \( G \) is 3-connected, \( G' \) is 2-connected. Moreover, \( |G^*| \geq 4 \) because \( 3 \leq |G'| < |G^*| \). Suppose for contradiction that \( S \) is a separator of \( G^* \) of order at most 2, and let \( v, w \) be in different connected components of \( G^* \setminus S \).

Suppose first that \( a \notin S \). Let \( P \) be a path from \( v \) to \( w \) in \( G \setminus S \). Such a path exists because \( G \) is 3-connected. Let \( v = v_0v_1v_2 \ldots v_nv_n = w \) be the walk corresponding to \( P \). Then \( W \cap \{v_1, \ldots, v_n\} \neq \emptyset \), because there is no path from \( v \) to \( w \) in \( G^* \setminus S \). Choose \( i, j \in [n] \) with \( i < j \) such that \( W \cap \{v_i, \ldots, v_j\} = \emptyset \) and \( W \cap \{v_j, \ldots, v_n\} = \emptyset \) and \( v_{i+1}, v_{j-1} \in W \). Then \( v_0e_1 \ldots e_i v_i \{v_i, a\} a \{a, v_j\} v_j e_{j+1} \ldots e_nv_n \) is a walk from \( v \) to \( w \) in \( G^* \), which is a contradiction.
Hence $a \in S$ and thus $|S \cap V(G')| \leq 1$. As $G'$ is 2-connected, there is a path from $v$ to $w$ in $G' \setminus S$ and hence in $G^* \setminus S$. This is a contradiction. \qed

**Corollary 15.2.7.** Let $G$ be a 3-connected graph, and let $Q$ be a nontrivial sps in $G$ without proper articulation vertices. Let $A_1, \ldots, A_m$ be connected components of $G \setminus V(Q)$, and let $G^* := G/A_1/A_2/\cdots/A_m$. For $i \in [m]$, let $a_i$ be the vertex of $G^*$ corresponding to $A_i$. Let $I_1, \ldots, I_n$ be a partition of $[m]$, and let $G^{**}$ be obtained from $G^*$ by identifying all vertices in $I_1$, all vertices in $I_2$, and et cetera.

Then $G^{**}$ is 3-connected.

**Lemma 15.2.8.** Let $G$ be a graph, $u, u' \in V(G)$, and let $Q_1, Q_2, Q_3 \subseteq G$ be shortest paths from $u$ to $u'$. Let $C \subseteq Q_1 \cup Q_2 \cup Q_3$ be a cycle. Then $||C|| \leq 2 \cdot ||Q_1||$.

**Proof.** Let $H := Q_1 \cup Q_2 \cup Q_3$, and let $Q$ be the sps generated by $Q_1, Q_2, Q_3$. That is, $Q$ is the set of all shortest paths from $u$ to $u'$ in $H$. Remember that every vertex $v$ of $H$ has a well defined height $ht(v) := ht_Q(v)$ and that $H$ only has edges between vertices whose height differs by 1. For an edge $e = vw \in E(H)$, let $ht(e) := \max\{ht(v), ht(w)\}$. Let $\ell := ||Q_1||$. Then for every $i \in \ell$ there are at most three vertices and at most 3 edges of height $i$ (one from each path $P_j$).

Let $k' := \max\{ht(v) \mid v \in V(C)\}$. Then there is a unique vertex $v' \in V(C)$ of height $k'$, because if there were two vertices of height $k'$ there would be four edges of height $k'$. Similarly, there is a unique vertex $v \in V(C)$ of height $k := \min\{ht(v) \mid v \in V(C)\}$. Let $P, P'$ be the two segments of $C$ from $v$ to $v'$. For every $i \in [k + 1, k']$, both $P$ and $P'$ contain at least one edge and one vertex of height $i$. Suppose for contradiction that there an $i$ such that $P$ contains more than one vertex of height $i$. Let $i$ be maximal with this property, and let $w, w' \in V(P)$ be the first and last vertex of height $i$. Then the vertices immediately before and after $w$ and the vertex immediately before $w'$ are all of height $(i - 1)$. Thus $P$ contains three vertices of height $(i - 1)$, and together with a vertex of height $(i - 1)$ in $P'$ this amounts to at least four vertices of height $i - 1$. But there only are three. Hence $P$ and, by symmetry $P'$ contain at most one vertex of height $i$.

Thus $|C| = ||C|| = 2(k' - k) \leq 2\ell$. \qed

We close this section by proving that canonical shortest path systems are IFP-definable. (Recall that for a graph $G$ and vertices $u, u' \in V(G)$ such that there is a path from $u$ to $u'$ in $G$, we denote the canonical sps from $u$ to $u'$ in $G$ by $Q^G(u, u')$.)

**Lemma 15.2.9.** There are IFP-formulae csps-vert$(x, x', y)$ and csps-edge$(x, x', y_1, y_2)$, csps-art$(x, x', y)$ and csps-height$(x, x', y_1, y_2)$ such for all graphs $G$ and vertices $u, u' \in V(G)$ such that there is a path from $u$ to $u'$ in $G$ the following holds:

\[
\begin{align*}
\text{csps-vert}[G, u, u', y] &= V\left(Q^G(u, u')\right), \\
\text{csps-edge}[G, u, u', y_1, y_2] &= E\left(Q^G(u, u')\right), \\
\text{csps-art}[G, u, u', y] &= \text{art}\left(Q^G(u, u')\right), \\
\text{csps-height}[G, u, u', y_1, y_2] &= \left\{(v_1, v_2) \in V\left(Q^G(u, u')\right)^2 \mid \text{ht}^G(u, u')(v_1) \leq \text{ht}^G(u, u')(v_2)\right\}.
\end{align*}
\]

**Proof.** Straightforward. \qed
15.3 Simplifying and Safe Subgraphs

Throughout Sections 15.3 and 15.4, we make Assumption 15.1.7 again. In addition, we make the following assumptions.

**Assumption 15.3.1.**

1. $D_1, \ldots, D_q \subseteq S$ are mutually disjoint closed disks such that for all $j \in [q]$ the disk $D_j$ protects the vortex $(R_j, \tau_j)$. Furthermore, $C_j \subseteq G_0$ is the cycle with $C_j = bd(D_j)$ and $\overline{R}_j := G \left[ V(R_j) \cup \{ \pi(v) \mid v \in V(G_0) \cap D_j \} \right]$.

2. $R$ is a class of graph such that $R$ is closed under taking subgraphs, contains all planar graphs, does not contain $G$, and contains for every $G_0$-normal simplifying curve $g$ in $S$ the $g$-reduction of $G$ with respect to the arrangement $(G_0, \pi, R_1, \tau_1, \ldots, R_q, \tau_q)$.

We think of $R$ as a class of graphs that are “simpler” than $G$ and that we already know how to handle. Let me remark that the class $R$ may depend on the specific graph $G$ and not just on the parameters $p, q, r$. Later, we will assume that we can define ordered treelike decompositions on the graphs in $R$ and show how to define an ordered treelike decomposition on $G$. We assume that all planar graphs are in $R$, that is, “simpler” than $G$.

**Definition 15.3.2.** Let $H \subseteq G$.

1. $H$ is **simplifying** if every connected component of $G \setminus H$ belongs to $R$.

2. $H$ is **safe** if $H \subseteq G_0$ and $H \cap D_i = \emptyset$ for all $i \in [q]$.

To see that safeness is well-defined, recall that by Assumption 15.1.7(3) we have

$$G_0 \setminus \bigcup_{i=1}^{q} \overline{\tau}_i = G_0 \setminus \bigcup_{i=1}^{q} bd(D_i) \subseteq G.$$

Clearly, if there is a cycle or path $Q \subseteq G_0$ such that $Q$ is simplifying curve and $\pi(Q) \subseteq H$ then $H$ is simplifying.

**Lemma 15.3.3.** Let $H \subseteq G$ be safe. Then for every $i \in [q]$ there is a connected component $A$ of $G \setminus H$ such that $\overline{R}_i \subseteq A$.

**Proof.** Let $i \in [q]$. As $H$ is safe, we have $\overline{R}_i \cap H = \emptyset$. As the disk $D_i$ protects the vortex $(R_i, \tau_i)$, the graph $\overline{R}_i$ is connected, and thus there is a connected component $A$ of $G \setminus H$ such that $\overline{R}_i \subseteq A$.

Note that if a subgraph $H \subseteq G$ is not simplifying, then there is a connected component $A^*$ of $G \setminus H$ that is not simpler than $G$. The following lemma implies that there is at most one such component; all other connected components of $G \setminus H$ are planar and hence simpler than $G$. We need additional notation: for subgraph $H \subseteq G$ we let $\pi^{-1}(H)$ be the subgraph of $G_0$ with vertex set

$$V(\pi^{-1}(H)) := \{ v \in V(G_0) \mid \pi(v) \in V(H) \}$$

and edge set

$$E(\pi^{-1}(H)) := \{ vw \in E(G_0) \mid \pi(v)\pi(w) \in E(H) \}.$$
Lemma 15.3.4. Let \( H \) be a safe subgraph of \( G \) that is not simplifying. Let \( A^* \) be a connected component of \( G \setminus H \) such that \( A^* \not\in \mathcal{R} \), and let \( A_0^* := \pi^{-1}(A^*) \). Let \( J_1, \ldots, J_m \) be the connected components of \( A_0^* \). Then \( J_i \subseteq G_0 \) for all \( i \in [m] \). Furthermore, there are mutually disjoint closed disks \( B_1, \ldots, B_m \subseteq S \) such that for each \( i \in [m] \) it holds that \( J_i \subseteq B_i \) and \( B_i \cap A_0^* = \emptyset \).

**Proof.** The \( p \)-arrangement of \( G \) in \( S \) induces a \( p \)-arrangement of \( A^* \) in \( S \).

**Claim 1.** For all \( i \in [q] \) it holds that \( \overline{R}^i \subseteq A^* \).

**Proof.** Let \( i \in [q] \) and suppose for contradiction that \( \overline{R}^i \nsubseteq A^* \). Then it follows from Lemma 15.3.3 that \( R_i \cap A^* = \emptyset \). Since \( S \not\cong S_{0,1} \), the boundary \( C^i \) of \( \overline{D} \) is a cuff-separating curve in \( S \). Thus \( \pi(C^i) \) is a simplifying subgraph of \( G \), and hence \( A^* \subseteq G \setminus \pi(C^i) \in \mathcal{R} \).

Claim 1 implies that \( G \setminus A^* \subseteq G_0 \). (Here we use the assumption that \( \pi \) is the identity on \( G_0 \setminus \bigcup_{i=1}^{q} \overline{R}^i \subset G_0 \setminus \bigcup_{i=1}^{q} \overline{R}^i \).) In particular, for all \( i \in [m] \) it holds that \( J_i \subseteq G_0 \). Moreover, \( J_i \cap \overline{D}^j = \emptyset \) if \( i \in [q] \) because \( J_i \cap \overline{R}^i = \emptyset \).

Let \( i \in [m] \). Viewing \( J_i \) as a graph embedded in the surface \( \overline{S} \) without boundary for a moment, by Fact 9.1.14 there either is a cycle \( C \subseteq J_i \) such that \( C \) is a noncontractible simple closed curve in \( \overline{S} \) or there is a disk \( B \subseteq \overline{S} \) such that \( J_i \subseteq B \). Suppose for contradiction that \( C \subseteq J_i \) is a cycle such that \( C \) is a noncontractible simple closed curve in \( \overline{S} \). Then \( C \) is a noncontractible simple closed curve in \( S \). Hence \( \pi(C) \) is simplifying, and \( A^* \subseteq G \setminus \pi(C) \in \mathcal{R} \), which is a contradiction.

Thus there is a disk \( B \subseteq \overline{S} \) such that \( J_i \subseteq B \). By shrinking the disk \( B \) to a small neighbourhood of the arcwise connected set \( J_i \), this yields the desired closed disk \( B_i \subseteq S \) with \( J_i \subseteq B_i \) and \( B_i \cap A^* = \emptyset \). As each \( J_i \) has an edge to \( A^* \), for \( j \neq i \) the set \( J_j \) belongs to the same arcwise connected component of \( S \setminus J_i \) as \( A^* \). Thus the disks \( B_1, \ldots, B_m \) can be chosen disjoint.

**Corollary 15.3.5.** Let \( H \) be a safe subgraph of \( G \) that is not simplifying. Then there is exactly one connected component \( A^* \) of \( G \setminus H \) such that \( A^* \not\in \mathcal{R} \), and all other connected components of \( G \setminus H \) are planar.

**Corollary 15.3.6.** Let \( H \) be a safe subgraph of \( G \) that is not simplifying. Let \( A^* \) be the connected component of \( G \setminus H \) such that \( A^* \not\in \mathcal{R} \), and let \( G' := G \setminus A^* \). Then there is a closed disk \( D' \subseteq S \) such that \( G' \subseteq \overline{D} \).

**Corollary 15.3.7.** Let \( H \) be a safe subgraph of \( G \) that is not simplifying. Let \( A^* \) be the connected component of \( G \setminus H \) such that \( A^* \not\in \mathcal{R} \). Then for every 2-connected subgraph \( J \subseteq G \setminus A^* \) there is a closed disk \( D \subseteq S \) such that \( J_0 \cap D = J \) and there is a cycle \( C \subseteq J \) such that \( bd(D) = C \).

**Lemma 15.3.8.** Let \( H \) be a safe subgraph of \( G \) that is not simplifying. Let \( A^* \) be the connected component of \( G \setminus H \) such that \( A^* \not\in \mathcal{R} \). Then \( G/A^* \) is planar.

**Proof.** Let \( J_1, \ldots, J_m \) be the connected components of \( G \setminus A^* \), and let \( B_1, \ldots, B_m \subseteq S \) be mutually disjoint closed disks such that \( J_i \subseteq B_i \) and \( B_i \cap A_0^* = \emptyset \) for each \( i \in [m] \). For every edge \( e \) from \( A_0^* \) to \( J_i \) we can choose an entry point \( x \in e \cap bd(B_i) \) such that the segment of \( e \) from \( x \) to its endpoint in \( J_i \) is contained in \( B_i \). Let \( e_1^i, \ldots, e_k^i \) be all edges from \( A_0^* \) to \( J_i \). For each \( j \in [k] \), let \( x_j^i \) be the entry point of \( e_j^i \), and let \( f_j^i \) be the segment of \( e_j^i \) from \( x_j^i \) to the endpoint in \( J_i \).
Let \( a^* \) be the vertex of \( G^* := G/A^* \) corresponding to \( A^* \). Observe that for every \( i \in [m] \) the induced subgraph \( G^*[V(J_i) \cup \{a^*\}] \) is planar: we place the disk \( B_i \) in the plane and the vertex \( a^* \) somewhere outside of \( B_i \). Then we draw internally disjoint curves \( g_i^1, \ldots, g_i^{k_i} \) from \( a^* \) to the entry points \( x_i^1, \ldots, x_i^{k_i} \) on \( bd(B_i) \) and embed the edges from \( a^* \) to \( J_i \) along the curves \( f_i^j \cup g_i^j \).

Since the graphs \( G^*[V(J_i) \cup \{a^*\}] \) for \( i = 1, \ldots, m \) pairwise only share the vertex \( a^* \) and \( G^* \) is the union of these graphs, it follows from Fact 9.1.26 that \( G^* \) is planar.

\[ \Box \]

### 15.4 Patches

We still make assumptions 15.1.7 and 15.3.1 throughout this section.

We call an sps \( Q \) in \( G \) simplifying if the subgraph \( G(Q) \subseteq G \) is simplifying, and we call \( Q \) safe if \( G(Q) \) is safe.

**Definition 15.4.1.** A **patch** in \( G \) is an sps \( Q \) in \( G \) such that:

(i) \( Q \) has no proper articulation vertices.

(ii) \( Q \) is safe.

(iii) There is a closed disk \( D \subseteq S \) such that \( G(Q) \subseteq D \).

Recall that an sps, and hence a patch, is nontrivial if it has more than two vertices. By Lemma 15.2.3 if \( Q \) is a nontrivial patch then the graph \( G(Q) \) is 2-connected.

**Lemma 15.4.2.** Let \( Q \) be a nontrivial patch. Then there is a (unique) closed disc \( D(Q) \subseteq S \) and a cycle \( C(Q) \subseteq G(Q) \) such that \( G(Q) \subseteq D(Q) \) and \( bd(D(Q)) = C(Q) \). Furthermore, for all \( i \in [q] \) it holds that \( \overline{D_i} \cap D(Q) = \emptyset \).

**Proof.** Let \( D' \subseteq S \) be a closed disk with \( G(Q) \subseteq D' \). Let \( S' \supseteq D' \) be a 2-sphere. We may view \( G(Q) \) as a graph embedded in \( S' \). By Fact 9.1.21 the boundaries of all faces of this embedded graph are cycles. Let \( f \) be the face of this embedding that contains \( S' \setminus D' \), and let \( D := S' \setminus f \). Then \( G(Q) \subseteq D \subseteq D' \subseteq S \), and \( bd_S(D) = bd_{S'}(D) = bd_{S'}(f) \) is a cycle in \( G(Q) \).

Let \( i \in [q] \). As \( Q \) is safe, we have \( \overline{D_i} \cap bd(D) = \emptyset \). Furthermore, we have \( \overline{D_i} \not\subseteq D \) because \( D \subseteq S \) and \( D_i \not\subseteq S \). Thus \( \overline{D_i} \cap D = \emptyset \).

\[ \Box \]

**Lemma 15.4.3.** Let \( Q \) be an sps of \( G \) that has no proper articulation vertices and is safe. Then \( Q \) is a patch if and only if there is no cycle \( C \subseteq G(Q) \) such that \( C \) is a noncontractible simple closed curve in \( S \).

**Proof.** Follows from Fact 9.1.14.

\[ \Box \]

Figure 15.3(a) shows the typical structure of a nontrivial patch \( Q \). It consists of several “parallel” shortest path systems with the same endvertices. Each of these is a sequence of smaller patches (the shaded regions in the figure), which may be trivial. Figure 15.3(b) shows the disk \( D(Q) \).

Let \( Q \) be a nontrivial patch of \( G \) and let \( D := D(Q) \). The graph \( G(Q) \) is embedded in \( S \). For simplicity, we call the faces of \( G(Q) \) faces of \( Q \) let \( F(Q) := F(G(Q)) \). In particular, \( S \setminus D \) is a face of \( Q \), the outer face. The cycle \( C(Q) \) is called the outer facial cycle. All other
faces of $Q$ are \textit{inner faces}. As $G(Q)$ is 2-connected, the boundaries of all inner faces are also cycles; we call them the \textit{inner facial cycles} of $Q$. An \textit{angle} of $Q$ is an angle of one of the facial cycles of $Q$, and the set of all angles of $Q$ is denoted by $\angle(Q)$.

\textbf{Lemma 15.4.4.} Let $Q$ be a nontrivial patch of $G$, and let $f$ be a face of $Q$. Then there is a unique vertex $v \in \text{bd}(f) \cap V(Q)$ of minimum height a unique vertex $v' \in \text{bd}(f) \cap V(Q)$ of maximum height. That is, for all $w \in \text{bd}(f) \cap V(Q) \setminus \{v, v'\}$ it holds that $\text{ht}^Q(v) < \text{ht}^Q(w) < \text{ht}^Q(v')$. Furthermore, there are paths $Q, Q' \in Q$ such that $\text{Bd}(f) = vQv' \cup vQ'v'$.

\textit{Proof.} Let $D := D(Q)$ and $S' \supseteq D$ a sphere. We view $G(Q)$ as a graph embedded in $S'$; this does not change the facial cycles. Let $C = \text{Bd}(f)$. Then $C$ is a cycle.

\textit{Claim 1.} $C$ contains a unique vertex $v$ of minimum height.

\textit{Proof.} Suppose for contradiction that there are distinct $v_1, v_2 \in V(C)$ with $h := \text{ht}^Q(v_1) = \text{ht}^Q(v_2) \leq \text{ht}^Q(w)$ for all $w \in V(C)$. Let $P_1, P_2$ be the two segments of $C$ connecting $v_1$ with $v_2$. Let $v_0$ be a greatest lower bound of $v_1, v_2$ with respect to the order $\leq^Q$, and let $Q_1, Q_2 \in Q$ such that $v_0, v_1 \in V(Q_1)$ and $v_0, v_2 \in V(Q_2)$. Then the segments $v_0Q_1v_1$ and $v_0Q_2v_2$ are internally disjoint. Let $P_3 := v_0Q_1v_1 \cup v_0Q_2v_2$ be their union. Then $P_3$ is a path from $v_1$ to $v_2$. Moreover, the path $P_3$ is internally disjoint from $P_1$ and $P_2$, because all vertices in $V(P_1) \cup V(P_2)$ have height at least $h$, whereas all internal vertices of $P_3$ have height less than $h$. Furthermore, $P_3 \cap f = \emptyset$, because $P_3 \subseteq G(Q)$.

Note that both $P_1$ and $P_2$ have length at least 2, because $E(Q)$ contains no edges between vertices of the same height. For $i = 1, 2$, let $w_i \in V(P_i) \setminus \{v_1, v_2\}$ be a vertex of maximum height. Without loss of generality we may assume that $h' := \text{ht}^Q(w_1) \leq \text{ht}^Q(w_2)$. Then $h' > h$. Let $u_0$ be a least upper of $w_1, w_2$. We can use $u_0$ to construct a path $P_4$ from $w_1$ to $w_2$ such that all internal vertices of $P_4$ have height greater than $h'$. Then $P_4$ is disjoint from $P_3$, because all vertices of $P_3$ have height at most $h$ and all vertices of $P_4$ have height at least $h'$. Furthermore, $V(P_4) \cap V(P_1) = \{w_1\}$, because all vertices of $P_1$ have height at most $h$ and all internal vertices of $P_4$ have height greater than $h'$. However, $P_4$ may have internal vertices in $P_2$. Let $w'_2$ be the first vertex of $P_4$ that is in $V(P_2)$ (possibly, $w'_2 = w_2$), and let $P'_4 = w_1P_4w'_2$. Then $P'_4$ is a path from $V(P_1)$ to $V(P_2)$ that is disjoint from $P_3$. Furthermore, $P'_4 \cap f = \emptyset$, because $P'_4 \subseteq G(Q)$. This contradicts Fact 9.1.1 and proves Claim 1. \hfill $\blacksquare$

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) A nontrivial patch $Q$ and (b) the disk $D(Q)$}
\end{subfigure}
\end{figure}
Similarly, we can prove that $C$ contains a unique vertex $v'$ of maximum height.

Let $P_1, P_2$ be the two segments of $C$ from $v$ to $v'$. Let $h := \text{ht}_Q(v)$ and $h' := \text{ht}_Q(v')$. For $i = 1, 2$ we have to prove that there is a path $Q_i \in \mathcal{Q}$ such that $P_i = vQ_i v'$. It suffices to prove that there are no two vertices of the same height on $P_i$, because if $P_i$ contains no two vertices of the same height, then it is a shortest path from $v$ to $v'$ in $G(Q)$, and by the definition of shortest path systems this is sufficient.

**Claim 2.** The vertices of $P_i$ have pairwise different heights.

**Proof.** Suppose for contradiction that $P_1$ contains two vertices of the same height, and choose $v_1, v_2$ of the same height such that $v_1$ is closest to $v$ and $v_2$ is closest to $v_1$ on $P_1$. Let $h_1 := \text{ht}_Q(v_1) = \text{ht}_Q(v_2)$. Then $h < h_1 < h'$. Furthermore, for all $j \in [h, h_1]$ there is exactly one vertex $w \in V(vP_1v_1)$ such that $\text{ht}_Q(w) = j$, and for all $w \in V(v_1P_1v_2) \setminus \{v_1, v_2\}$ it holds that $\text{ht}_Q(w) > h_1$, and for all $w \in V(v_2P_1v')$ it holds that $\text{ht}_Q(w) \geq h_1$. All this follows from the choice of $v_1$ and $v_2$.

Let $v_0$ be the greatest lower bound of $v_1$ and $v_2$, and let $Q_1, Q_2 \in \mathcal{Q}$ such that $v_0, v_1 \in V(Q_1)$ and $v_0, v_2 \in V(Q_2)$. Then the segments $v_0Q_1v_1$ and $v_0Q_2v_2$ are internally disjoint. Let $P_3 := v_0Q_1v_1 \cup v_0Q_2v_2$ be their union. All internal vertices of the path $P_3$ have height less than $h_1$. This implies that no internal vertex of $P_3$ appears on the segment $v_1P_1v'$ of $P_1$, because otherwise there would be a vertex in the segment $vP_1v_1$ and a vertex in the segment $v_1P_1v'$ of the same height, which would contradict the choice of $v_1$. However, there may be internal vertices of $P_3$ in the segment $vP_1v_1$ or on the path $P_2$. Let $v_1'$ be the last vertex of $P_3$ in $V(vP_1v_1 \cup P_2)$ (possibly, $v_1' = v_1$). Let $P_3' := v_1'P_3v_2$. Then $V(C) \cap V(P_3') = \{v_1', v_2\}$. Let $P_1'$ be the segment of $C$ from $v_1'$ to $v_2$ such that $v_1P_1v_2 \subseteq P_1'$, and let $P_2'$ be the other segment of $C$ from $v_1'$ to $v_2$. Then $P_1', P_2', P_3'$ are three internally disjoint paths from $v_1'$ to $v_2$. Note that $v'$ is an internal vertex of $P_3'$. As $\text{ht}_Q(v_1) = \text{ht}_Q(v_2)$ and there are no edges in $E(G(Q))$ between vertices of the same height, the segment $v_1P_1v_2 \subseteq P_1'$ has at least one internal vertex $w$. Let $w' \in V(P_1')$ be a vertex of maximum height. Then $\text{ht}_Q(w') > h_1$.

Let $x'$ be a least upper bound of $w', v'$ with respect to the partial order $\preceq^Q$. Let $Q_v, Q_w \in \mathcal{Q}$ such that $v', x' \in V(Q_v)$ and $w', x' \in V(Q_w)$. Then $P_4 := v'Q_vx' \cup w'Q_wx'$ is a path from $v'$ to $w'$, and all vertices of $P_4$ have height greater than $h_1$. Let $w''$ be the last vertex of $P_4$ in $P_2'$, and let $w'''$ be the first vertex after $v''$ in $P_1'$. Let $P_4' := v''P_4w'''$. Then $P_4'$ is a path from an internal vertex of $P_3'$ to an internal vertex of $P_1'$ that is internally disjoint from $P_1' \cup P_2'$. Furthermore, $P_4'$ is disjoint from $P_3'$, because all vertices of $P_4'$ have height at most $h_1$ and all vertices of $P_3'$ have height greater than $h_1$. As $P_3', P_4' \subseteq G(Q)$, we have $P_3' \cap f = P_4' \cap f = \emptyset$.

Again, this contradicts Fact 9.1.1. \(\square\)

Completely analogously, we can prove that the vertices of $P_2$ have pairwise different heights, and as pointed out above, this completes the proof of the lemma.

15.4.1 Non-Simplifying Patches

In this section, we will study non-simplifying patches, that is, patches $Q$ such that the graph $G(Q) \subseteq G$ is not simplifying in the sense of Definition 15.3.2. Note that, by Corollary 15.3.5, if $Q$ is a non-simplifying patch then there is a unique connected component $A^*(Q)$ of $G[V(Q)]$ such that $A^*(Q) \not\subseteq \mathcal{R}$, which intuitively means that $A^*(Q)$ is not simpler than $G$. Furthermore, all other components of $G \setminus V(Q)$ are planar.

**Corollary 15.4.5.** Let $Q$ be a non-simplifying patch and $A^* := A^*(Q)$. Then $G \setminus A^*$ is planar. Furthermore, $G \setminus A^*$ is either 3-connected or a triangle.
Proof. The planarity follows from Lemma 15.3.8. If $Q$ is nontrivial then 3-connectedness follows from Corollary 15.2.5. Suppose $Q$ is trivial, that is, consists of a single edge. Then the 3-connectedness of $G$ implies that $A^*$ is the only connected component of $G \setminus V(Q)$ and that $G/A^*$ is a triangle.

Let $Q$ be a nontrivial non-simplifying patch and $A^* := A^*(Q)$. Here and in the following, we let $a^* := a^*(Q)$ be the vertex of $G/A^*$ corresponding to $A^*$. By Corollary 15.4.5 and Whitney’s Theorem (Fact 9.1.23), $G/A^*$ has a unique embedding $\Pi$ in the sphere (up to homeomorphism). Let $\Pi'$ be the restriction of $\Pi$ to the subgraph $G' := G \setminus A^*$. As $G' = (G/A^*) \setminus \{a^*\}$ is 2-connected, all $\Pi'$-facial subgraphs are cycles. Let $f^*$ be the face of $\Pi'$ that contains $\Pi(a^*)$, and let $C^* := C^*(Q) := Bd_{\Pi'}(f^*)$. We call $C^*$ the outer cycle of $G'$. Note that $C^*$ does not depend on the choice of the embedding $\Pi$, and that, up to homeomorphism, $G'$ has a unique embedding in a closed disk that maps $C^*$ to the boundary of the disk.

Corollary 15.4.6. Let $Q$ be a nontrivial non-simplifying patch and $A^* := A^*(Q)$. Let $G' := G \setminus A^*$, and let $C^*$ be the outer cycle of $G'$. Then there is a closed disk $D^* := D^*(Q) \subseteq S$ such that $C^* = \partial D^*$ and $G_0 \cap D^* = G'$.

Proof. This follows from Corollary 15.3.7.

Let $Q$ be a nontrivial non-simplifying patch and $A^* := A^*(Q)$, $D^* := D^*(Q)$. Let $G' := G \setminus A^*$. Note that the faces of $G'$ viewed as a graph embedded in $S$ are $S \setminus D^*$ and all faces $f \in F(G_0)$ with $f \subseteq D^*$. We call $S \setminus D^*$ the outer face of $G'$ and the faces $f \in F(G_0)$ with $f \subseteq D^*$ the inner faces. Note that we distinguish between the outer and inner faces of $Q$ and the outer and inner faces of $G'$. The two sets of faces coincide if $G(Q) = G'$, that is, if $A^*$ is the only connected component of $G \setminus V(Q)$.

Lemma 15.4.7. Let $Q$ be a nontrivial non-simplifying patch and $A^* := A^*(Q)$. Then a cycle $C \subseteq G \setminus A^*$ is an inner facial cycle of $G \setminus A^*$ if and only if it is chordless and nonseparating in $G/A^*$.

Proof. Let $G' := G \setminus A^*$, $C^* := C^*(Q)$ and $D^* := D^*(Q)$.

By Whitney’s Theorem (Fact 9.1.23), the 3-connected planar graph $G/A^*$ has a unique embedding in the sphere, up to homeomorphism. It follows that, again up to homeomorphism, the graph $G'$ has a unique embedding in a closed disk $D^*$ that maps the outer cycle $C^*$ to the boundary of the disk. Thus a cycle $C \subseteq G \setminus A^*$ is a facial cycle of $G/A^*$ when embedded to the sphere if and only if it is a facial cycle of $G'$, viewed as a graph embedded in the closed disk $D^* \subseteq S$. By Fact 9.1.22, these facial cycles are precisely the chordless and nonseparating cycles of $G/A^*$.

As usual, the 1FP-formulae defined in the following lemmas do not depend on the graph $G$. Actually, they do not even depend on the parameters $p$, $q$, $r$, even though it would not be a problem if they did.

Lemma 15.4.8. There is an 1FP-formula $a^*-\text{vert}(x, x', y)$ (not depending on $G$ and $R$) such that for all $u, u' \in V(G)$ the following holds: If the canonical sps $Q := Q^G(u, u')$ from $u$ to $u'$ is a non-simplifying patch then

$$a^*-\text{vert}[G, u, u', y] = V(A^*(Q)).$$

M. Grohe, Definable Graph Structure Theory
Lemma 15.4.10. There are IFP-formulae defined in Lemmas 15.4.10, 15.4.12, 15.4.15, 15.4.20, 15.4.23, 15.5.3, 15.5.7 and Corollaries 15.4.11, 15.4.13, 15.4.14 (that is, all IFP-formulae defined in this chapter).

Remark 15.4.9. It will be important later that the IFP-formula in Lemma 15.4.8 neither depend on the specific graph $G$ nor on the specific surface $S$, the local $p$-arrangement of $G$ in $S$, the surface $S'$, or the disks $D_i$, as long as they satisfy Assumptions 15.1.7 and 15.3.1. The same holds for the IFP-formulae defined in Lemmas 15.4.10, 15.4.12, 15.4.15, 15.4.20, 15.4.23, 15.5.3, 15.5.7 and Corollaries 15.4.11, 15.4.13, 15.4.14 (that is, all IFP-formulae defined in this chapter).

Lemma 15.4.10. There are IFP-formulae
\[
\text{in-angle}(x, x', y_1, y_2), \quad \text{in-aligned}(x, x', y_1, y_2, y_3, y_4),
\]
\[
\text{out-angle}(x, x', y_1, y_2), \quad \text{out-aligned}(x, x', y_1, y_2, y_3, y_4)
\]
(not depending on $G$ and $R$) such that for all $u, u' \in V(G)$ the following holds. If the canonical sps $Q := Q^G(u, u')$ from $u$ to $u'$ is a nontrivial non-simplifying patch then for all $v_1, v_2, v_3, v_4 \in V(G)$,
\[
G \models \text{in-angle}[u, u', v_1, v_2, v_3] \iff (v_1, v_2, v_3) \text{ is an angle of some inner face of } G',
\]
\[
G \models \text{in-aligned}[u, u', v_1, v_2, v_3, v_4] \iff (v_1, v_2, v_3, (v_2, v_3, v_4) \text{ are aligned angles of an inner face of } G',
\]
\[
G \models \text{out-angle}[u, u', v_1, v_2, v_3] \iff (v_1, v_2, v_3) \text{ is an angle of the outer face of } G',
\]
\[
G \models \text{out-aligned}[u, u', v_1, v_2, v_3, v_4] \iff (v_1, v_2, v_3, (v_2, v_3, v_4) \text{ are aligned angles of the outer face of } G'.
\]

Proof. Follows from Lemma 15.4.8 and Lemma 9.3.6 (the definability of the angles of a 3-connected planar graph) by means of the Transduction Lemma (Fact 2.4.6).

Corollary 15.4.11. There are IFP-formulae
\[
c^*-\text{vert}(x, x', y), \quad c^*-\text{edge}(x, x', y_1, y_2)
\]
(not depending on $G$ and $R$) such that for all $u, u' \in V(G)$ the following holds. If the canonical sps $Q := Q^G(u, u')$ from $u$ to $u'$ is a nontrivial non-simplifying patch then for all $v_1, v_2 \in V(G)$,
\[
G \models c^*-\text{vert}[u, u', v] \iff v \in V(C^*(Q)),
\]
\[
G \models c^*-\text{edge}[u, u', v_1, v_2] \iff v_1v_2 \in E(C^*(Q)).
\]

With a little additional effort, we can also prove that the outer facial cycle of $Q$, and not only $G \setminus A^*(Q)$, is also IFP-definable:

Lemma 15.4.12. There are IFP-formulae
\[
b-\text{vert}(x, x', y), \quad x-\text{vert}(x, x', y), \quad i-\text{vert}(x, x', y),
\]
\[
b-\text{edge}(x, x', y_1, y_2), \quad x-\text{edge}(x, x', y_1, y_2), \quad i-\text{edge}(x, x', y_1, y_2)
\]
Proof. Let \( G \) and \( \mathcal{R} \) be a nontrivial nonsimplifying patch in \( G \) such that for all \( u, u' \in V(G) \) the following holds. If the canonical sps \( Q := Q^G(u, u') \) from \( u \) to \( u' \) is a nontrivial nonsimplifying patch then for all \( v, v_1, v_2 \in V(G) \),

\[
\begin{align*}
G \models b\text{-}vert[u, u', v] & \iff v \in V(C(Q)) = V(G_0 \cap \partial(D(Q))), \\
G \models x\text{-}vert[u, u', v] & \iff v \in V(G \setminus (G_0 \cap D(Q))), \\
G \models i\text{-}vert[u, u', v] & \iff v \in V(G_0 \cap \text{int}(D(Q))), \\
G \models b\text{-}edge[u, u', v_1, v_2] & \iff v_1v_2 \in E(C(Q)), \\
G \models x\text{-}edge[u, u', v_1, v_2] & \iff v_1v_2 \in E(G \setminus (G_0 \cap D(Q))), \\
G \models i\text{-}edge[u, u', v_1, v_2] & \iff v_1v_2 \in E(G_0 \cap \text{int}(D(Q))).
\end{align*}
\]

Observe that if \( u, u' \in V(G) \) is a nontrivial nonsimplifying patch then for all \( v, v_1, v_2 \in V(G) \),

\[
\begin{align*}
G \models b\text{-}vert[u, u', v] & \iff v \in V(C(Q)) = V(G_0 \cap \partial(D(Q))), \\
G \models x\text{-}vert[u, u', v] & \iff v \in V(G \setminus (G_0 \cap D(Q))), \\
G \models i\text{-}vert[u, u', v] & \iff v \in V(G_0 \cap \text{int}(D(Q))), \\
G \models b\text{-}edge[u, u', v_1, v_2] & \iff v_1v_2 \in E(C(Q)), \\
G \models x\text{-}edge[u, u', v_1, v_2] & \iff v_1v_2 \in E(G \setminus (G_0 \cap D(Q))), \\
G \models i\text{-}edge[u, u', v_1, v_2] & \iff v_1v_2 \in E(G_0 \cap \text{int}(D(Q))).
\end{align*}
\]

Proof. Let \( u, u' \in V(G) \) such that \( Q := Q^G(u, u') \) is a nontrivial nonsimplifying patch. Let \( G' := G \setminus A^*(Q) \), \( C := C(Q) \), \( D := D(Q) \), \( C^* := C^*(Q) \) and \( D^* := D^*(Q) \). As \( D \subseteq D^* \), a point \( x \in D \) belongs to \( C = \partial(D) \) if and only if \( x \in \partial(D^*) \) or there is a simple curve \( g \subseteq D^* \) from a point in \( C^* = \partial(D^*) \) to \( x \) with \( g \cap D = g \cap C = \{ x \} \). Hence a vertex \( v \in V(Q) \) is in \( V(C) \) if and only if there is a path \( P \) from a vertex \( w \in V(C^*) \) to \( v \) (possibly of length 0) such that \( V(P) \cap V(Q) = \{ v \} \). Using this observation and the formula \( \text{c}^*\text{-vert}(x, x', y) \) of Corollary \( \text{15.4.11} \), we can easily construct an \text{IFP}-formula \( b\text{-vert}(x, x', y) \) such that \( b\text{-vert}[G, u, u', y] = V(C) \). The formulae \( x\text{-vert}(x, x', y) \) and \( i\text{-vert}(x, x', y) \) can be constructed similarly.

It is slightly more difficult to define the formulae for the edges. We do this by “peeling of” faces of \( G' \) starting from the outer face \( f^* \) until we reach the boundary of \( D \). We let \( F_0 := \{ f^* \} \) and \( E_0 := E(\partial(f^*)) = E(C^*) \). For \( i \geq 0 \), we let \( F_{i+1} \) be the union of \( F_i \) with the set of all faces \( f \in F(G') \) such that there is an edge \( e \in \partial(f) \) with \( e \in E_i \setminus E(Q) \), and we let \( E_{i+1} := \bigcup_{f \in F_{i+1}} E(\partial(f)) \). We let \( F_{\infty} := \bigcup_{i \geq 0} F_i \) and \( E_{\infty} := \bigcup_{i \geq 0} E_i \). It is not hard to see that \( F_{\infty} \) is the set of all face \( f \in F(G') \) with \( f \cap D = \emptyset \) and that \( E_{\infty} \) is the set of all boundary edges of these faces. Hence

\[
E_{\infty} = E\left(G_0 \cap (D^* \setminus \text{int}(D^*))\right).
\]

Using the previous lemmas, it is easy to formalise the inductive definition of \( E_{\infty} \) in \text{IFP}. Observe that

\[
\begin{align*}
E(C(Q)) &= E_{\infty} \cap E(Q), \\
E\left(G \setminus (G_0 \cap D(Q))\right) &= E_{\infty} \setminus E(Q) \cup \{ vw \in E(G) \mid \{ v, w \} \cap V(A^*) \neq \emptyset \}, \\
E\left(G_0 \cap \text{int}(D(Q))\right) &= E(G) \setminus \left( E(C(Q)) \cup E\left(G \setminus (G_0 \cap D(Q))\right)\right).
\end{align*}
\]

Combined with the previous lemma, these observations yield the desired formulae for the edge sets. \( \square \)

Let \( Q \) be a non-simplifying patch in \( G \). The internal graph of \( Q \) is the graph

\[
I(Q) := \begin{cases} 
G_0 \cap D(Q) & \text{if } Q \text{ is nontrivial,} \\
G(Q) & \text{if } Q \text{ is trivial.}
\end{cases}
\]

Observe that if \( Q \) is nontrivial then \( I(Q) \) is a 2-connected planar graph embedded in the surface \( S \).

The following two corollaries follow immediately from Lemma \( \text{15.4.12} \) and Lemma \( \text{15.4.10} \).
Corollary 15.4.13. There are IFP-formulae

\[ \text{int-vert}(x, x', y), \quad \text{int-edge}(x, x', y_1, y_2) \]

(not depending on \( G \) and \( R \)) such that for all \( u, u' \in V(G) \) the following holds. If the canonical sps \( Q := Q^G(u, u') \) from \( u \) to \( u' \) is a non-simplifying patch then for all \( v, v_1, v_2 \in V(G) \),

\[
G \models \text{int-vert}[u, u', v] \iff v \in V(I(Q)),
G \models \text{int-edge}[u, u', v_1, v_2] \iff v_1v_2 \in E(I(Q)).
\]

Corollary 15.4.14. There are IFP-formulae

\[ \text{int-angle}(x, x', y_1, y_2, y_3), \quad \text{int-aligned}(x, x', y_1, y_2, y_3, y_4) \]

(not depending on \( G \) and \( R \)) such that for all \( u, u' \in V(G) \) the following holds. If the canonical sps \( Q := Q^G(u, u') \) from \( u \) to \( u' \) is a nontrivial non-simplifying patch then for all \( v_1, v_2, v_3, v_4 \in V(G) \),

\[
G \models \text{int-angle}[u, u', v_1, v_2, v_3] \iff (v_1, v_2, v_3) \text{ is an angle of some face of } I(Q),
G \models \text{int-aligned}[u, u', v_1, v_2, v_3, v_4] \iff (v_1, v_2, v_3, v_2, v_3, v_4) \text{ are aligned angles of some face of } I(Q).
\]

Lemma 15.4.15. There is an IFP-formula \( \text{int-ord}(x, x', x'', y_1, y_2) \) (not depending on \( G \) and \( R \)) such that for all \( u, u', u'' \in V(G) \) the following holds. If the canonical sps \( Q := Q^G(u, u') \) from \( u \) to \( u' \) is a non-simplifying patch and \( u'' \in V(C(Q)) \setminus \{u, u'\} \), or \( u'' = u' \) if \( Q \) is trivial, then

\[ \text{int-ord}[G, u, u', u'', y_1, y_2] \]

is a linear order of \( V(I(Q)) \).

\[ \square \]

Proof. Without loss of generality we assume that \( Q \) is nontrivial. By Lemma 15.4.12, we can define the vertex set and edge relation of the cycle \( C := C(Q) \), which is a facial cycle of the internal graph \( I(Q) \). Let \( P \) be the segment of \( C \) from \( u \) to \( u' \) that contains \( u'' \). Let \( v_1 = u, v_2, v_3 \) be the first three vertices on \( P \). Then \( (v_1, v_2, v_3) \) is an angle of the facial cycle \( C \). It is easy to construct an IFP-formula \( \text{first-angle}(x, x', x'', x_1, x_2, x_3) \) that defines this angle, that is, \( \text{first-angle}(G, u, u', u'', x_1, x_2, x_3) = \{v_1, v_2, v_3\} \). Now can apply the First Angle Lemma 9.2.3 with the formulae \( \text{int-angle}(x, x', y_1, y_2, y_3) \) and \( \text{int-aligned}(x, x', y_1, y_2, y_3, y_4) \) of Corollary 15.4.14.

15.4.2 Simplifying Patches

Definition 15.4.16. Let \( Q \) be a patch in \( G \).

1. Let \( Q' = Q[v, v'] \) be a segment of \( Q \). If \( Q' \) is also a patch, we call it a subpatch of \( Q \), and if in addition \( Q' \neq Q \), we call it a proper subpatch.

2. \( Q \) is a minimal simplifying patch if \( Q \) is simplifying, but all proper subpatches of \( Q \) are non-simplifying.

Observe that a segment \( Q' \) is a subpatch if and only if it has no proper articulation vertex.
Lemma 15.4.17. Let $Q$ be a simplifying patch, and let $Q'$ be a subpatch of $Q$ that is nontrivial and non-simplifying.

1. All inner faces of $Q'$ are also inner faces of $Q$.
2. The outer face of $Q'$ is not a face of $Q$.

Proof. Statement (1) is trivial, because $D(Q') \subseteq D(Q)$ and $G(Q') = D(Q') \cap G(Q)$.

To prove (2), suppose for contradiction that the outer face of $Q'$ is also a face of $Q$. Then the component $A^*(Q')$, which is attached to the outer face of $Q'$, is also a connected component of $G \setminus V(Q)$. But this means that $Q$ is non-simplifying, a contradiction. \qed

Let $Q$ be a patch in $G$. The internal region of $Q$ is the set

$$R(Q) := \bigcup_{e \in E(Q)} e \cup \bigcup_{Q' \text{ nontrivial non-simplifying subpatch of } Q} D(Q').$$

Note that $R(Q) \subseteq D(Q)$ and that there is a subgraph $B(Q) \subseteq G(Q)$ such that $\beta_d(R(Q)) = B(Q)$. If $Q$ is a nontrivial non-simplifying patch then $R(Q) = D(Q)$ and $B(Q) = C(Q)$. If $Q$ is a trivial patch then $R(Q) = B(Q) = G(Q)$. The internal graph for $Q$ is the graph

$$I(Q) := G_0 \cap R(Q).$$

Note that for non-simplifying patches $Q$, this coincides with the definition of $I(Q)$ given in the previous subsection. A face $f \in F(I(Q))$ is internal if $f \subseteq R(Q)$; otherwise $f$ is external. In this subsection it will be more convenient to work with bridges (cf. p. 22) of $G(Q)$, which we call $Q$-bridges, than with connected components of $G \setminus V(Q)$. The reason is that $G(Q)$ is not necessarily an induced subgraph of $G$ and thus there may be bridges that consist of a single edge in $E(G) \setminus E(Q)$; such bridges do not correspond to any connected components of $G \setminus V(Q)$. A $Q$-bridge $H$ is internal if $H \subseteq I(Q)$; otherwise, it is external. Observe that $I(Q)$ is the union of $G(Q)$ with all internal $Q$-bridges. A connected component $A$ of $G \setminus V(Q)$ is internal if it is contained in an internal bridge, or equivalently, if $A \subseteq \text{int}(R(Q))$, and it is external otherwise. We let $I^*(Q)$ be the graph obtained from $G$ by contracting all external connected components of $Q$ to single vertices. That is, if $A_1, \ldots, A_m$ are the external components of $G \setminus V(Q)$, then

$$I^*(Q) := G/A_1/\cdots/A_m.$$ 

Observe that if $Q$ is trivial, then $I(Q) = G(Q)$ is a single edge, and all connected components of $G \setminus V(Q)$ are external. Then $m = 1$, because $G$ is 3-connected and not planar. Hence $I^*(Q)$ is a triangle.

Lemma 15.4.18. Let $Q$ be a patch. Then $I^*(Q)$ is embeddable in $S$ and thus $I^*(Q) \in \mathcal{E}_r$.

Proof. Let $A_1, \ldots, A_m$ be the external connected components of $G \setminus V(Q)$. Since $Q$ is safe, for each $i \in [q]$ it holds that $R(Q) \cap D_i = \emptyset$ and thus $V(I(Q)) \cap V(\overline{R}) = \emptyset$. Hence there is a $j \in [m]$ such that $\overline{R}_j \subseteq A_j$. It follows that $I^*(G, Q)$ is a minor of the graph $G^* := (\cdots((G/\overline{R}_1)/\overline{R}_2)/\cdots)/\overline{R}_m)$, which is embeddable in $\overline{S}$ and thus in $S$. \qed

Lemma 15.4.19. Let $Q$ be a nontrivial patch. Then $I(Q)$ is a 2-connected planar graph embedded in $S$, and it holds that $I(Q) \subseteq D(Q)$.
Proof. $I(Q)$ is 2-connected, because $G(Q)$ is 2-connected and $G$ is 3-connected, and thus the union of $G(Q)$ with some of its bridges is 2-connected. As $I(Q) \subseteq G_0$, it is embedded in $S$, and we have $I(Q) \subseteq R(Q) \subseteq D(Q)$. Hence $I(Q)$ is planar.

Lemma 15.4.20. There are IFP-formulae int-vert($x, x', y$) and int-edge($x, x', y_1, y_2$) (not depending on $G$ and $R$) such that for all $u, u' \in V(G)$ the following holds. If the canonical sps $Q := Q^G(u, u')$ from $u$ to $u'$ is a patch then for all $v, v_1, v_2 \in V(I(Q))$,

$$G \models \text{int-vert}[u, u', v] \iff v \in V(I(Q)),$$

$$G \models \text{int-edge}[u, u', v_1, v_2] \iff v_1 v_2 \in E(I(Q)).$$

Proof. This follows directly from the definition of $I(Q)$ and Corollary 15.4.13.

Lemma 15.4.21. Let $Q$ be a nontrivial minimal simplifying patch from $u$ to $u'$, and let $f$ be a face of $I(Q)$. Then $f$ is external if and only if $u, u' \in \text{bd}(f)$.

Furthermore, if $f$ is external then there is a cycle $C \subseteq G(Q)$ such that $\text{bd}(f) = C$.

Proof. For the forward direction, suppose that $f$ is external. Then $\text{Bd}(f) \subseteq B(Q) \subseteq G(Q)$. As $I(Q)$ is 2-connected, there is a cycle $C \subseteq G(Q)$ such that $C = \text{bd}(f)$. Suppose for contradiction that $\{u, u'\} \not\subseteq V(C)$. By Lemma 15.4.4 there is a unique vertex $v \in V(C)$ of minimum height and a unique vertex $v' \in V(C)$ of maximum height, and the two segments of $C$ from $v$ to $v'$ are segments of paths in $Q$. But then $Q' := Q[v, v']$ is a proper subpatch of $Q$ and $f$ a face of $Q'$. As $Q$ is minimal simplifying, $Q'$ is non-simplifying. By Lemma 15.4.17, $f$ is an inner face of $Q'$, which is a contradiction.

For the backward direction, just note that no proper subpatch $Q'$ of $Q$ contains both $u$ and $u'$. Therefore, no face of $Q'$ has both $u$ and $u'$ in its boundary.

Lemma 15.4.22. Let $Q$ be a nontrivial minimal simplifying patch from $u$ to $u'$, and let $I := I(Q)$. Let $A_1, \ldots, A_m$ be the connected components of $I \setminus \{u, u'\}$, and for each $i \in [m]$, let $H_i := I[V(A_i) \cup \{u, u'\}]$.

(1) If $I \setminus \{u, u'\}$ is connected, then $R(Q) = D(Q)$, and the only external face of $I$ is $S \setminus D(Q)$, the outer face of $Q$.

(2) If $I \setminus \{u, u'\}$ is not connected, then there is a permutation $\pi$ of $[m]$ and an enumeration $f_0, \ldots, f_{m-1}$ of the external faces of $I$ such that $\text{Bd}(f_0) \subseteq H_{\pi^{-1}(m)} \cup H_{\pi^{-1}(1)}$, and $\text{Bd}(f_i) \subseteq H_{\pi^{-1}(i)} \cup H_{\pi^{-1}(i+1)}$ for all $i \in [m - 1]$.

As an illustration of the lemma, recall Figure 15.3(a) (on page 378). The four sequences of subpatches, indicated by the shaded regions of the Figure, correspond to the graphs $H_i$.

Proof. Let $D := D(Q)$, $R := R(Q)$, $C := C(Q)$, and $B := B(Q)$. Let $f_0 := S \setminus D$ be the outer face of $Q$. As $f_0$ is external, by Lemma 15.4.21 we have $u, u' \in V(C) = \text{bd}(f_0) \cap V(G)$. All other faces, external and internal, are contained in $D$. Let $f_1, \ldots, f_{\ell-1}$ be the external faces of $I$ contained in $D$. As $u, u' \in \text{bd}(f_i)$ for each $i \in [1, \ell - 1]$ by Lemma 15.4.4 there is a simple curve $g_i \subseteq D$ with endpoints $u, u'$ and all internal points in $f_i$. By Lemma 15.4.4 there are paths $Q, Q' \in Q$ such that $C = Q \cup Q'$. Thus $\text{bd}(D) = C = Q \cup Q'$. As $Q$ is nontrivial, the paths $Q, Q'$ have length at least 2. Hence there are vertices $w \in V(Q) \setminus \{u, u'\}$ and $w' \in V(Q') \setminus \{u, u'\}$. To simplify the notation, we let $g_0 := Q$ and $g_\ell := Q'$.
To prove (1), suppose that \( I(\mathcal{Q}) \setminus \{u, u'\} \) is connected. Then there is a path \( P \subseteq I(\mathcal{Q}) \setminus \{u, u'\} \) from \( w \) to \( w' \), and \( P \) is a simple closed curve from \( w \) to \( w' \) in \( D \setminus \{u, u'\} \). Suppose for contradiction that \( \ell \geq 2 \). Then \( g_1 \) is a simple closed curve in \( D \) from \( u \) to \( u' \). As \( P \subseteq I \setminus \{u, u'\} \) and \( g \cap I = \{u, u'\} \), the curves \( P \) and \( g_1 \) are disjoint, and this contradicts Corollary \ref{cor:9.1.2}.

Hence \( \ell = 1 \), that is, \( f_0 = S \setminus D \) is the only external face of \( I \). This immediately implies that all faces of \( I \) in \( D \) are internal and thus that \( R = D \).

Let us turn to the proof of (2). It is not hard to see, by repeatedly applying Fact \ref{fact:9.1.1} that \( D \setminus (g_1 \cup \ldots \cup g_{\ell-1}) \) has \( \ell \) arcwise connected components \( F_1, \ldots, F_\ell \) and that the indices can be chosen in such a way that \( bd(F_i) = g_{i-1} \cup g_i \) for all \( i \in [\ell] \). Note the irregularity at the boundary: We have \( g_0 \setminus \{u, u'\} \subseteq F_1 \) and \( g_\ell \setminus \{u, u'\} \subseteq F_\ell \), but \( F_i \cap g_j = \emptyset \) for all \( i \in [\ell], j \in [\ell - 1] \).

Recall the definition of the graphs \( H_j \), for \( j \in [m] \). For each \( i \in [\ell - 1] \) and \( j \in [m] \) it holds that \( g_i \cap H_j = \{u, u'\} \). Thus \( H_i \setminus \{u, u'\} \subseteq F_i \) for some \( i \in [\ell] \). The following claim implies (2).

**Claim 1.** For all \( i \in [\ell] \) there is exactly one \( j \in [m] \) such that \( H_j \setminus \{u, u'\} \subseteq F_i \).

**Proof.** Let \( i \in [\ell] \). Suppose first that there is no \( j \in [m] \) such that \( H_j \setminus \{u, u'\} \subseteq F_i \). Then \( F_i \cap I = \emptyset \). Then \( 2 \leq i \leq \ell \), because \( Q \setminus \{u, u'\} \subseteq F_i \) and \( Q' \setminus \{u, u'\} \subseteq F_i \). Furthermore, there is a simple curve \( h \subseteq cl(F_i) \) from a point \( x \in g_{i-1} \) to a point \( y \in g_i \) with \( h \cap I = \emptyset \). But then the set \( f_{i-1} \cup f_i \) is arcwise connected, which is a contradiction.

Suppose next that there are \( j, j' \in [m] \) with \( j \neq j' \) such that \( H_j \setminus \{u, u'\} \subseteq F_i \) and \( H_{j'} \setminus \{u, u'\} \subseteq F_i \). Without loss of generality, we may assume that \( j = 1 \) and \( j' = 2 \). As \( I \) is 2-connected, for \( k = 1, 2 \) there is a path \( P_k \subseteq H_k \) from \( u \) to \( u' \). The paths \( P_1, P_2 \) are internally disjoint. Thus \( F_1 \setminus (P_1 \cup P_2) \) has three arcwise connected components \( F^1, F^2, F^3 \), and by possibly renumbering the components \( A_1, A_2 \), we may assume that \( bd(F^1) = g_{i-1} \cup P_1 \), \( bd(F^2) = P_1 \cup P_2 \), and \( bd(F^3) = P_2 \cup g_i \). Let us consider \( F^2 \). As \( H_1 \cap H_2 = \{u, u'\} \) there is a simple curve \( h \) from \( u \) to \( u' \) with all internal points in \( F^2 \) and with \( h \cap I = \{u, u'\} \). To find such a curve, we closely follow the boundary of \( H_1 \) in \( F_1 \). Let \( f \) be the face of \( I \) that contains \( h \setminus \{u, u'\} \). Then \( f \subseteq F^2 \subseteq F_i \), and thus \( f \neq f_{j'} \) for all \( i' \in [m] \), because \( P_i \) separates \( f \) from \( g_0, \ldots, g_{i-1} \) and \( P_j \) separates \( f \) from \( g_i, \ldots, g_{\ell} \). Moreover, \( f \) is an external face of \( I \), because \( u, u' \in bd(f) \). This is a contradiction. \( \square \)

**Lemma 15.4.23.** There is an \( \text{lFP-formula bridge-ord}(x, x', \overline{y}, z_1, z_2) \) (not depending on \( G \) and \( \mathcal{R} \)) such that for all \( u, u' \in V(G) \), if the canonical sps \( \mathcal{Q} \) from \( u \) to \( u' \) is a minimal simplifying patch then there is a \( \overline{v} \in V(G)^\overline{\mathcal{R}} \) such that for every external \( \mathcal{Q} \)-bridge \( B \), the restriction of the binary relation \( \text{bridge-ord}(G, u, u', \overline{v}, z_1, z_2) \) to the vertices of attachment of \( B \) is a linear order.

**Proof.** Let \( u, u' \in V(G) \) such that the canonical sps \( \mathcal{Q} \) from \( u \) to \( u' \) is a minimal simplifying patch. Without loss of generality we may assume that \( \mathcal{Q} \) is nontrivial. Let \( D := D(\mathcal{Q}), R := R(\mathcal{Q}), I := I(\mathcal{Q}) \).

**Case 1:** \( I \setminus \{u, u'\} \) is connected.

Then by Lemma \ref{lem:15.4.22} (1) we have \( D = R \), and all faces of \( I \) contained in \( D \) are internal. It follows from Lemmas \ref{lem:15.4.20} and \ref{lem:15.4.10} that we can define the angles of all faces of \( I \) and the alignment relation between the angles in \( \text{lFP} \). Then by the First Angle Lemma \ref{lem:9.2.3} we can define a linear order on \( V(I) \), and of course this linear order induces a linear order on the vertices of attachment of any external \( \mathcal{Q} \)-bridge.

M. Grohe, Definable Graph Structure Theory
Case 2: \( I \setminus \{u, u'\} \) is not connected.

Let \( A_1, \ldots, A_m \) be the connected components of \( I \setminus \{u, u'\} \). For all \( i \in [m] \), let \( H_i := I[V(A_i) \cup \{u, u'\}] \). Let \( f_0, \ldots, f_{\ell-1} \) be the external faces of \( I \). It follows from Lemma 15.4.22(2) that we can choose the indices in such a way that \( \ell = m \) and \( \text{bd}(f_i) \subseteq H_i \cup H_{i+1} \) (15.4.1)

for all \( i \in [m-1] \). To simplify the notation, we let \( f_m := f_0 \) and \( A_0 := A_m, A_{m+1} := A_1, H_0 := H_m, H_{m+1} := H_1 \). Then (15.4.1) holds for all \( i \in [m] \). By Lemma 15.4.4 for all \( i \in [m] \) there are paths \( Q_i, Q'_i \in Q \) such that \( \text{Bd}(f_i) = Q_i \cup Q'_i \). We choose such paths in such a way that \( Q_i \subseteq H_i \) and \( Q'_i \subseteq H_{i+1} \).

Observe that there is at most one \( i \in [m] \) such that there is no path \( P \subseteq G \setminus \{u, u'\} \) from a vertex \( w \in V(H_i) \) to \( w' \in V(H_{i+1}) \) with \( P \setminus \{w, w'\} \subseteq f_i \), because otherwise \( \{u, u'\} \) would be a separator of \( G \). Let \( i^* \in [m] \) such that for all \( i \in [m] \setminus \{i^*\} \) there is a path \( P_i \subseteq G \setminus \{u, u'\} \) from a vertex \( w_i \in V(H_i) \) to \( w'_i \in V(H_{i+1}) \) with \( P_i \setminus \{w_i, w'_i\} \subseteq f_i \).

Claim 1. There are IFP-formulae \( \text{same-H}(x, x', z_1, z_2) \) and \( \text{adj-H}(x, x', z_1, z_2) \) such that for all \( w_1, w_2 \in V(G) \) we have:

\[
G \models \text{same-H}[u, u', w_1, w_2] \iff \text{there is an } i \in [m] \text{ such that } w_1, w_2 \in V(H_i),
G \models \text{adj-H}[u, u', w_1, w_2] \iff \text{there is an } i \in [m] \text{ such that } w_1 \in V(A_i) \text{ and } w_2 \in V(A_{i-1}) \cup V(A_{i+1}).
\]

Proof. To define the formula \( \text{same-H}(x, x', z_1, z_2) \), remember that there is an \( i \in [m] \) such that \( w_1, w_2 \in V(H_i) \) if and only if \( w_1, w_2 \in V(I) \) and either \( \{w_1, w_2\} \cap \{u, u'\} \neq \emptyset \) or \( w_1 \) and \( w_2 \) belong to the same connected component of \( I \setminus \{u, u'\} \). Recall that Lemma 15.4.20 the vertex set and edge relation of \( I \) is definable from \( u, u' \). Now it is easy to construct \( \text{same-H}(x, x', z_1, z_2) \) by means of the Transduction Lemma (Fact 2.4.6).

To define \( \text{adj-H}(x, x', z_1, z_2) \), we first define a formula \( \text{adj-H}'(x, x', z_1, z_2) \) stating that \( z_1 \) and \( z_2 \) do not belong to the same \( A_i \), but there are \( z'_1 \) and \( z'_2 \) in their respective \( A_i \) and there is a path from \( z'_1 \) to \( z'_2 \) that has no internal vertex in \( I \). This formula almost has the desired meaning, except possibly for vertices in \( A_{i^*} \) and \( A_{i^*+1} \). But we can easily detect this and fix the formula.

Claim 2. There is an IFP-formula \( \text{ord-H}(x, x', y_1, y_2, z_1, z_2) \) such that for all \( v_1 \in V(A_1), v_2 \in V(A_2) \) and \( w_1, w_2 \in V(G) \),

\[
G \models \text{ord-H}[u, u', v_1, v_2, w_1, w_2] \iff \text{there are } i, j \in [m] \text{ such that } i \leq j \text{ and } w_1 \in V(H_i) \text{ and } w_2 \in V(H_j).
\]

Proof. Follows easily from Claim 1.

Remember that in Lemma 15.2.9 we defined a formula \( \text{cspfs-height}(x, x', z_1, z_2) \) such that for all \( w_1, w_2 \in V(Q) \) it holds that

\[
G \models \text{cspfs-height}[u, u', w_1, w_2] \iff \text{ht}(w_1) \leq \text{ht}(w_2). \quad (15.4.2)
\]
Now we let
\[
\text{bridge-ord}(x, x', y_1, y_2, z_1, z_2) := \left( \neg \text{same-H}(x, x', z_1, z_2) \land \text{ord-H}(x, x', y_1, y_2, z_1, z_2) \right) \\
\lor \left( \text{same-H}(x, x', z_1, z_2) \land \text{csps-height}(x, x', z_1, z_2) \right).
\]

The following claim completes the proof.

Claim 3. Let \( v_1 \in V(A_1) \) and \( v_2 \in V(A_2) \), and let \( J \) be an external \( Q \)-bridge. Then
\[
\text{bridge-ord}[G, u, u', v_1, v_2, z_1, z_2]
\]
induces a linear order on the vertices of attachment of \( J \).

Proof. As \( J \) is an external \( Q \)-bridge, it is embedded in some external face of \( I \). Let \( i \in [m] \) such that \( J \subseteq cI(f_i) \). Then all vertices of attachment of \( J \) are in \( V(Q_i) \cup V(Q'_i) \).

Let \( w_1, w_2 \) be two vertices of attachment of \( J \). If \( w_1, w_2 \in V(Q_i) \), then \( \text{ht}^Q(w_1) \neq \text{ht}^Q(w_2) \), because \( Q_i \subseteq Q \) contains exactly one vertex of each height. Similarly, if \( w_1, w_2 \in V(Q'_i) \), then \( \text{ht}^Q(w_1) \neq \text{ht}^Q(w_2) \). Otherwise, either \( w_1 \in V(Q_i) \subseteq V(A_i) \) and \( w_2 \in V(Q'_i) \subseteq V(A_{i+1}) \) or vice versa. Thus the claim follows from Claims 1 and 2 and from \([15.4.2]\). \( \square \)

### 15.5 Belts

We can use simplifying patches in the inductive step of the proof of the Definable Structure Theorem for Almost Embeddable Graphs \([16.3.1]\) (see the introduction to this chapter for an outline): we delete a simplifying patch, then define an ordered treelike decomposition on the resulting simpler graph, and finally extend the decomposition to the original graph using a generalisation of the Ordered Extension Lemma \([7.3.2]\). However, simplifying patches do not always exist. If there are no simplifying patches, we need more complicated simplifying subgraphs, which we will call “belts.”

Throughout this section, we make Assumptions \([15.1.7]\) and \([15.3.1]\) and, in addition, the following assumptions:

**Assumption 15.5.1.**

1. The representativity of \( G_0 \) is at least 5.
2. There is no canonical simplifying patch in \( G \).

**Definition 15.5.2.** A belt in \( G \) is a tuple \( B := (u^0, Q^1, u^1, Q^2, u^2) \), where \( u^0, u^1, u^2 \in V(G) \) and \( Q^i = Q^i_c(u_{i-1}, u_i) \) is the canonical sps from \( u_{i-1} \) to \( u_i \) for \( i = 1, 2 \), such that the following conditions are satisfied.

(B.1) \( u^0, u^1, u^2 \) are pairwise distinct.

(B.2) \( V(Q^1) \cap V(Q^2) = \{ u^1 \} \).

(B.3) \( Q^1 \) and \( Q^2 \) are safe.

(B.4) For \( i \in [2] \) there is a disk \( D_i \subseteq S \) such that \( G(Q^i) \subseteq D_i \).

We introduce some additional notation and terminology for belts and make a few observations. Let \( B := (u^0, Q^1, u^1, Q^2, u^2) \) be a belt in \( G \). We let \( V(B) := V(Q^1) \cup V(Q^2) \) and \( E(B) := E(Q^1) \cup E(Q^2) \) and \( G(B) := (V(B), E(B)) = G(Q^1) \cup G(Q^2) \). The length of \( B \) is
the sum of the lengths of $Q^1, Q^2$. We call $u^0$ and $u^2$ the endvertices of $B$, and we say that $B$ is a belt from $u^0$ to $u^2$. For $i \in [2]$, let $u_i - 1 = u_{i0}, u_{i1}, \ldots, u_{in_i} = u^i$ be the articulation vertices of $Q^i$, ordered by height. We call the vertices $u^0, u^1, u^2$ the major articulation vertices of $B$ and the vertices $u^i_j$ for $i \in [2], j \in [n_i - 1]$ the minor articulation vertices. We denote the set of all articulation vertices of $B$, both major and minor, by $\text{art}(B)$. All articulation vertices except $u^0$ and $u^2$ are proper. Note that all proper articulation vertices are separators of the graph $G(B)$. For all $j \in [n_i]$, let $Q^i_j := Q^i[u^i_{j-1}, u^i_j]$. As $Q^i$ is a canonical sps, $Q^i_j$ is canonical as well. By (B.3) and (B.4) $Q^i_j$ is a path. We call the $Q^i_j$, for $i \in [2]$ and $j \in [n_i]$, the patches of $B$. Note that all these patches are non-simplifying by Assumption [15.5.1](2). If the patch $Q^i_j$ is trivial, we denote its unique edge by $e^i_j$. We let $I(B) := \bigcup_{j=1}^{n_i} I(\mathcal{Q}^i_j)$ and

$$R(B) := \bigcup_{i=1}^{2} \left( \bigcup_{1 \leq j \leq n_i} D(Q^i_j) \right) \bigcup_{1 \leq j \leq n_i, \mathcal{Q} \text{ trivial}} e^i_j.$$ 

The following lemma collects a few definability results for belts.

**Lemma 15.5.3.** There are IFP-formulae

- $\text{belt-vert}(x^0, x^1, x^2, y)$,
- $\text{belt-edge}(x^0, x^1, x^2, y)$,
- $\text{belt-art}(x^0, x^1, x^2, y)$,
- $\text{belt-int-ord}(x^0, x^1, x^2, y_1, y_2)$,
- $\text{belt-int-vert}(x^0, x^1, x^2, y)$,
- $\text{belt-int-edge}(x^0, x^1, x^2, y_1, y_2)$.

(not depending on $G$ and $\mathcal{R}$) such that for all $u^0, u^1, u^2 \in V(G)$ the following holds. If $B := (u^0, Q^G(u^0, u^1), u^1, Q^G(u^1, u^2), u^2)$ is a belt in $G$, then

1. $\text{belt-vert}[G, u^0, u^1, u^2, y] = V(B)$;
2. $\text{belt-edge}[G, u^0, u^1, u^2, y] = E(B)$;
3. $\text{belt-art}[G, u^0, u^1, u^2, y] = \text{art}(B)$;
4. $\text{belt-int-ord}[G, u^0, u^1, u^2, y_1, y_2]$ is the linear order of $\text{art}(B)$ that orders the articulation vertices of both $Q^G(u^0, u^1)$ and $Q^G(u^1, u^2)$ by increasing height and puts the articulation vertices of $Q^G(u^0, u^1)$ before those of $Q^G(u^1, u^2)$;
5. $\text{belt-int-vert}[G, u^0, u^1, u^2, y] = V(I(B))$;
6. $\text{belt-int-edge}[G, u^0, u^1, u^2, y_1, y_2] = E(I(B))$.

**Proof.** The first four formulae can easily be constructed using Lemma 15.2.9. For the last two formulae we use Corollary 15.4.13.

**Definition 15.5.4.** A belt $B$ in $G$ with endvertices $u^0$ and $u^2$ is reducing if there is a simple curve $g \subseteq R(B)$ with endpoints $u^0, u^2$ such that one of the following two conditions is satisfied.

1. For $i = 0, 2$ there is a $G_0$-normal simple curve $g'$ from $u^i$ to a point in $\text{bd}(S)$ such that $g' \cap R(B) = \{ u^i \}$ and $|g' \cap V(G_0)| = 2$ and $b := g^0 \cup g \cup g^2$ is a noncontractible loop or a link.
2. There is a $G_0$-normal simple curve $g' \subseteq S$ with endpoints $u^0, u^2$ such that $g' \cap R(B) = \{ u^0, u^2 \}$ and $|g' \cap V(G_0)| = 3$ and $b := g' \cup g$ is a proper noncontractible curve in $S$. 

Preliminary Version
We call the curve $b$ a simplifying curve through $B$.

**Remark 15.5.5.** We use the term “reducing belt” instead of “simplifying belt”, because a belt $B$ for which the graph $G(B)$ is simplifying is not necessarily reducing.

Observe that for each of the curves $g^i$ in Definition 15.5.4(i) there is a $j_i \in [q]$ such that the endpoint of $g^i$ in $bd(S)$ is in $c^{j_i}$, and there is a vertex $v^i \in bd(D^{j_i})$ such that $g^i \cap G_0 = \{u^i, v^i\}$. This follows from the condition $|g^i \cap V(G_0)| = 2$ and the facts that there is a cycle $C^{j_i} \subseteq G_0$ such that $bd(D^{j_i}) = C^{j_i}$ and that $bd(D^{j_i})$ separates $c^{j_i}$ from $S \setminus D^{j_i}$. We call $v^0$ and $v^2$ the exit vertices of the loop or link $g^0 \cup g \cup g^2$. Note that we may have $j_0 = j_2$ (if $b = g^0 \cup g \cup g^2$ is a loop) and even $v^0 = v^2$ (if $b$ is a closed loop).

It follows from Assumption 15.3.1(2) that for every reducing belt $B$ there are vertices $v^0, v^2$ (possibly equal) such that $G \setminus (V(B) \cup \{v^0, v^2\}) \in \mathcal{R}$. We need a slightly stronger version of this observation. Let $B$ be a belt and $R := R(B)$. We define the outside of $B$ to be the graph $O(B)$ defined by

\[
V(O(B)) := V(G) \setminus int(R), \\
E(O(B)) := E(G) \setminus \{e \in E(G_0) \mid e \cap int(R) \neq \emptyset\}.
\]

We define the graph $Cut(B)$ obtained from $G$ by cutting through $B$ by letting

\[Cut(B) := O(B) \setminus \text{art}(B).\]

Note that we can also define $O(B)$ and thus $Cut(B)$ in terms of the patches of $B$ and without explicit reference to the region $R$. For $i \in [2]$, let $u^i_0 = u^{i-1}, u^i_1, \ldots, u^i_{n_i} = u^i$ be the articulation vertices of $Q^i$ ordered by height. For $i \in [2]$ and $j \in [n_i]$, let $Q^i_j := Q^i(u^i_{j-1}, u^i_j)$. If the patch $Q^i_j$ is nontrivial, let $C^i_j := C(Q^i_j)$, $I^i_j := I(Q^i_j)$, and $J^i_j := I(Q^i_j) \setminus C^i_j$. Note that $J^i_j = G_0 \cap \text{int}(D(Q^i_j))$ and thus $V(G) \cap \text{int}(R) = \bigcup_{i=1}^{2} \bigcup_{\substack{1 \leq j \leq n_i \leq n_j \leq n_i \nontrivial}} V(J^i_j)$. Hence we have

\[
V(O(B)) = V(G) \setminus \bigcup_{i=1}^{2} \bigcup_{\substack{1 \leq j \leq n_i \leq n_j \leq n_i \nontrivial}} V(J^i_j),
\]

\[
E(O(B)) = E(G) \setminus \bigcup_{i=1}^{2} \bigcup_{\substack{1 \leq j \leq n_i \leq n_j \leq n_i \nontrivial}} (E(I^i_j) \setminus E(C^i_j)).
\]

**Lemma 15.5.6.** Let $B$ be a reducing belt in $G$. Then there are vertices $v^0, v^2 \in V(G)$ such that $Cut(B) \setminus \{v^0, v^2\} \in \mathcal{R}$.

**Proof.** Let $R := R(B)$. Define the articulation vertices $u^i_j$ and the patches $Q^i_j$ of $B$ as above. If $Q^i_j$ is nontrivial, let $D^i_j := D(Q^i_j)$.

Let $b$ be a reducing curve through $B$. Then $R \cap b$ is a simple curve from $u^0$ to $u^2$, and for all $i \in [2]$, $j \in [n_i]$ such that $Q^i_j$ is nontrivial, the intersection $b^j := D^i_j \cap b$ is a simple curve in the disk $D^i_j$ with endpoints $u^i_{j-1}$ and $u^i_j$. By slightly perturbing $b$ (moving it away from the boundaries of the disks $D^i_j$), we obtain a homotopic curve $b'$ such that $b' \cap bd(D^i_j) = \{u^i_{j-1}, u^i_j\}$ and $b' \setminus R(B) = b \setminus R(B)$. Then $b'$ intersects $bd(R)$ in the articulation.
vertices $u^j_i$ of $B$ and in the edges $e^j_i$ for the trivial patches $Q^j_i$. It follows from Definition \[15.5.4\] that $| (b \setminus R) \cap G_0 | = | (b \setminus R) \cap V(G_0) | \leq 2$. Thus there are vertices $v^0, v^2 \in V(G_0)$ (possibly equal) such that

\[(b' \setminus R) \cap G_0 = (b \setminus R) \cap G_0 \subseteq \{ v^0, v^2 \}. \quad (15.5.2)\]

This implies that for $H := \text{Cut}(B) \setminus \{ v^0, v^2 \}$ we have $b' \cap H = \emptyset$. Thus $H$ is a subgraph of the $b'$-reduction of $G$ with respect to the arrangement $(G_0, \pi, \tau_1, \ldots, \tau^q, \tau^q)$. By Assumption \[15.3.1(2)\], this implies $H \in \mathcal{R}$.

**Lemma 15.5.7.** There are IFP-formulae

\[
\begin{align*}
\text{belt-out-vert}(x^0, x^1, x^2, y), & \quad \text{belt-out-edge}(x^0, x^1, x^2, y), \\
\text{belt-cut-vert}(x^0, x^1, x^2, y), & \quad \text{belt-cut-edge}(x^0, x^1, x^2, y_1, y_2)
\end{align*}
\]

(not depending on $G$ and $\mathcal{R}$) such that for all $u^0, u^1, u^2 \in V(G)$ the following holds: if $B := (u^0, Q^G(u^0, u^1), u^1, Q^G(u^1, u^2), u^2)$ is a belt in $G$, then

1. $\text{belt-out-vert}(G, u^0, u^1, u^2, y) = V(O(B))$;
2. $\text{belt-out-edge}(G, u^0, u^1, u^2, y_1, y_2) = E(O(B))$;
3. $\text{belt-cut-vert}(G, u^0, u^1, u^2, y) = V(\text{Cut}(B))$;
4. $\text{belt-cut-edge}(G, u^0, u^1, u^2, y_1, y_2) = E(\text{Cut}(B))$.

**Proof.** This follows from Lemmas \[15.5.3\] and \[15.4.12\].

15.5.1 Existence of Reducing Belts

In this section, we prove that, under the Assumptions \[15.1.7\] \[15.3.1\] and \[15.5.1\] our graph $G$ has a reducing belt. The proof is based on the following intuitive reasoning. Take a shortest path or cycle $H$ such that $H$ is a simplifying curve in $S$. Such an $H$ exists by Lemma \[15.1.3\]. If $H$ is a cycle, remove one vertex, and if $H$ is a path, remove the two endvertices. It can be shown that this will make $H$ safe. Split the remaining path in half and let $P^1, P^2$ be the parts. Consider the two canonical shortest path systems $Q^1, Q^2$ between the endvertices of the two paths $P^1, P^2$. Then $G(Q^i)$ is contained in a disk, because otherwise $G(Q^i)$ would contain a simplifying curve that is shorter than $H$. The intersection between $V(Q^1)$ and $V(Q^2)$ is empty, because otherwise we could find a simplifying curve shorter than $H$. Finally, $P^i \in Q^i$, and this implies that the belt consisting of $Q^1$ and $Q^2$ is a reducing belt.

**Lemma 15.5.8 (Belt Lemma).** $G$ has a reducing belt.

It requires some efforts to make the argument sketched above precise. As a matter of fact, we will not follow the proof sketch very closely.

**Lemma 15.5.9.** Let $Q$ be a safe sps in $G$ such that there is no disk $D \subseteq S$ with $G(Q) \subseteq D$, but for every proper segment $Q'$ of $Q$ there is a disk $D' \subseteq S$ with $G(Q') \subseteq D'$. Then there are internally disjoint paths $Q, Q' \in Q$ such that $Q \cup Q'$ is a noncontractible simple closed curve in $\text{int}(S)$.
Proof. Let \( u, u' \) be the source and sink of \( Q \). Clearly, \( Q \) is nontrivial.

**Claim 1.** \( Q \) has no proper articulation vertices.

**Proof.** Let \( u_0 = u, u_1, \ldots, u_n = u' \) be the articulation vertices of \( Q \), ordered by increasing height, and suppose for contradiction that \( n \geq 2 \). Then the segments \( Q_i := Q[u_{i-1}, u_i] \), for \( i \in [n] \) are patches. Let \( D_i := D(Q_i) \). Then \( D_{i+1} \cap D_i = \{ u_i \} \) for \( i \geq 2 \), and \( D_i \cap D_{i+1} = \emptyset \) for all \( j \in [n] \setminus \{ i - 1, i, i + 1 \} \). It follows inductively from Fact 9.1.9 that there is a disk \( D \subseteq S \) such that \( G(Q) \subseteq \bigcup_{i=1}^n D_i \subseteq D \). This is a contradiction. Similarly, we can prove that \( Q \) is nontrivial.

Hence by Lemma 15.2.3 \( G(Q) \) is a 2-connected graph, and by Lemma 15.2.4 there are internally disjoint paths \( Q, Q' \in Q \). Suppose for contradiction that for all internally disjoint paths \( Q, Q' \in Q \) the simple closed curve \( Q \cup Q' \) is contractible in \( S \).

We shall prove inductively that there is an enumeration \( Q_1, \ldots, Q_n \) of \( Q \) such that for all \( k \in [2, n] \) there are a disk \( D_k \subseteq S \) and paths \( Q, Q' \in \{ Q_1, \ldots, Q_k \} \) such that:

1. \( \bigcup_{j=1}^k Q_j \subseteq D_k \).
2. \( \text{bd}(D_k) = Q \cup Q' \).

Once we have proved this we are done, because then \( G(Q) \subseteq D_n \), which is a contradiction.

For the base step \( k = 2 \), recall that by Lemma 15.2.4 there are two internally disjoint paths \( Q_1, Q_2 \in Q \). Then by our assumption, the simple closed curve \( Q_1 \cup Q_2 \) is contractible in \( S \). Thus there is a disk \( D_2 \subseteq S \) such that \( Q_1 \cup Q_2 = \text{bd}(D_2) \).

For the inductive step, suppose that \( Q_1, \ldots, Q_k, D_k \) and \( Q, Q' \) satisfy (i) and (ii). If \( \{ Q_1, \ldots, Q_k \} = \emptyset \), there is nothing left to do. Otherwise, we choose a path \( Q_{k+1} \subseteq Q \setminus \{ Q_1, \ldots, Q_k \} \) with the least possible number of edges in \( E(G) \setminus E(Q \cup Q') \). If \( Q_{k+1} \subseteq D_k \), we let \( D_{k+1} := D_k \). Suppose otherwise. Then \( Q_{k+1} \) has at least one edge not contained in \( D_k \).

**Claim 2.** There are vertices \( w, w' \in V(Q_{k+1}) \) such that

\[
(uQ_{k+1}w = uQw \quad \text{or} \quad uQ_{k+1}w = uQ'w)
\]

and

\[
(w'Q_{k+1}u' = w'Qu' \quad \text{or} \quad w'Q_{k+1}u' = w'Q'u'),
\]

and the segment \( Q'' := wQ_{k+1}w' \) is internally disjoint from \( D_k \), that is, \( Q'' \cap D_k = \{ w, w' \} \).

**Proof.** Let \( e = xx' \in E(Q_{k+1}) \) such that \( e \not\subseteq D_k \). Let \( w \) be the last vertex on the segment \( uQ_{k+1}x \) in \( V(Q \cup Q') \) (possibly \( w = u \) or \( w = x \)), and let \( w' \) be the first vertex on the segment \( x'Q_{k+1}u' \) in \( V(Q \cup Q') \) (possibly \( w' = x' \) or \( w' = u' \)). Let \( Q'' := wQ_{k+1}w' \). Then \( Q'' \cap D_k = \{ w, w' \} \), because \( e \not\subseteq D_k \) and \( \text{bd}(D_k) = Q \cup Q' \).

We may assume without loss of generality that \( w \in V(Q) \). We shall prove that then \( uQ_{k+1}w = uQw \). Suppose for contradiction that this is not the case. Then \( E(uQ_{k+1}w) \setminus E(Q \cup Q') \neq \emptyset \), because \( Q \) and \( Q' \) are internally disjoint. Let \( Q_{k+1} := uQw \cup uQ_{k+1}u' \). Then \( Q_{k+1} \) is a shortest path from \( u \) to \( u' \) with fewer edges in \( E(G) \setminus E(Q \cup Q') \) than \( Q_{k+1} \). However, \( Q'_{k+1} \) is still distinct from \( Q_1, \ldots, Q_k \), because \( e \not\subseteq E(Q'_{k+1}) \) and \( e \not\subseteq D_k \subseteq Q_1 \cup \ldots \cup Q_k \). This is a contradiction. Similarly, we can prove that \( w'Q_{k+1}u' = w'Qu' \) if \( w' \in V(Q) \) and \( w'Q_{k+1}u' = w'Q'u' \) if \( w' \in V(Q') \).

We choose \( w, w' \in V(Q_{k+1}) \) according to the claim and let \( Q'' := wQ_{k+1}w' \).

**Case 1:** \( w, w' \in V(Q) \).

If \((w, w') = (u, u')\), then \( Q, Q'' \in Q \) are internally disjoint, and by our assumption they...
Figure 15.4. Proof of Lemma 15.5.9 (Case 3)

bound a disk $D \subseteq S$. If $(w, w') \neq (u, u')$, then the paths $wQw'$ and $Q''$ are internally
disjoint paths in a proper segment of $Q$, and hence they also bound a disk $D$ by the
minimality of $Q$. In both cases, the intersection of the two disks $D_k$ and $D$ is the
segment $wQw'$ of their boundaries. Hence by Fact 9.1.9 their union is a disk. We let
$D_{k+1} := D_k \cup D$. Then $Q_i \subseteq D_{k+1}$ for all $i \in [k+1]$, and $bd(D_{k+1}) = Q' \cup Q_{k+1}$.

Case 2: $w, w' \in V(Q')$.
Symmetric to Case 1.

Case 3: $w \in V(Q) \setminus V(Q')$ and $w' \in V(Q') \setminus V(Q)$ (see Figure 15.4).
Then $uQ_{k+1}w = uQw$ and $w'Q_{k+1}u' = w'Q'u'$. Consider the two cycles $C := uQ_{k+1}w' \cup uQ'w'$ and $C' := wQ_{k+1}u' \cup wQ'u'$. These cycles are contained in the proper segments
$Q[u, w]$ and $Q[w, u']$ of $Q$ and hence they bound closed disks $D, D'$. Moreover, the
intersection of these two disks is $Q''$. Hence their union is a closed disk as well. However,
it holds that $bd(D \cup D') = vQv' \cup vQ'v' = bd(D_k)$, and as $S$ is not a sphere this
implies $D \cup D' = D_k$. Thus $Q_{k+1} \subseteq D_k$, and this is a contradiction.

Case 4: $w \in V(Q') \setminus V(Q)$ and $w' \in V(Q) \setminus V(Q')$.
Symmetric to Case 3.

Proof of the Belt Lemma 15.5.8 Let $S' := S \setminus \bigcup_{i=1}^{q} \text{int}(D_i)$. Then $S' \simeq S$. Furthermore,
if $g' \subseteq S'$ is a proper non-contractible curve or a cuff-separating curve in $S'$ then it is a
proper non-contractible curve or a cuff-separating curve, respectively, in $S$. If $g' \subseteq S'$ is a
noncontractible closed loop in $S'$, then it is either a noncontractible closed loop or a proper
noncontractible curve in $S$. If $g' \subseteq S'$ is a noncontractible open loop or a link in $S'$ then there
is a noncontractible open loop or link $g \subseteq S$ in $S$ such that $g \cap S = g'$ and $g \cap G_0 = g' \cap G_0$.
This follows from the fact that for every point $x \in bd(S') = \bigcup_{i=1}^{q} bd(D_i)$, if $x \notin bd(S)$ then there is a simple curve $h$ from $x$ to a point in $bd(S)$ with $h \setminus \{x\} \subseteq S' \setminus S'$ and $h \cap G_0 \subseteq \{x\}$ (by Definition 15.1.9(iii)). For every $i \in [q]$, let $C_i \subseteq G_0$ such that $bd(D_i) = C_i$. Let
$G'_0 := G_0 \cap S'$. Observe that $G'_0$ is connected.
Suppose first that \( q \geq 2 \), that is, \( S \) has at least two cuffs. Choose \( i, j \in [q] \) with \( \neq j \) such that \( \text{dist}^{\mathbf{G}}(C^i, C^j) \) is least possible. Without loss of generality we may assume that \( i = 1 \) and \( j = 2 \). Let \( P \) be a shortest path from \( C^1 \) to \( C^2 \), and let \( v_0 \in V(C_1) \) and \( v_2 \in V(C_2) \) be the endvertices of \( P \). By Assumption [15.5.11] that the representativity of \( G_0 \) be at least 5 it holds that \( |P| \geq 4 \). For \( i = 0, 2 \), let \( u^i \) be the neighbour of \( v^i \) on \( P \), and let \( u^1 \) be a vertex in the middle of \( P \), that is, \( u^1 \in V(P) \) such that 

\[
||v^0Pu^1||, \ ||u^1Pv^2|| \geq \left \lceil \frac{|P|}{2} \right \rceil .
\]

Then \( u^1 \neq u^0, u^2 \), because \( |P| \geq 4 \). Note that for \( i = 1, 2 \), the segment \( u^{i-1}Pu^i \) is a shortest path from \( u^{i-1} \) to \( u^i \), and we have

\[
||u^{i-1}Pu^i|| \leq \left \lceil \frac{|u^0Pu^2|}{2} \right \rceil < \frac{|P|}{2}. \tag{15.5.3}
\]

Let \( Q^i \) be the canonical sps from \( u^{i-1} \) to \( u^i \). Observe that \( Q^i \) is safe, because otherwise there would be a path between two cycles \( C^k, C^l \) that is shorter than \( P \). For the same reason, we have \( V(Q^1) \cap V(Q^2) = \{u^1\} \). Hence if there were disks \( D_1, D_2 \subseteq S \) such that \( G(Q^1) \subseteq D_1 \), for \( i = 1, 2 \), then \( B := (u^0, Q^1, u^1, Q^2, u^2) \) would be a belt. Actually, \( B \) would be a reducing belt, because we can find a link through \( B \) by taking a \( G_0 \)-normal curve that closely follows the path \( P \).

Hence without loss of generality we may assume that there is no disk \( D_1 \subseteq S \) such that \( G(Q^1) \subseteq D_1 \). We choose a minimal segment of \( Q^1 \) that is not covered by a disk. That is, we choose \( u, u' \in V(Q^1) \) with \( \text{ht}^Q(u) < \text{ht}^Q(u') \) such that there is no disk \( D \subseteq S \) such that \( G(Q^1[u, u']) \subseteq D \), but for all \( v, v' \in V(Q^1[u, u']) \) with \( \text{ht}^Q(v) < \text{ht}^Q(v') \) and \( (u, u') \neq (v, v') \) there is a disk \( D \subseteq S \) such that \( G(Q^1[v, v']) \subseteq D \). To simplify the notation, let \( Q := Q^1[u, u'] \). Then \( Q \) is the canonical sps from \( u \) to \( u' \). By Lemma [15.5.9] there are internally disjoint paths \( Q, Q' \in \mathcal{Q} \) such that \( \mathcal{Q} \cup Q' \) is a noncontractible simple closed curve in \( S \). Choose such paths \( Q, Q' \). Let \( u_0 \) be the successor of \( u \) on \( Q \), and let \( u_2 \) be the successor of \( u \) on \( Q' \). Let \( u_1 := u' \), and for \( i = 1, 2 \), let \( Q_i \) be the canonical sps from \( u_{i-1} \) to \( u_i \). Note that \( Q_1 = Q[u_0,u_1] = Q^1[u_0,u_1] \), and \( Q_2 \) is the reversal of \( Q[u_2,u_1] = Q^1[u_2,u_1] \). Hence for \( i = 1, 2 \) we have \( G(Q_i) \subseteq G(Q) \). Thus \( Q_i \) is safe. By the minimality of \( Q \), there is a disk \( D_1 \) such that \( G(Q_i) \subseteq D_1 \). Furthermore, \( u_0Q_1u_1 \in Q_1 \) and \( u_1Q'2u_2 \in Q_2 \). Thus all paths in \( Q_1 \) are homotopic to \( u_0Q_1u_1 \) and all paths in \( Q_2 \) are homotopic to \( u_1Q'2u_2 \).

**Claim 1.** \( V(Q_1) \cap V(Q_2) = \{u_1\} \).

**Proof.** Let \( Q_1 \in Q_1 \). We shall first prove that \( Q_1 \cap Q' = \{u_1\} \). Let \( w \) be the first vertex of \( Q_1 \) in \( V(Q') \). Suppose that \( w \neq u_1 \). As the segment \( wQ'u_1 \) is a shortest path, the path \( Q'' := u_0Q_1w \cup wQ'u_1 \) is a shortest path from \( u_0 \) to \( u_1 \). Thus \( Q'' \in Q_1 \), and hence \( Q'' \) is homotopic to \( u_0Q_1u_1 \). Let \( C' := u_0Q_1wQ'u_2u_0 \). Then \( C' \) is homotopic to the noncontractible simple closed curve \( Q \cup Q' \). But \( C' \) is contained in the proper segment \( Q[u, w] \) of \( Q \), which contradicts the minimality of \( Q \). Hence indeed \( Q_1 \cap Q' = \{u_1\} \).

Let \( Q' := u_0Q_1u_1 \). Then \( Q' \cup Q' \) is also a noncontractible simple closed curve. A similar argument to the one above shows that for every \( Q_2 \in Q_2 \) it holds that \( Q_2 \cap Q'' = \emptyset \) and thus \( Q_2 \cap Q_1 = \emptyset \). This proves the claim.

Hence \( (u_0, Q_1, u_1, Q_2, u_2) \) is a belt, and the noncontractible simple closed curve \( Q \cup Q' \), which is either proper noncontractible curve or a cuff-reducing curve, witnesses that it is a reducing belt.
It remains to deal with the case that \( q \leq 1 \). To simplify the discussion, let us assume that \( q = 1 \) in the following. The case \( q = 0 \) is simpler and only requires a special case of the following argument. As \( S \not\cong S_0, S_{0,1} \), it follows that \( \text{eg}(S) = \text{eg}(S') = \text{eg}(S) > 0 \). By Lemma \[15.1.3\] applied to the graph \( G' \), which is 2-cell embedded in \( S' \) by Lemma \[15.1.8\] there is a subgraph \( H \subseteq G'_0 \) such that \( H \) is either a proper noncontractible curve or a noncontractible loop in \( S' \). We choose \( H \subseteq G'_0 \) such that

(i) \( H \) is either a proper noncontractible curve or a noncontractible loop in \( S' \);

(ii) \( ||H|| \) is minimum subject to (i).

(iii) \( |V(H) \cap V(C^1)| \) is maximum subject to (i) and (ii). (That is, we prefer open loops over closed loops over proper noncontractible curves.)

Note that \( ||H|| \geq 4 \) by Assumption \[15.5.1(1)\].

If \( H \) is an open loop and hence \( H \) is a path, we let \( v^0, v^2 \in V(C^1) \) be the endvertices of \( H \). For \( i = 0, 2 \), we let \( u^i \) be the neighbour of \( v^i \) on \( H \), and we let \( P := u^0 H u^2 \). If \( H \) is a closed loop, we let \( v^0 \) be the unique vertex in \( V(H) \cap V(C^1) \), and we let \( v^2 := v^0 \). If \( H \) is a proper noncontractible curve, we let \( v^0 \) be an arbitrary vertex of \( H \) and again \( v^2 := v^0 \). We let \( P := H \setminus \{v^0\} \), and we let \( v^0, v^2 \) be the endpoints of \( P \). In any case, let \( u^1 \in V(P) \) such that

\[ ||u^0 P u^1||, ||u^1 P u^2|| \geq \left\lfloor \frac{||P||}{2} \right\rfloor, \]

and let \( P^1 := u^0 P u^1, P^2 := u^1 P u^2 \). Note that for \( i = 1, 2 \) we have

\[ ||P_i|| \geq \left\lfloor ||P||/2 \right\rfloor = \left\lfloor (||H|| - 2)/2 \right\rfloor \geq 1 \] \[ (15.5.4) \]

and

\[ ||P_i|| \leq \left\lceil \frac{||P||}{2} \right\rceil < ||H||/2. \] \[ (15.5.5) \]

For \( i = 1, 2 \) we let \( Q^i \) be the canonical sps from \( u^{i-1} \) to \( u^i \). Let \( B := (u^0, Q^1, u^1, Q^2, u_2) \). We shall now prove a sequence of claims which will establish that \( B \) is a reducing belt.

**Claim 2.** For \( i = 1, 2 \) the system \( Q^i \) is safe.

**Proof.** By symmetry, it suffices to prove the claim for \( i = 1 \). Let \( Q \in Q^1 \), and suppose for contradiction that \( Q \) is not safe. Then \( Q \cap C^1 \neq \emptyset \).

**Case 1:** \( H \) is an open loop.

Let \( v \) be the last vertex of \( Q \) in \( V(C^1) \), and let \( v' \) be the first vertex of \( Q \) in \( V(H) \) after \( v \) (see Figure \[15.5\]). Such vertices exist, because \( Q \cap C^1 \neq \emptyset \) and the endvertex of \( Q \) is \( u^1 \in V(H) \setminus V(C^1) \). Let \( Q' := v Q v' \). Note that all internal vertices of \( Q' \) are in \( V(G) \setminus (V(H) \cup V(C^1)) \). If \( v \notin \{v^0, v^2\} \), let \( B \) be the segment of \( C^1 \) from \( v^0 \) to \( v^2 \) that contains \( v \). Otherwise, let \( B \) be an arbitrary segment of \( C^1 \) from \( v^0 \) to \( v^2 \). Let \( C := B \cup H \). As \( H \) is a noncontractible loop, the simple closed curve \( C \) is noncontractible in \( S \). Let \( P', P'' \) be the two segments of \( C \) from \( v \) to \( v' \). It follows from from Fact \[9.1.9\] that one of the simple closed curves \( Q' \cup P' \) and \( Q' \cup P'' \) is noncontractible. Without loss of generality we may assume that \( Q' \cup P'' \) is noncontractible. Let \( H'' := v Q' v' P'' v^2 \). If \( v \neq v^2 \), then \( H' \) is a noncontractible open loop in \( S' \), and if \( v = v^2 \) then \( H' \) is a noncontractible closed loop in \( S \).
Chapter 15. Almost Embeddable Graphs

Let $\text{Claim } 3.$

Case 2: $H$ contain a noncontractible cycle, and by Fact 9.1.14 there is a closed disk $Q$ such that $\|Q\| \leq \|P^1\|$, because $Q$ is a shortest path from $u^0$ to $u^1$. If $v' \in V(P^2)$ then $\|H'\| \leq \|Q\| + \|P^2\| + 1 < \|P_1\| + \|P_2\| + 2 = \|H\|$. So suppose that $v' \in V(P_1)$. Then $\|Q'\| < \|u^0 Q v'\| \leq \|u^0 P^1 v'\|$, because $Q$ is a shortest path. Thus $\|H'\| = \|v Q' v'\| + \|v' P^1 u^1\| + \|P^2\| + 1 < \|P^1\| + \|P^2\| + 1 < \|H\|$.

Case 2: $H$ is a closed loop.

Then $v^0 = v^2 \in \text{bd}(S')$. Arguing very similarly as in the first case, we can construct a path $H'$ or cycle with $\|H'\| < \|H\|$ such that $H'$ is a noncontractible loop in $S'$. This leads to a contradiction.

Case 3: $H \subseteq \text{int}(S')$ is a proper noncontractible curve.

Let $v$ be the first vertex of $Q$ in $V(C^1)$, and let $v'$ be the last vertex of $Q$ in $V(C^1)$. (Possibly, $v = v'$.) Let $w$ be the last vertex of the segment $u^0 Q v$ in $V(H)$, and let $w'$ be the first vertex of the segment $v Q u^1$ in $V(H)$. Let $P', P''$ be the two segments of the cycle $H$ from $w$ to $w'$. Let $H' := v Q w P' w' Q v'$ and $H'' := v Q w P'' w' Q v'$. Similarly to Case 1, we can argue that one of $H', H''$ is a noncontractible loop in $S'$. Say, $H'$ is a noncontractible loop. Note that $\|P''\| \geq \|w Q w'\|$, because $Q$ is a shortest path. Thus $\|H'\| \leq \|H\| - \|P''\| + \|w Q w'\| \leq \|H\|$.

Furthermore, $V(H') \cap V(C^1) \neq \emptyset$. This contradicts condition (iii) on the choice of $H.$

Claim 3. Let $i \in [2]$. Let $Q, Q' \subseteq G$ be safe paths from $u^{i-1}$ to $u^i$ such that $\|Q\|, \|Q'\| \leq \|P^i\|$. Then there is a disk $D \subseteq S'$ such that $Q \cup Q' \subseteq \text{int}(D)$.

Proof. We have $\|Q \cup Q'\| \leq \|Q\| + \|Q'\| \leq 2 \|P_i\| < \|H\|$. Thus the graph $Q \cup Q'$ does not contain a noncontractible cycle, and by Fact 9.1.14 there is a closed disk $D' \subseteq S'$ such that $Q, Q' \subseteq D'$. As $Q \cup Q' \cap \text{bd}(S') = \emptyset$, we can slightly increase the disk $D'$ to get a disk $D$ such that $Q, Q' \subseteq \text{int}(D)$.

M. Grohe, Definable Graph Structure Theory
Claim 4. Let $i \in [2]$, and let $Q \in Q^i$. Then $V(Q) \cap V(P^{3-i}) = \{u^1\}$.

Proof. By symmetry, it suffices to prove the claim for $i = 1$. Let $Q \in Q^1$. Suppose for contradiction that $V(Q) \cap V(P^2) \supset \{u^1\}$. Then there is a segment of $Q$ with one endvertex in $V(P^1) \setminus \{u^1\}$ and one endvertex in $V(P^2) \setminus \{u^1\}$ and no internal vertex in $V(P^1) \cup V(P^2)$.

(It may be that the endvertex of the segment in $P^1$ is $u^0$.) Let $B$ be the last such segment that appears on $Q$. Let $w \in V(P_1) \setminus \{u^1\}$ and $w' \in V(P_2) \setminus \{u^1\}$ be the endvertices of $B$. Note that $V(B) \cap V(H) = \{w, w'\}$ and that $B \cap C^1 = \emptyset$, because $Q$ is safe. Moreover,

$$||wQu^1|| \leq ||wP^1u^1|| \quad \text{(15.5.6)}$$

and

$$||w'Qu^1|| \leq ||w'P^2u^1||, \quad \text{(15.5.7)}$$

because $Q$ is a shortest path. Let $C := B \cup wP^1u^1 \cup w'P^2u^1$. Then $C$ is a cycle. As either $B \subseteq wQu^1$ or $B \subseteq w'Qu^1$, it follows from (15.5.6) and (15.5.7) that $||B|| < ||wP^1u^1 \cup w'P^2u^2||$. Let

$$H' := v^0u^0P^1wu'Bw'P^2u^2v^2.$$ 

Then $||H'|| < ||H||$. If the simple closed curve $C$ is contractible then $H'$ is homotopic to $H$ and thus a noncontractible loop as well. This contradicts condition (ii) on the choice of $H$. Thus $C$ is noncontractible. We shall prove that $||C|| < ||H||$. This will lead to a contradiction.

Case 1: $w$ occurs before $w'$ on $Q$.

Then the segment $w'Qu^1$ has an empty intersection with $P^1 \setminus \{u^1\}$. Let $P' := u^2P^2w'Qu^1$. Then $V(P') \cap V(P^1) = \{u^1\}$. By (15.5.7) it holds that $||P'|| \leq ||P^2||$, and by Claim 3 $P'$ is homotopic to $P^2$. Let

$$H'' := v^0u^0P^1u^1P'u^2v^2.$$ 

Then $H''$ is homotopic to $H$, and by (ii) we have $||H''|| \geq ||H||$. This implies $||w'Qu^1|| = ||wP^2u^1||$ and thus, by (15.5.6),

$$||B \cup w'P^2u^1|| = ||wQu^1|| \leq ||wP^1u^1||.$$

For the cycle $C$ this yields

$$||C|| = ||B \cup w'P^2u^1|| + ||wP^1u^1|| \leq 2 \cdot ||wP^1u^1|| \leq 2 \cdot ||P^1|| < ||H||.$$

This is a contradiction.

Case 2: $w'$ occurs before $w$ on $Q_1$.

We argue similarly as in the first case, but with $P^1$ and $P^2$ exchanged.

Claim 5. For $i = 1, 2$ it holds that $P^i \in Q^i$.

Proof. Let $i \in [2]$, and let $Q_i \in Q^i$. By Claim 3 $Q_i$ is homotopic to $P^i$, and by Claim 4 we have $V(Q^i) \cap V(P^{3-i}) = \{u^1\}$. Let

$$H' := v^iu'u^1P^{3-i}u^3-iv^{3-i}.$$ 

Then $H'$ is homotopic to $H$, and by (ii) this implies $||H'|| \geq ||H||$. It follows that $||Q_i|| \geq ||P^i||$. Hence $P^i$ is a shortest path.
Claim 6. For $i = 1, 2$, let $Q_i \in Q^i$. Then $V(Q_1) \cap V(Q_2) = \{u^1\}$.

Proof. Let $H' := v^0u^0Q_1u^1P^2u^2v^2$. By Claims 3 and 4, $H'$ is homotopic to $H$, and thus $||H'|| = ||H||$. This means that we can apply all previous claims to $H'$ instead of $H$. In particular, it follows from Claim 4 applied to $H'$ and $Q = Q_2$ that $V(Q_1) \cap V(Q_2) = \{u^1\}$. ⊥

Claim 7. Let $i \in [2]$. Then there is a disk $D \subseteq S$ such that $G(Q^i) \subseteq \text{int}(D)$.

Proof. This follows from Claim 6 and Lemma 15.5.9. ⊥

Claims 6, 2, 7 imply that $B$ is a belt, and Claim 5 implies that this belt is reducing. ⊓⊔
Chapter 16

Decompositions of Almost Embeddable Graphs

In this chapter, we take the last big step of the proof of our main theorem by proving a Definable Structure Theorem for Almost Embeddable Graphs, stating that such graphs admit definable ordered treelike decompositions, and a Completion Theorem, stating that a pre-decomposition that has a star completion whose centre has an almost embeddable torso admits an ordered completion. These theorems generalise the corresponding theorems for almost planar graphs, Theorem 13.5.1 and Theorem 14.1.3.

The proofs of both theorems are by induction on the surface, where almost planar graphs and graphs embeddable in a surface form the base case. In the inductive step we either delete a simplifying patch or a reducing belt from our graph. By the induction hypothesis, we have a decomposition of the resulting simpler graph, and then we have to extend this decomposition to the original graph. To be able to do this, we need two tailor-made extension lemmas, the Combination Lemma 16.1.1 and the Last Extension Lemma 16.2.1. Proving these two lemmas (in Sections 16.1 and 16.2) is the main technical challenge of this chapter.

16.1 The Combination Lemma

In this section we prove an extension lemma that will allow us, in a specific situation, to combine definable ordered treelike decompositions into a new decomposition. The proof of the lemma is not entirely trivial and certainly tedious, but in some sense straightforward. Technically, it does not go beyond the proofs in Chapters 4 and 7.

Lemma 16.1.1 (Combination Lemma). Let $\Lambda^1, \Lambda^2$ be od-schemes, and let $\varphi(\bar{x}, y)$ and $\psi(\bar{x}, y_1, y_2)$ be $\text{IFP}$-formulae. Then there exists an od-scheme $\Lambda$ such that the following holds.

Let $G$ be a graph and $v \in V(G)^{[\bar{x}]}$. Let $W := \varphi[G, v, y]$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus W$. Suppose that the following conditions are satisfied.

(i) The od-scheme $\Lambda^1$ defines an ordered treelike decomposition on $G/A_1/\cdots/A_m$.
(ii) For every $i \in [m]$, the od-scheme $\Lambda^2$ defines an ordered treelike decomposition on $A_i$.
(iii) For every $i \in [m]$, the restriction of the binary relation $\psi[G, v, y_1, y_2]$ to the vertices of attachment of $A_i$ is a linear order.

Then $\Lambda$ defines an ordered treelike decomposition on $G$. 
Before we prove the lemma, we give its main application. Remember the Assumptions [15.1.7] [15.3.1] and [15.5.1] we made in the previous chapter. When we say that a graph $G$ satisfies these assumptions (for some $p, q, r \in \mathbb{N}$), it means that there is a suitable surface $S$, a local $p$-arrangement $(G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q)$ of $G$ in $S$, et cetera, such that the assumptions are satisfied. We use other variants of this terminology, for example, saying that a graph $G$ and a class $\mathcal{R}$ satisfy the assumptions. The meaning of all these should be clear.

**Lemma 16.1.2.** Let $p, q, r \in \mathbb{N}$ with $(r, q) >^* (0, 1)$. Let $\mathcal{R}$ be a class of graphs and $\Lambda^2$ an od-scheme that defines an ordered treelike decomposition of every graph in $\mathcal{R}$. Then there is an od-scheme $\Lambda$ such that for all graphs $G$, if $G$ and $\mathcal{R}$ satisfy Assumptions [15.1.7] and [15.3.1] and $G$ has a canonical simplifying patch, then $\Lambda$ defines an ordered treelike decomposition on $G$.

**Proof.** Let $\Lambda^1$ be an od-scheme with $\mathcal{E}_r \subseteq \mathcal{OT}_{\Lambda^1}$, that is, $\Lambda^1$ defines an ordered treelike decomposition on all graphs embeddable in a surface of Euler genus at most $r$. Such an od-scheme exists by Theorem [9.4.1]. Let $k$ be the length of the tuple $\overline{y}$ of variables of the formula $\text{bridge-ord}(x, x', \overline{y}, z_1, z_2)$ of Lemma [15.4.23] and let $\overline{\pi} := (x_1, \ldots, x_k+2)$. Let $\varphi(\overline{\pi}, y) := \text{int-vert}(x_1, x_2, y)$ (of Lemma [15.4.20]) and $\psi(\overline{\pi}, y_1, y_2) := \text{bridge-ord}(\overline{\pi}, y_1, y_2)$. Let $\Lambda$ be the od-scheme obtained by applying the Combination Lemma [16.1.1] to $\Lambda^1, \Lambda^2, \varphi, \psi$.

Let $G$ be a graph that, together with $\mathcal{R}$, satisfies the conditions of Assumptions [15.1.7] and [15.3.1] and has a canonical simplifying patch $Q'$. Then there are $v_1, v_2 \in V(Q)$ such that $Q := Q'^G(v_1, v_2) = Q'[v_1, v_2]$ is a minimal simplifying patch. Let $(v_3, \ldots, v_{k+2})$ be such that for every external $Q$-bridge $B$ $\text{bridge-ord}(G, \overline{\pi}, z_1, z_2)$ is a linear order on the vertices of attachment of $B$. Let $W := \varphi[G, \overline{\pi}, y] = \text{int-vert}[G, v_1, v_2, y] = V(I(Q))$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus W$. Then $G/A_1/\cdots/A_m = I^*(Q)$. By Lemma [15.4.18] we have $I^*(Q) \in \mathcal{E}_r$. Thus condition (i) of the Combination Lemma is satisfied. As $Q$ is a simplifying patch, all components $A_i$ are simpler than $G$, and thus condition (ii) is satisfied as well. Condition (iii) is satisfied by the choice of $(v_3, \ldots, v_{k+2})$. Hence $\Lambda$ defines an ordered treelike decomposition on $G$. 

**16.1.1 Proof of the Combination Lemma**

To explain the proof, we fix a graph $G$ and a tuple $\overline{\pi} \in V(G)[\overline{\pi}]$. As usual, the od-scheme we shall define will not depend on $G, \overline{\pi}$. Let $W := \varphi[G, \overline{\pi}, y]$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus W$. Suppose that (i)–(iii) holds. Let $G^* := G/A_1/\cdots/A_m$, and for $i \in [m]$ let $a_i$ be the vertex of $G^*$ corresponding to $A_i$. Let $W_i := N^G(a_i) = N^{G^*}(a_i)$ be the set of vertices of attachment of $A_i$. Let $R$ be the binary relation $\psi[G, \overline{\pi}, y_1, y_2]$, and for every $i \in [m]$ let $R_i := R \cap W_i^2$ be the restriction of $R$ to $W_i$. By (iii), $R_i$ is a linear order on $W_i$. Let $\Delta^1 := (D^1, \sigma^1, \alpha^1, \leq^1) := \Lambda^1[G^*]$.

**Step 1.** Modifying the decomposition of $G^*$.

We define a decomposition $(D', \sigma', \alpha')$ of $G^*$ as follows:

- We let $D' := D^1$.
- For every $t \in V(D')$, we let $\sigma'(t) := \sigma^1(t) \cup \bigcup_{i \in [m]} W_i$.
Claim 1. \( (D', \sigma', \alpha') \) is a treelike decomposition of \( G^* \).

Proof. \((TL.1)\) is trivial, because the decomposition \( \Delta^1 \) is treelike.

To prove \((TL.2)\), let \( t \in V(D') \). It follows immediately from \( \sigma^1(t) \cap \alpha^1(t) = \emptyset \) that \( \sigma'(t) \cap \alpha'(t) = \emptyset \). Let \( v \in \alpha'(t) \) and \( w \in N^G_G(v) \). Then \( v \in \alpha^1(t) \) and hence \( w \in \gamma^1(t) \subseteq \gamma'(t) \) by \((TL.2)\) for \( \Delta^1 \) and \((16.1.1)\).

To prove \((TL.3)\), let \( t \in V(D') \) and \( u \in N^D_G(t) \). We first prove \( \gamma'(u) \subseteq \gamma'(t) \). Let \( w \in \gamma'(u) \). If \( w \in \gamma^1(u) \), then \( w \in \gamma^1(t) \) because \( \sigma^1(u) \subseteq \gamma^1(u) \subseteq \gamma^1(t) \). As \( \sigma^1(u) \subseteq \gamma^1(u) \subseteq \gamma^1(t) \), we have \( \gamma^1(u) = \gamma^1(t) = \sigma^1(t) \cap \alpha^1(t) \). If \( a_i \in \sigma^1(t) \) then \( w \in W_i \). If \( a_i \in \alpha^1(t) \) then \( w \in W_i = N^G_G(a_i) \subseteq \sigma^1(t) \cap \alpha^1(t) = \gamma^1(t) \subseteq \gamma'(t) \), because \( N^G_G(a_i) \subseteq \sigma^1(t) \) by \((16.1.1)\). This completes the proof that \( \gamma'(u) \subseteq \gamma'(t) \). To prove \( \alpha'(u) \subseteq \alpha'(t) \), let \( w \in \alpha'(u) \). Then \( w \in \alpha^1(u) \subseteq \alpha^1(t) \), and \( w \notin W_i \) for any \( i \in [m] \) with \( a_i \in \sigma^1(u) \). Suppose for contradiction that \( w \notin \alpha^1(t) \). Then \( w \in W_i \) for some \( i \in [m] \) with \( a_i \in \sigma^1(t) \). It follows that \( a_i \in N^G_G(u) \). As \( N^G_G(\alpha^1(u)) \subseteq \sigma^1(u) \), either \( a_i \in \sigma^1(u) \) or \( a_i \in \alpha^1(u) \). We have already ruled out \( a_i \in \sigma^1(u) \). Hence \( a_i \in \alpha^1(u) \). But by the \( \beta\gamma\alpha\)-Lemma 4.2.9, this implies \( a_i \in \sigma^1(u) \), which is a contradiction.

To prove \((TL.4)\), let \( t \in V(D') \) and \( u_1, u_2 \in N^D_G(t) \). If \( u_1 \parallel \Delta^1 u_2 \) then \( u_1 \parallel \Delta^1' u_2 \). So suppose that \( \gamma^1(u_1) \cap \gamma^1(u_2) = \sigma^1(u_1) \cap \sigma^1(u_2) \). Suppose for contradiction that \( \sigma^1(u_1) \cap \alpha^1(u_2) \neq \emptyset \). Then there is an \( i \in [m] \) such that \( a_i \in \sigma^1(u_1) \) and \( W_i \cap \alpha^1(u_2) \neq \emptyset \). Let \( w \in W_i \cap \alpha^1(u_2) \). As \( N^G_G(\alpha^1(u_2)) \subseteq \sigma^1(u_2) \), it holds that \( a_i \in \gamma^1(u_2) \). Thus \( a_i \in \gamma^1(u_2) \), by \((16.1.1)\) and because \( a_i \notin \bigcup_{j=1}^m W_j \). By the \( \beta\gamma\alpha\)-Lemma 4.2.9, it follows that \( a_i \in \gamma^1(u_2) \cap \sigma^1(u_1) \subseteq \gamma^1(t) \cap \beta^1(t) = \sigma^1(u_2) \). Hence \( w \in W_i \subseteq \sigma^1(u_2) \), which is a contradiction, because \( \sigma^1(u_2) \cap \alpha^1(u_2) = \emptyset \). This proves \( \sigma^1(u_1) \cap \alpha^1(u_2) = \emptyset \), and by symmetry, we also have \( \sigma^1(u_2) \cap \alpha^1(u_1) = \emptyset \). Since \( \alpha^1(u_1) \cap \alpha^1(u_2) \) for \( i = 1, 2 \) we have \( \alpha^1(u_1) \cap \alpha^1(u_2) = \emptyset \) and thus \( \gamma^1(u_1) \cap \gamma^1(u_2) = \sigma^1(u_1) \cap \sigma^1(u_2) \).

Claim 2. Let \( t \in V(D') \). Then:

(1) \( \beta'(t) \subseteq \beta^1(t) \cup \bigcup_{i \in [m]} W_i \).

(2) For all \( i \in [m] \), if \( a_i \in \beta'(t) \) then \( W_i \subseteq \beta'(t) \).

Proof. We have

\[
\beta'(t) = \gamma'(t) \setminus \bigcup_{u \in N^D_G(t)} \alpha^1(u)
\]
are all disjoint from the treelike decomposition defined by $\Lambda$ decompositions $\Delta$. For every $i \in [m]$ let $\alpha^1(u)$ \( u \in N^D_+(t) \)

\[ \alpha^1(u) \cap \bigcup_{a_i \in \sigma^1(u)} a_i \in \sigma^1(u) \]

\[ \leq \left( \gamma^1(t) \cup \bigcup_{i \in [m]} W_i \right) \setminus \bigcup_{a_i \in \sigma^1(u)} \left( \alpha^1(u) \cup \bigcup_{i \in [m]} W_i \right) \]

\[ = \beta^1(t) \cup \bigcup_{i \in [m]} W_i \bigcup \bigcup_{i \in [m]} W_i \bigcup \bigcup_{i \in [m]} W_i \]

To prove (2), let $i \in [m]$ such that $a_i \in \beta^1(t)$. If $a_i \in \sigma^1(u)$ then $W_i \subseteq \sigma^1(u) \subseteq \beta^1(t)$, and similarly if $a_i \in \sigma^1(t)$ then $W_i \subseteq \sigma^1(t) \subseteq \beta^1(t)$. Otherwise, we have $W_i = N^G(a_i) \in \beta^1(t)$ because the sets $\sigma^1(t)$ and $\sigma^1(u)$ for $u \in N^D_+(t)$ separate $\beta^1(t)$ from $V(G^*) \setminus \beta^1(t)$.

To turn the decomposition $(D', \sigma', \alpha')$ into an ordered decomposition, for every $t \in V(D')$ we define a linear order $\leq^1_t$ on $\beta^1(t)$ as follows. Let $\beta^1(t) \cap \{a_1, \ldots, a_m\} = \{a_{i_1}, \ldots, a_{i_\ell}\}$. To simplify the notation, we assume that $i_j = j$ for all $j \in [\ell]$ and that $a_1 \leq^1_t a_2 \leq^1_t \ldots \leq^1_t a_\ell$.

Let $U_0 := \emptyset$ and $U_i := \bigcup_{j \leq i} W_j$ for $1 \leq i \leq \ell$. By Claim 2, we have $\beta^1(t) \leq^{\ell-1}_t \beta^1(t)$. For $v, w \in \beta^1(t)$, we let

\[ v \leq^1_t w :\iff \begin{cases} v, w \in \beta^1(t) 	ext{ and } v \leq^1_t w, \\
 or v \in U_i \cup \beta^1(t) \text{ and } w \notin U_i \cup \beta^1(t) \text{ for some } i \in [0, \ell - 1], \\
 or v, w \in W_{i+1} \setminus (U_i \cup \beta^1(t)) \text{ for some } i \in [0, \ell - 1] \text{ and } (v, w) \in R_{i+1}. \end{cases} \]

As $\leq^1_t$ and $R_i$ for $i \in [\ell]$ are linear orders, $\leq^1_t$ is indeed a linear order on $\beta^1(t)$. We let $\Delta' := (D', \sigma', \alpha', \leq^1_t)$. Then $\Delta'$ is an ordered treelike decomposition of $G^*$.

**Step 2.** Extending the decomposition to $G$.
For every $i \in [m]$, let $M_i \subseteq V(D')$ be the set of all nodes of $D'$ that are $\leq^{D'}$-minimal among all $t \in V(D')$ with $a_i \in \beta^1(t)$. Furthermore, let $\Delta_i := (D_i, \sigma_i, \alpha_i, \leq_i) := \Delta^2[A_i]$ be the ordered treelike decomposition defined by $\Delta^2$ on $A_i$. Without loss of generality we assume that the decompositions $\Delta_i$ are normal, that the node sets $V(D_i)$ are mutually disjoint, and that they are all disjoint from $V(D')$. We define a decomposition $(D'', \sigma'', \alpha'')$ of $G$ as follows:

- $V(D'') := V(D') \cup \bigcup_{i=1}^m V(D_i)$;
- $E(D'') := E(D') \cup \bigcup_{i=1}^m \left( E(D_i) \cup \{ (s, t) \mid s \in M_i, t \in V(D_i) \ \leq^{D_i} \text{-minimal} \} \right)$;
- $\sigma''(t) := \begin{cases} \sigma'(t) \setminus \{a_1, \ldots, a_m\} & \text{if } t \in V(D'), \\
 \sigma_i(t) \cup W_i & \text{if } t \in V(D_i) \text{ for some } i \in [m]. \end{cases}$
- $\alpha''(t) := \begin{cases} \left( \alpha'(t) \setminus \{a_1, \ldots, a_m\} \right) \cup \bigcup_{i \in [m]} V(A_i) & \text{if } t \in V(D'), \\
 \alpha_i(t) & \text{if } t \in V(D_i) \text{ for some } i \in [m]. \end{cases}$

M. Grohe, *Definable Graph Structure Theory*
As $A_i$ is connected and $\Delta_i$ is normal, we have $\alpha''(t) = \alpha_i(t) = \nu(A_i)$ for all $\preceq_{D_1}$-minimal $t \in V(D_i)$.

**Claim 3.** $(D'', \sigma'', \alpha'')$ is a treelike decomposition of $G$.

**Proof.** The directed graph $D''$ is acyclic because $D'$ and the $D_i$ are, and there are only edges from $D'$ to $D_i$, but no edges among the $D_i$ or from $D_i$ back to $D'$.

To verify (TL.2), let $t \in V(D'')$. Remember that $W = V(G^*) \setminus \{a_1, \ldots, a_m\} = V(G) \setminus \bigcup_{i=1}^m V(A_i)$.

**Case 1:** $t \in V(D')$.

Then clearly we have $\sigma''(t) \cap \alpha''(t) = \emptyset$. To see that $N^G(\alpha''(t)) \subseteq \sigma''(t)$, let $v \in \alpha''(t)$ and $w \in N^G(v)$. We shall prove that $w \in \gamma''(t)$.

**Case 1a:** $v, w \in W$.

Then $v \in \alpha'(t)$ and hence $w \in \gamma'(t) \setminus \{a_1, \ldots, a_m\} \subseteq \gamma''(t)$.

**Case 1b:** $v \in W$ and $w \in V(A_i)$ for some $i \in [m]$.

Then $v \in W_i$. As $v \in \alpha'(t)$, we have $a_i \in \gamma'(t)$. If $a_i \in \beta'(s)$ for some $s \in V(D')$ with $s \preceq_{D'} t$, then by the $\beta$-$\gamma$-$\sigma$-Lemma 4.2.9 we have $a_i \in \sigma'(t)$ and therefore $v \in W_i \subseteq \sigma'(t)$, which contradicts $v \in \alpha'(t)$. As $a_i \in \gamma'(t)$, it follows that there is some $u \in M_i$ such that $t \preceq_{D'} u$. Thus $w \in V(A_i) \subseteq \alpha''(t)$.

**Case 1c:** $v \in V(A_i)$ for some $i \in [m]$.

Then there is a $u \in M_i$ with $t \preceq_{D'} u$. Hence $V(A_i) \cup W_i \subseteq \gamma''(t)$. Moreover, $w \in N^G(v) \subseteq V(A_i) \cup W_i$.

**Case 2:** $t \in V(D_i)$ for some $i \in [m]$.

Then (TL.2) follows from (TL.2) for $(D_i, \sigma_i, \alpha_i)$ and the fact that $W_i \subseteq \sigma''(t)$.

To prove (TL.3) let $t \in V(D'')$ and $u \in N^D(t)$.

**Case 1:** $t \in V(D_i)$ for some $i \in [m]$.

Then $u \in V(D_i)$ as well, and we have $\gamma''(u) = \gamma_i(u) \cup W_i \subseteq \gamma_i(t) \cup W_i = \gamma''(t)$ and $\alpha''(u) = \alpha_i(u) \subseteq \alpha_i(t) = \alpha''(t)$.

**Case 2:** $t \in V(D')$ and $u \in V(D_i)$ for some $i \in [m]$.

Then $t \in M_i$. Hence $W_i \subseteq \gamma''(t)$ and $V(A_i) \subseteq \alpha''(t)$. Thus $\gamma''(u) \subseteq W_i \cup V(A_i) \subseteq \gamma''(t)$ and $\alpha''(u) \subseteq V(A_i) \subseteq \alpha''(t)$.

**Case 3:** $t, u \in V(D')$.

Then for all $i \in [m]$, if $V(A_i) \subseteq \alpha''(u)$ then $V(A_i) \subseteq \alpha''(t)$. Hence $\gamma''(u) \subseteq \gamma''(t)$ and $\alpha''(u) \subseteq \alpha''(t)$ follow from (TL.3) for $(D', \sigma', \alpha')$.

To prove (TL.4) let $t \in V(D'')$ and $t_1, t_2 \in N^D(t)$.

**Case 1:** $t \in V(D_i)$ for some $i \in [m]$.

Then $t_1, t_2 \in V(D_i)$, and either $\gamma''(t_1) \cap \gamma''(t_2) = (\gamma_i(t_1) \cup W_i) \cap (\gamma_i(t_2) \cup W_i) = (\sigma_i(t_1) \cup W_i) \cap (\sigma_i(t_2) \cup W_i) = \sigma''(t_1) \cap \sigma''(t_2)$ or $\alpha''(t_1) = \alpha_i(t_1) = \alpha_i(t_2) = \alpha''(t_2)$ and $\sigma''(t_1) = \sigma_i(t_1) \cup W_i = \sigma_i(t_2) \cup W_i = \sigma''(t_2)$.

**Case 2:** $t \in V(D')$ and $t_1, t_2 \in V(A_i)$ for some $i \in [m]$.

Then $t_1, t_2$ are $\preceq_{D_i}$-minimal, and we have $\alpha''(t_1) = V(A_i) = \alpha''(t_2)$ and $\sigma''(t_1) = W_i = \sigma''(t_2)$.
Case 3: \( t \in V(D') \) and \( t_1 \in V(A_i), t_2 \in V(A_j) \) for distinct \( i, j \in [m] \).
Then \( \gamma''(t_1) \cap \gamma''(t_2) = W_i \cap W_j = \sigma''(t_1) \cap \sigma''(t_2) \).

Case 4: \( t, t_1 \in V(D') \) and \( t_2 \in V(A_i) \) for some \( i \in [m] \).
Then \( t \in M_i \). Thus \( W_i \subseteq \beta'(t) \) by Claim 2(2) and hence \( \gamma'(t_1) \cap W_i = \sigma'(t_1) \cap W_i \). It follows that \( \gamma''(t_1) \cap \gamma''(t_2) = \gamma'(t_1) \cap W_i = \sigma'(t_1) \cap W_i = \sigma''(t_1) \cap \sigma''(t_2) \).

Case 5: \( t, t_2 \in V(D') \) and \( t_1 \in V(A_i) \) for some \( i \in [m] \).
Symmetric to Case 4.

Case 6: \( t, t_1, t_2 \in V(D') \).
By [TL.4] for \( \Delta' \), either \( t_1 \perp_{\Delta'} t_2 \) or \( t_1 \parallel_{\Delta'} t_2 \).

Case 6a: \( t_1 \perp_{\Delta'} t_2 \).
Suppose for contradiction that \( \gamma''(t_1) \cap \gamma''(t_2) \neq \sigma''(t_1) \cap \sigma''(t_2) \). Then there is an \( i \in [m] \) such that \( V(A_i) \subseteq \gamma''(t_1) \cap \gamma''(t_2) \). Hence for \( j = 1, 2 \), there is a \( u_j \in M_i \) such that \( t_j \leq_{D'} u_j \). Then for \( j = 1, 2 \) we have \( a_i \in \beta'(u_j) \subseteq \gamma'(t_j) \) and thus \( a_i \in \gamma'(t_1) \cap \gamma'(t_2) \subseteq \sigma'(t_1) \cap \sigma'(t_2) \subseteq \beta'(t) \). As \( t \not<_{D'} u_j \), this contradicts the minimality of \( u_j \).

Case 6b: \( t_1 \parallel_{\Delta'} t_2 \).
Then \( \sigma''(t_1) = \sigma''(t_2) \). To prove that \( \alpha''(t_1) = \alpha''(t_2) \), we have to prove that for all \( i \in [m] \) there is a \( u_1 \in M_i \) such that \( t_1 \leq_{D'} u_1 \) if and only if there is a \( u_2 \in M_i \) such that \( t_2 \leq_{D'} u_2 \). By symmetry, it suffices to prove the forward direction of this equivalence. So let \( i \in [m] \) and \( u_1 \in M_i \) such that \( t_1 \leq_{D'} u_1 \). Then \( a_i \in \alpha'(u_1) \subseteq \alpha'(t_1) = \alpha'(t_2) \). Hence there is a \( u_2 \in V(D') \) such that \( t_2 \leq_{D'} u_2 \) and \( a_i \in \beta'(u_2) \). As \( a_i \in \alpha'(t_2) \), by the \( \beta\gamma\sigma\)-Lemma [TL.5] there is no \( s \leq_{D'} t_2 \) such that \( a_i \in \beta'(s) \). Hence there is a \( u_2 \in M_i \) such that \( t_2 \leq_{D'} u_2 \).

To prove [TL.5] let \( A \) be a connected component of \( G \). Without loss of generality, we may assume that \( A_1, \ldots, A_\ell \subseteq A \) and \( A_{\ell+1}, \ldots, A_m \not\subseteq A \). Let \( A^* = A/A_1/\ldots/A_\ell \). Then \( A^* \) is a connected component of \( G^* \), and by [TL.5] there is a node \( t \in V(D') \) such that \( \gamma'(t) = \alpha'(t) = V(A^*) \). Moreover, for all \( i \in [\ell] \) there is a node \( u \in M_i \) with \( t \leq_{D'} u \). Thus \( V(A_i) \subseteq \gamma''(t) \) for all \( i \in [\ell] \), and it follows that \( V(A) = \alpha''(t) \). Furthermore, \( \sigma''(t) \subseteq \sigma'(t) = \emptyset \). \( \blacksquare \)

Claim 6. For every \( t \in V(D') \), it holds that \( \beta''(t) = \beta'(t) \setminus \{a_1, \ldots, a_m\} \), and for every \( i \in [m] \) and \( t \in V(D_i) \), it holds that \( \beta''(t) = \beta_i(t) \cup W_i \).

Proof. For \( t \in V(D') \), just note that if \( V(A_i) \subseteq \alpha''(t) \) then there is a \( u \in V(D_i) \) such that \( t \not<_{D''} u \) and \( \alpha''(u) = V(A_i) \). This implies that \( \beta(t) \cap V(A_i) = \emptyset \). For \( t \in V(D_i) \), the claim follows immediately from the definitions. \( \blacksquare \)

For every \( t \in V(D'') \), we define a linear order \( \leq''_i \) on \( \beta''(t) \) as follows:

- If \( t \in V(D') \), then we let \( \leq''_i \) be the restriction of \( \leq'_i \) to \( \beta''(t) \setminus \{a_1, \ldots, a_m\} \).
- If \( t \in V(D_i) \) for some \( i \in [m] \), then for \( v, w \in \beta(t) = W_i \cup \beta_i(t) \) we let

\[
v \leq''_i w :\iff \begin{cases} v, w \in W_i \text{ and } (v, w) \in R_i, \\ v \in W_i \text{ and } w \in \beta_i(t), \\ v \in \beta_i(t) \text{ and } w \in \beta_i(t). \end{cases}\]

M. Grohe, Definable Graph Structure Theory
Then $\Delta'' := (D'', \sigma'', \alpha'', \leq''')$ is an ordered treelike decomposition of $G$. Using the Transduction Lemma (Fact 2.4.6), it is straightforward to define an od-scheme $\Lambda$ such that $\Lambda[G] = \Delta''$. We leave the details to the reader.

This completes the proof of the Combination Lemma 16.1.1. 

16.2 The Last Extension Lemma

In this section we prove the last and arguably most complicated extension lemma of this book. Essentially, it says that if we simplify a graph by cutting through a belt and we can define an ordered treelike decomposition on the simplified graph, then we can extend this decomposition to the original graph.

Lemma 16.2.1 (Last Extension Lemma). Let $p, q, r \in \mathbb{N}$ with $(r, q) >^* (0, 1)$. Then for every od-scheme $\Lambda'$ there is an od-scheme $\Lambda$ such that for all graphs $G$ and classes $\mathcal{R}$ of graphs, the following holds. Suppose that

(i) $G$ and $\mathcal{R}$ satisfy Assumptions 15.1.7, 15.3.1, and 15.5.1;

(ii) for all $G' \in \mathcal{R}$ the o-decomposition $\Lambda'[G']$ is an ordered treelike decomposition of $G'$.

Then $\Lambda[G]$ is an ordered treelike decomposition.

To explain the difficulty of the proof, let $G$ be a graph and $\mathcal{R}$ a class of graphs satisfying the assumptions, and let $B$ be a reducing belt in $G$. (Such a belt exists by the Belt Lemma 15.5.8.) Let $Q_1, \ldots, Q_n$ be the patches of the belt. Let us assume for a moment that the patches $Q_h$ are all nontrivial. For every $h \in [n]$, let $C_h$ be the outer facial cycle of $Q_h$. Let $I_h$ be the interior graph and $J_h := I_h \setminus C_h$. If we cut through the belt $B$, all of $J_h$ is deleted, and the cycle $C_h$ is divided into two paths, say, $P^l_h$ and $P^r_h$. Let $H'$ be the graph obtained by cutting through $B$. Suppose we have an ordered treelike decomposition $\Delta$ of $H'$.

We would like to extend this decomposition to $G$ by re-inserting the subgraphs $J_h$. If we have a bag of $\Delta$ that contains the cycle $C_h$, we can just add $J_h$ to this bag. However, it may be that the two pieces $P^l_h$ and $P^r_h$ of $C_h$ never appear together in a bag of $\Delta$. Thus we first have to modify the decomposition by “pulling the two paths $P^l_h$ and $P^r_h$ together” to a single bag. Then we can add $J_h$ to this bag. We can do this, simultaneously for all $h \in [n]$, and obtain a treelike decomposition of $G$, but it is not yet an ordered decomposition. Now we want to use Lemma 15.4.4 which allows us to define a linear order on every $I_h$. However, this linear order is not canonical, as its definition requires an additional parameter. Essentially, we can restrict our attention to two linear orders $\leq^l_h$ and $\leq^r_h$, obtained by choosing the parameter from $P^l_h$ and $P^r_h$, respectively. Since there is no canonical way to choose one of these two orders, we simply replace each node of the decomposition whose bag contains $I_h$ by two copies of this node, and in each copy we use one of $\leq^l_h$ and $\leq^r_h$ to order $I_h$. This would work if we only had to deal with one patch $Q_h$, but we have to deal with all $n$ patches simultaneously, and we cannot replace a node by $2^n$ copies. To handle this problem, we cluster the patches into components of dependent patches. Then each component only requires two copies of a node, and different components can be dealt with independently. This last part of the proof is reminiscent of the proof of Lemma 10.3.2, another unpleasant extension lemma.

In the proof, we shall repeatedly use the construction described in the following lemma. The reader may notice that we have used essentially the same construction in the proof of the Combination Lemma 16.1.1.
Lemma 16.2.2. Let \( \Delta = (D, \sigma, \alpha) \) be a treelike decomposition of a graph \( H \), and let \( I \subseteq H \) be a connected subgraph. Let \( \Delta' = (D', \sigma', \alpha') \) be the decomposition defined as follows:

- \( D' := D \).
- For all \( t \in V(D) \),
  \[ \sigma'(t) := \begin{cases} \sigma(t) & \text{if } V(I) \cap \sigma(t) = \emptyset, \\ \sigma(t) \cup V(I) & \text{if } V(I) \cap \sigma(t) \neq \emptyset. \end{cases} \]
- For all \( t \in V(D) \),
  \[ \alpha'(t) := \begin{cases} \alpha(t) & \text{if } V(I) \cap \sigma(t) = \emptyset, \\ \alpha(t) \setminus V(I) & \text{if } V(I) \cap \sigma(t) \neq \emptyset. \end{cases} \]

Then \( \Delta' \) is a treelike decomposition of \( H \), and for all \( t \in V(D) \) it holds that

\[ \beta'(t) := \begin{cases} \beta(t) & \text{if } V(I) \cap \beta(t) = \emptyset, \\ \beta(t) \cup V(I) & \text{if } V(I) \cap \beta(t) \neq \emptyset. \end{cases} \]

Proof. Note that for all \( t \in V(D) = V(D') \) we have

\[ \sigma'(t) \supseteq \sigma(t) \quad \text{and} \quad \alpha'(t) \subseteq \alpha(t) \quad \text{and} \quad \gamma'(t) \supseteq \gamma(t). \]

\( \Delta' \) satisfies [TL.1] because \( \Delta \) does.

To prove [TL.2] let \( t \in V(D) \). Since \( \sigma(t) \cap \alpha(t) = \emptyset \) we have \( \sigma'(t) \cup \alpha'(t) = \emptyset \). Moreover, if \( v \in \alpha'(t) \subseteq \alpha(t) \) then \( N^H(v) \subseteq \gamma(t) \subseteq \gamma'(t) \). Thus \( N^H(\alpha'(t)) \subseteq \sigma(t) \).

To prove [TL.3] let \( t \in V(D) \) and \( u \in N_D^+(t) \). We first prove \( \alpha'(u) \subseteq \alpha'(t) \). Let \( v \in \alpha'(u) \subseteq \alpha(u) \subseteq \alpha(t) \). Suppose for contradiction that \( v \notin \alpha'(t) \). Then \( v \in V(I) \) and \( V(I) \cap \alpha(t) \neq \emptyset \). Thus the connected set \( V(I) \) has a nonempty intersection with \( \alpha(u) \) and with \( \sigma(t) \subseteq V(H) \setminus \alpha(u) \). It follows that \( V(I) \) has a nonempty intersection with \( \sigma(u) \), which implies that \( V(I) \cap \alpha'(u) = \emptyset \). This is a contradiction. Hence \( \alpha'(u) \subseteq \alpha'(t) \). To prove that \( \gamma'(u) \subseteq \gamma'(t) \), it suffices to prove that \( \sigma'(u) \subseteq \gamma'(t) \). If \( \sigma'(u) \subseteq \gamma'(t) \), there is nothing to prove. Otherwise, let \( \gamma \in \sigma'(u) \setminus \gamma(t) \). Then \( \exists v \in \sigma'(u) \setminus \gamma(t) \). If \( V(I) \cap \sigma(t) = \emptyset \), then \( V(I) \cap \sigma(u) \subseteq \alpha(t) \). Thus \( V(I) \subseteq \alpha(t) \), because \( V(I) \) is connected. Hence \( v \in \alpha(t) \subseteq \gamma(t) \subseteq \gamma'(t) \). If \( V(I) \cap \sigma(t) \neq \emptyset \) then \( V(I) \subseteq \sigma'(t) \) and thus \( v \in \sigma'(t) \subseteq \gamma'(t) \).

To prove [TL.4] let \( t \in V(D) \) and \( u_1, u_2 \in N_D^+(t) \). If \( u_1 \parallel^\Delta u_2 \), then \( u_1 \parallel^{\Delta'} u_2 \). So suppose that \( u_1 \perp^\Delta u_2 \). We shall prove that \( u_1 \perp^{\Delta'} u_2 \). Note first that \( \alpha'(u_1) \cap \alpha'(u_2) \subseteq \alpha(u_1) \cap \alpha(u_2) = \emptyset \). Suppose for contradiction that \( \alpha'(u_1) \cap \alpha'(u_2) \neq \emptyset \). Let \( v \in \alpha'(u_1) \setminus \alpha'(u_2) \). Then \( v \in V(I) \cap \alpha(u_1) \) and \( V(I) \cap \alpha(u_2) = \emptyset \). This is impossible, because \( \sigma(u_1) \) separates \( \alpha(u_1) \) from \( \alpha(u_2) \setminus \gamma(u_1) \) and \( I \) is connected. Thus \( \alpha'(u_1) \cap \alpha'(u_2) = \emptyset \), and by symmetry \( \alpha'(u_1) \cap \alpha'(u_2) = \emptyset \). This implies \( \gamma'(u_1) \cap \gamma'(u_2) = \alpha'(u_1) \cap \alpha'(u_2) \).

To prove [TL.5] let \( A \) be a connected component of \( H \). Let \( t \in V(D) \) with \( \sigma(t) = \emptyset \) and \( \alpha(t) = V(A) \). Then \( \sigma'(t) = \sigma(t) = \emptyset \) and \( \alpha'(t) = \alpha(t) = V(A) \). Thus \( \Delta' \) is a treelike decomposition.

To prove the assertion about the bags, let \( t \in V(D) \). Note that we have \( \gamma(t) \subseteq \gamma'(t) \subseteq \gamma(t) \cup V(I) \) and \( \alpha(u) \supseteq \alpha'(u) \supseteq \alpha(u) \setminus V(I) \) for all \( u \in N_D^+(t) \). This implies \( \beta(t) \subseteq \beta'(t) \subseteq \beta(t) \cup V(I) \).
Case 1: $V(I) \cap \beta(t) = \emptyset$.
Then $\sigma(t) \cap V(I) = \emptyset$ and thus $\gamma'(t) = \gamma(t)$. Moreover, for all $u \in N^D_+(t)$ we also have $\sigma(u) \cap V(I) \subseteq \beta(t) \cap V(I) = \emptyset$ and thus $\alpha'(u) = \alpha(u)$. This implies $\beta'(t) = \beta(t)$.

Case 2: $V(I) \cap \beta(t) \neq \emptyset$.

Case 2a: $V(I) \cap \sigma(t) \neq \emptyset$.
Then $V(I) \subseteq \sigma'(t) \subseteq \beta'(t)$.

Case 2b: $V(I) \cap \sigma(u) \neq \emptyset$ for some $u \in N^D_+(t)$.
Then $V(I) \subseteq \sigma(u) \subseteq \beta'(t)$.

Case 2c: $V(I) \cap \beta(t) \neq \emptyset$ and $V(I) \cap \sigma(t) = \emptyset$ and $V(I) \cap \sigma(u) = \emptyset$ for all $u \in N^D_+(t)$.
Then $V(I) \subseteq \beta(t)$, because $I$ is connected and $\sigma(t) \cup \bigcup_{u \in N^D_+(t)} \sigma(u)$ separates $\beta(t)$ from $V(H) \setminus \beta(t)$.

Thus $\beta'(t) = \beta(t) \cup V(I)$.

Proof of the Last Extension Lemma 16.2.1. To explain the proof, we fix a graph $G$ and a class $\mathcal{R}$ satisfying (i) and (ii). As usual, the od-scheme we shall define will not depend on $G$ or $\mathcal{R}$. Let $u^0, u^1, u^2 \in V(G)$ and $Q^1 := Q^G(u^0, u^1)$, $Q^2 := Q^G(u^1, u^2)$ such that $B := (u^0, Q^1, u^1, Q^2, u^2)$ is a reducing belt in $G$. Such a belt exists by the Belt Lemma 15.5.8. Let $u_0 := u^0, u_1, \ldots, u_n := u^1$ be the articulation vertices of $Q^1$ and $u_{n_1} = u^1, u_{n_1+1}, \ldots, u_{n_1+n_2} = u^2$ the articulation vertices of $Q^2$, ordered by height. Let $n := n_1 + n_2$. Recall that the vertex set $V(B)$ and edge set $E(B)$ are definable by IFP-formulae from the three parameters $u^0, u^1, u^2$.

The set $\mathrm{art}(B)$ of articulation vertices is definable as well. Furthermore, there is an IFP-formula that defines the natural linear order (corresponding to the enumeration $u_0, \ldots, u_n$) on the articulation vertices. For each $i \in [n]$, let $Q_i := Q^G(u_{i-1}, u_i)$. Then $Q_1, \ldots, Q_n$ are the patches of $B$. For each $i \in [n]$ such that $Q_i$ is a nontrivial patch, let $I_i := I(Q_i)$, $C_i := C(Q_i)$, and $J_i := I_i \setminus C_i$. Let $P'_i, P''_i$ be the two components of $C_i \setminus \{u_{i-1}, u_i\}$, and let $u^c_i \in V(P'_i), u^d_i \in V(P''_i)$ be the neighbours of $u_i$ on $C_i$. To unify the notation, if the patch $Q_i$ is trivial, we let $I_i := C_i := G(Q_i) = \{\{u_{i-1}, u_i\}, \{u_{i-1} u_i\}\}$ and $J_i := P'_i := P''_i := \emptyset$ and $u^c_i := u^d_i := u_{i-1}$. Recall that by Lemma 15.4.12 and Corollary 15.4.13 there are IFP-formulae defining the vertex and edge sets of the graphs $I_i, C_i, J_i$ from the two parameters $u_{i-1}, u_i$. Furthermore, by Lemma 15.4.15 there is an IFP-formula $\mathrm{int-ord}(x, x^c, x^d, y^c, y^d)$ such that for all $u^c, u^d \in V(C_i) \setminus \{u_{i-1}, u_i\}$ the binary relation $\mathrm{int-ord}(G, u_{i-1}, u_i, u^c, u^d, y^c, y^d)$ is a linear order of $V(I_i)$. If $Q_i$ is trivial, we may assume that $\mathrm{int-ord}(G, u_{i-1}, u_i, u_{i-1}, y^c, y^d)$ is the linear order that puts $u_{i-1}$ before $u_i$. We let $\leq^c_i := \mathrm{int-ord}(G, u_{i-1}, u_i, u^c_i, y^c, y^d)$ and $\leq^d_i := \mathrm{int-ord}(G, u_{i-1}, u_i, u^d_i, y^c, y^d)$.

Let $H' := \text{Cut}(B)$ and recall that $H' \cap I_i = P'_i \cup P''_i$. By Lemma 15.5.6 there are vertices $u^3, u^4$ such that $H' \setminus \{u^3, u^4\} \in \mathcal{R}$. We choose such vertices $u^3, u^4$. Then $\Lambda'$ defines an ordered treelike decomposition on $H' \setminus \{u^3, u^4\}$. It follows from Lemma 15.5.7 that the vertex set and edge set of $H' \setminus \{u^3, u^4\}$ are IFP-definable with parameters $u_0, \ldots, u_4$. Hence by the Transduction Lemma for Definable O-Decompositions 7.2.3 there is a parametrised odscheme $\Lambda^1(x^0, \ldots, x^4)$ that defines an ordered treelike decomposition $\Lambda^1 := (D^1, \sigma^1, \alpha^1)$ on $H' \setminus \{u^3, u^4\}$ within $(G, u^0, \ldots, u^4)$.

Step 1. Extension to $O(B)$.
Let $H := O(B)$ (see (15.5.1)). In this step, we shall construct an ordered treelike decomposition $\Delta^2$ of $H$. Note that $H$ is connected. Recall that $V(H) \setminus V(H') = \text{art}(B) = \{u_0, \ldots, u_n\}$.

Preliminary Version
To simplify the notation, we let $u_{n+1} := u^3$ and $u_{n+2} := u^4$. Recall that we may have $u^3 = u^4$. If this is the case, we let $n' := n + 1$ and if $u^3 \neq u^4$ we let $n' := n + 2$.

We let $art(B) := \{u_0, \ldots, u_{n'}\}$, and we let $\leq_{art}$ be the natural order on $art(B)$, that is, we let $u_i \leq_{art} u_j$ if and only if $i \leq j$. Our new decomposition $\Delta^2$ will be the ordered decomposition obtained from $\Delta^1$ by adding $art'(B)$ to all bags and order it by $\leq_{art}$ appended to the order $\leq^1$ on the bag. To achieve this, we first define a decomposition $(D^2, \sigma^2, \alpha^2)$ as follows:

- $D^2$ is obtained from $D^1$ by adding a new node $r$ and edges from $r$ to all nodes of $D^1$ of in-degree 0.
- $\sigma^2(t) := \emptyset$ and $\sigma^2(t) := \sigma^1(t) \cup art'(B)$ for every $t \in V(D^1)$.
- $\alpha^2(t) := V(H)$ and $\alpha^2(t) := \alpha^1(t)$ for every $t \in V(D^1)$.

**Claim 1.**

1. $(D^2, \sigma^2, \alpha^2)$ is a treelike decomposition of $H$.
2. $\beta^2(r) := art'(B)$ and $\beta^2(t) := \beta^1(t) \cup art'(B)$ for every $t \in V(D^1)$.

**Proof.** Straightforward.

We let $\leq^2 := \leq_{art}$, and for every $t \in V(D^1)$ we define the order $\leq^2_t$ by

$$v \leq^2_t w \iff \begin{cases} v, w \in \beta^1(t) \text{ and } v \leq^1_t w \\ \text{or } v \in \beta^1(t) \text{ and } w \in art'(B) \\ \text{or } v, w \in art'(B) \text{ and } v \leq_{art} w, \end{cases}$$

for all $v, w \in \beta^2(t)$.

**Claim 2.** $\Delta^2 := (D^2, \sigma^2, \alpha^2, \leq^2)$ is an ordered treelike decomposition of $H$.

**Proof.** Straightforward.

**Claim 3.** There is a parametrised od-scheme $\Lambda^2(x^0, \ldots, x^4)$ that defines the ordered decomposition $\Delta^2$ of $H$ within $(G, x^0, \ldots, x^4)$.

**Proof.** We need to be more specific about how the vertices of $V(D^2)$ are represented. Let $k \in \mathbb{N}$ such that $V(D^1) \subseteq V(G)^k$. The nodes of $D^2$ are $(k+1)$-tuples, where we use the last place to indicate whether a tuple is the root node $r$ or a node of $D^1$. We let

$$V(D^2) := \{ (\underbrace{u^0, \ldots, u^0}_{(k+1) \text{ times}}) \} \cup \{ \overline{v}u^1 \mid \overline{v} \in V(D^1) \}.$$ 

We let $r := (u^0, \ldots, u^0)$ and identify each node $\overline{v} \in V(D^1)$ with the tuple $\overline{v}u^1$. With this representation it is straightforward to define $\Delta^1$ in IFP.

Our goal is to extend the decomposition $\Delta^2$ from $H$ to $G$. Recall that $G \setminus H = \bigcup_{h=1}^m J_h$. Let us call the indices $h \in [n]$ such that $J_h \neq \emptyset$ the holes of $H$. Note that for all holes $h$ the patch $Q_h$ is nontrivial. In Steps 2–6, we shall construct a sequence of ordered treelike decompositions $\Delta^3, \ldots, \Delta^7$ of $H$ such that in $\Delta^7$ for each hole $h$ the circle $C_h$, which we call the frame of the hole, is contained in the bags of all $\leq_{D^7}$-maximal nodes $t$ whose cone contains...
Proof. This follows from Lemma 16.2.2.

Step 2. Adding the sides of the frames.
In this step, we slightly modify the decomposition $\Delta^2$ in such a way that for each hole $h$, if some bag has a nonempty intersection with $P^f_h$ or $P^r_h$ then it contains $P^f_h$, $P^r_h$, respectively.

For all $t \in V(D^2)$ we define sets $\Sigma^f(t), \Sigma^r(t), B^f(t), B^r(t)$ of holes as follows:
\[
\Sigma^f(t) := \{ h \mid V(P^f_h) \cap \sigma^2(t) \neq \emptyset \},
\]
\[
\Sigma^r(t) := \{ h \mid V(P^r_h) \cap \sigma^2(t) \neq \emptyset \},
\]
\[
B^f(t) := \{ h \mid V(P^f_h) \cap \beta^2(t) \neq \emptyset \},
\]
\[
B^r(t) := \{ h \mid V(P^r_h) \cap \beta^2(t) \neq \emptyset \}.
\]

We define a decomposition $(D^3, \sigma^3, \alpha^3)$ of $H$ as follows:

- $D^3 := D^2$.
- For all $t \in V(D^3)$, we let
\[
\sigma^3(t) := \sigma^2(t) \cup \bigcup_{h \in \Sigma^f(t)} V(P^f_h) \cup \bigcup_{h \in \Sigma^r(t)} V(P^r_h).
\]

- For all $t \in V(D^3)$, we let
\[
\alpha^3(t) := \alpha^2(t) \setminus \left( \bigcup_{h \in \Sigma^f(t)} V(P^f_h) \cup \bigcup_{h \in \Sigma^r(t)} V(P^r_h) \right).
\]

Claim 4.

1. $(D^3, \sigma^3, \alpha^3)$ is a treelike decomposition of $H$.
2. For all $t \in V(D)$ it holds that
\[
\beta^3(t) = \beta^2(t) \cup \bigcup_{h \in B^f(t)} V(P^f_h) \cup \bigcup_{h \in B^r(t)} V(P^r_h).
\]

Proof. This follows from Lemma 16.2.2.

To complete the definition of $\Delta^3$, we need to define the linear orders $\leq^2_h$ on the bags. We need a few preliminary definitions. Let $t \in V(D^3)$. For each hole $h$, we let
\[
\beta^+_h(t) := \begin{cases}
\emptyset & \text{if } h \notin B^f(t) \cup B^r(t), \\
V(P^f_h) & \text{if } h \in B^f(t) \setminus B^r(t), \\
V(P^r_h) & \text{if } h \in B^r(t) \setminus B^f(t), \\
V(P^f_h) \cup V(P^r_h) & \text{if } h \in B^f(t) \cap B^r(t).
\end{cases}
\]

For each $h \in B^f(t) \cup B^r(t)$, we define a linear order $\leq^2_h$ on $\beta^+_h(t)$ as follows. If the smallest element in $\beta^2(t) \cap (V(P^f_h) \cup V(P^r_h))$ with respect to the linear order $\leq^2_h$ is in $V(P^f_h)$, we let $\leq^2_h$ be the restriction of $\leq_h$ to $\beta^+_h(t)$. If the smallest element in $\beta^2(t) \cap (V(P^f_h) \cup V(P^r_h))$ with respect to the linear order $\leq^2_h$ is in $V(P^r_h)$, we let $\leq^2_h$ be the restriction of $\leq^2_h$ to $\beta^+_h(t)$. Note that $\beta^3(t) = \beta^2(t) \cup \bigcup_{h \text{ hole}} \beta^+_h(t)$. We are now ready to define $\leq^2_h$.
• For every $t \in V(D^3)$ and $v, w \in \beta^3(t)$ we let:

$$v \leq^t w :\Leftrightarrow \begin{cases} 
  v, w \in \beta^2(t) \text{ and } v \leq^t w \\
  v \in \beta^2(t) \text{ and } w \in \beta^3(t) \setminus \beta^2(t) \\
  v \in \beta^+_h(t) \text{ and } w \in \beta^+_i(t) \text{ for holes } h, i \text{ with } h < i \\
  v, w \in \beta^+_h(t) \text{ for some hole } h \text{ and } v \leq_h w.
\end{cases}$$

Claim 5. $\Delta^3 = (D^3, \sigma^3, \alpha^3, \leq^3)$ is an ordered treelike decomposition of $H$.

Proof. Straightforward. \hfill \qed

Claim 6. There is a parametrised od-scheme $\Lambda^3(x^0, \ldots, x^4)$ that defines the ordered decomposition $\Delta^3$ of $H$ within $(G, u^0, \ldots, u^4)$.

Proof. Straightforward. \hfill \qed

Claim 7.

1. For each $t \in V(D^3)$ and for each hole $h$, if $V(P'^t_h) \cap \sigma^3(t) \neq \emptyset$ then $V(P'^t_h) \subseteq \sigma^3(t)$ and if $V(P'^t_h) \cap \sigma^3(t) = \emptyset$ then $V(P'^t_h) \subseteq \sigma^3(t)$.

2. For each $t \in V(D^3)$ and for each hole $h$, either $\beta^3(t) \cap V(C_h) = \{u_{h-1}, u_h\}$ or $\beta^3(t) \cap V(C_h) = \{u_{h-1}, u_h\} \cup V(P'^t_h)$ or $\beta^3(t) \cap V(C_h) = \{u_{h-1}, u_h\} \cup V(P'^r_h)$ or $\beta^3(t) \cap V(C_h) = V(C_h)$.

Proof. Straightforward. \hfill \qed

Step 3. Completing the half-frames.

Let $h$ be a hole. A node $s$ is $h$-critical in a decomposition $\Delta$ of $H$ if $V(C_h) \subseteq \gamma^\Delta(s)$, but $V(C_h) \not\subseteq \gamma^\Delta(t)$ for any $t \in N^\Delta_C(s)$. If $\Delta$ is treelike, then for every $s \in V(\Delta)$ with $V(C_h) \subseteq \gamma^\Delta(s)$ there is at least one $t \supseteq^\Delta s$ that is $h$-critical. By [TL.5] this implies that there is at least one $h$-critical node. We call a hole $h$ framed in a node $t \in V(\Delta)$ if $V(C_h) \subseteq \beta^\Delta(t)$. We call $h$ left-framed in $t$ if $\beta^\Delta(t) \cap V(C_h) = \{u_{h-1}, u_h\} \cup V(P'^t_h)$, and we call $h$ right-framed if $\beta^\Delta(t) \cap V(C_h) = \{u_{h-1}, u_h\} \cup V(P'^r_h)$. Otherwise, we call $h$ unframed in $t$. By Claim 7(2), if $h$ is unframed in a node $t$ of $\Delta^3$ then $\beta^3(t) \cap V(C_h) = \{u_{h-1}, u_h\}$.

We let $\circ^\Delta(h), \circ^3(h), \oslash^\Delta(h), \oslash^3(h)$ be the sets of all $h$-critical nodes $t$ of $\Delta$ such that $h$ is framed, left-framed, right-framed, unframed, respectively, in $s$. We write $\circ^\Delta$ instead of $\circ^\Delta(h)$, and $\oslash^3$ instead of $\oslash^3(h)$, and et cetera, and we use similar notations for the decompositions $\Delta^4, \Delta^5$ et cetera that we shall define later.

Our goal in this step is to define an ordered treelike decomposition $\Delta^4$ of $H$ such that for all holes $h$ and all $h$-critical nodes $s$ either $h$ is framed in $s$ or $h$ is unframed in $s$.

Let $h$ be a hole and $s \in \oslash^3(h)$. A node $t \in V(D^3)$ is an $(h, s)$-extension node if $t \triangleright D^3 s$ and $V(P'^r_h) \subseteq \gamma^3(t)$. Observe that there is at least one $(h, s)$-extension node, because $V(P'^r_h) \subseteq V(C_h) \subseteq \gamma^3(s)$ and $V(P'^r_h) \not\subseteq \beta^3(s)$, which by Claim 7 implies $V(P'^r_h) \cap \beta^3(s) = \emptyset$. Let $X(h, s)$ be the set of all $(h, s)$-extension nodes and

$$X^t(h) := \bigcup_{s \in \oslash^3(h)} X(h, s).$$

M. Grohe, Definable Graph Structure Theory
We define \((h, s)\)-extension nodes for nodes \(s \in \mathcal{Y}^3(h)\) symmetrically and let
\[
X_r(h) := \bigcup_{s \in \mathcal{Y}^3(h)} X(h, s).
\]

For every \(s \in \mathcal{Y}^3(h)\), we let
\[
Y(h, s) := \{tu \in E(D^3) \mid t, u \in X(h, s)\} \cup \{su \in E(D^3) \mid u \in X(h, s)\}.
\]

We let \(Y_t(h) := \bigcup_{s \in \mathcal{Y}^3(h)} Y(h, s)\) and \(Y_r(h) := \bigcup_{s \in \mathcal{Y}^3(h)} Y(h, s)\). Hence \(Y_t(h)\) is the edge set of the induced subgraph of \(D^3\) with vertex set \(\mathcal{Y}^3(h) \cup X_t(h)\), and similarly for \(Y_r(h)\).

**Claim 8.** Let \(h\) be a hole. Then \(X_t(h) \cap X_r(h) = \emptyset\) and \(Y_t(h) \cap Y_r(h) = \emptyset\).

**Proof.** To prove that \(X_t(h) \cap X_r(h) = \emptyset\), suppose for contradiction that \(t \in X(h, s) \cap X(h, s')\) for some \(s \in \mathcal{Y}^3(h), s' \in \mathcal{Y}^3(h)\). Then \(V(P_h^t) \subseteq \gamma^3(t)\) and \(V(P_h^t) \subseteq \gamma^3(t)\) and thus \(V(C_h) \subseteq \gamma^3(t)\). As \(s \notin D^3\), this contradicts \(s\) being \(h\)-critical.

\[
Y_t(h) \cap Y_r(h) = \emptyset
\]
follows, because if \(tu \in Y_t(h) \cap Y_r(h)\) then \(u \in X_t(h) \cap X_r(h)\).

It is our goal to add \(V(P_h^t)\) to the bags of the nodes \(s \in \mathcal{Y}^3(h)\) and to add \(V(P_h^t)\) to the bags of the nodes \(s \in \mathcal{Y}^3(h)\). We could achieve this by adding \(V(P_h^t)\) to the separator of all nodes \(t \in X_t(h)\) and by adding \(V(P_h^t)\) to the separator of all nodes \(t \in X_r(h)\). The problem is that this would also affect the bags of nodes \(s' \notin \mathcal{Y}^3(h)\) with \(N^3_+(s') \cap X_t(h) \neq \emptyset\) and nodes \(s' \notin \mathcal{Y}^3(h)\) with \(N^3_+(s') \cap X_r(h) \neq \emptyset\). This is something we need to avoid, because otherwise it may happen that we add both \(V(P_h^t)\) and \(V(P_h^t)\) to the bag of a node \(s'\) such that \(h\) is unframed in \(s'\), and then we do not know how to order the extended bag of \(s'\). We can avoid such problems with a very simple trick. We first subdivide all edges. Then if we have a node \(s' \notin \mathcal{Y}^3(h)\) with a child \(t \in N^3_+(s') \cap X_t(h)\), and we add \(V(P_h^t)\) to the separator of \(t\) then this will not affect the bag of \(s'\) anymore, but only the bag of the new node on the subdivided edge from \(s'\) to \(t\).

We define a decomposition \((D^4, \sigma^4, \alpha^4)\) as follows:

- The digraph \(D^4\) is defined by
  \[
  V(D^4) := V(D^3) \cup \{z_{tu} \mid tu \in E(D^3)\}
  \]
and
  \[
  E(D^4) := \{tz_{tu}, z_{tu}u \mid tu \in E(D^3)\}.
  \]
  Here we assume that \(z_{tu} \notin V(D^3)\) for any \(tu \in E(D^3)\).
- For all \(t \in V(D^3)\), we let
  \[
  \sigma^4(t) := \sigma^3(t) \cup \bigcup_{h \text{ hole with } t \in X_t(h)} V(P_h^t) \cup \bigcup_{h \text{ hole with } t \in X_r(h)} V(P_h^t).
  \]
For all \(tu \in E(D^3)\) we let
\[
\sigma^4(z_{tu}) := \sigma^3(u) \cup \bigcup_{h \text{ hole with } tu \in Y_t(h)} V(P_h^t) \cup \bigcup_{h \text{ hole with } tu \in Y_r(h)} V(P_h^t).
\]

Preliminary Version
For all \( t \in V(D^3) \), we let
\[
\alpha^4(t) := \gamma^3(t) \setminus \sigma^4(t).
\]

For all \( tu \in E(D^3) \) we let
\[
\alpha^4(z_{tu}) := \gamma^3(u) \setminus \sigma^4(z_{tu}).
\]

Observe that for all \( t \in V(D^3) \) we have
\[
\gamma^4(t) = \gamma^3(t) \quad \text{and} \quad \sigma^4(t) \supseteq \sigma^3(t) \quad \text{and} \quad \alpha^4(t) \subseteq \alpha^3(t).
\]

For all \( tu \in E(D^3) \) we have
\[
\gamma^4(z_{tu}) = \gamma^3(u) \quad \text{and} \quad \sigma^4(z_{tu}) \supseteq \sigma^3(u) \quad \text{and} \quad \alpha^4(z_{tu}) \subseteq \alpha^3(u).
\]

Claim 9. \((D^4, \sigma^4, \alpha^4)\) is a treelike decomposition of \( H \).

Proof. [TL.1] is immediate, because \( D^4 \) is a subdivision of \( D^3 \). [TL.2] follows from the definition of \( \alpha^4 \) and [16.2.1] and [16.2.2].

To prove [TL.3], let \( t \in V(D^3) \) and \( u \in N^+_4(t) \). Let \( z := z_{tu} \). Then \( \gamma^4(u) = \gamma^4(z) \subseteq \gamma^4(t) \) by [16.2.2] and [16.2.1] and [TL.3] for \( \Delta^3 \). We have \( \sigma^4(u) \supseteq \sigma^4(z) \) because \( tu \in Y_t(h) \) implies \( u \in X_t(h) \), and \( tu \in Y_r(h) \) implies \( u \in X_r(h) \). Thus \( \alpha^4(u) \subseteq \alpha^4(z) \). It remains to prove that \( \alpha^4(z) \nsubseteq \alpha^4(t) \). Suppose for contradiction that \( \alpha^4(z) \nsubseteq \alpha^4(t) \). Let \( v \in \alpha^4(z) \setminus \alpha^4(t) \). Then \( v \in \alpha^3(u) \subseteq \alpha^3(t) \). As \( v \notin \alpha^3(t) \), this implies that there is a hole \( h \) such that either \( t \in X_t(h) \) and \( v \in P^h_t \) or \( t \in X_r(h) \) and \( v \in P^h_r \). Say, \( t \in X_t(h) \) and \( v \in P^h_t \). Then \( V(P^h_t) \cap \alpha^3(u) \neq \emptyset \) and thus \( V(P^h_t) \subseteq \gamma^4(u) \). Let \( s \in \gamma^3(h) \) such that \( t \in X(h,s) \). Since \( s \preceq D^3 t \preceq D^3 u \) we have \( u \in X(h,s) \). This implies \( tu \in Y(h,s) \) and thus \( V(P^h_t) \subseteq \gamma^4(z) \), which contradicts \( V(P^h_t) \cap \gamma^3(u) \supseteq \emptyset \).

To prove [TL.4] note first that nodes \( z_{tu} \) have only one child, thus there is nothing to prove for them. So let \( t \in V(D^3) \) and \( z_1, z_2 \in N^+_4(t) \). Then there are \( u_1, u_2 \in N^+_4(t) \) such that \( z_i = z_{tu} \), for \( i = 1, 2 \).

Case 1: \( u_1 \perp \perp^3 u_2 \).

Then \( \gamma^4(z_1) = \gamma^4(z_2) \) by [16.2.2]. We shall prove that \( \sigma^4(z_1) = \sigma^4(z_2) \), then \( \alpha^4(z_1) = \alpha^4(z_2) \) as well. Suppose for contradiction that \( \sigma^4(z_1) \neq \sigma^4(z_2) \). Say, \( \sigma^4(z_1) \setminus \sigma^4(z_2) \neq \emptyset \). Then there is a hole \( h \) such that either \( tu_1 \in Y_t(h) \) and \( tu_2 \notin Y_t(h) \) or \( tu_1 \in Y_r(h) \) and \( tu_2 \notin Y_r(h) \). Say, \( tu_1 \in Y_t(h) \) and \( tu_2 \notin Y_t(h) \). Let \( s \in \gamma^3(h) \) such that \( tu_1 \in Y(h,s) \). Then \( u_1 \in X(h,s) \) and thus \( V(P^h_t) \subseteq \gamma^3(u_1) \). Furthermore, since \( tu_2 \notin Y_t(h) \) we have \( u_2 \notin X(h,s) \) and thus \( V(P^h_t) \subseteq \gamma^3(u_2) \). Hence \( \gamma^3(u_1) \neq \gamma^3(u_2) \), which is a contradiction.

Case 2: \( u_1 \perp ^3 u_2 \).

We shall prove that \( z_1 \perp^4 z_2 \). Observe that by [16.2.2] we have \( \alpha^4(z_1) \cap \alpha^4(z_2) \subseteq \alpha^3(u_1) \cap \alpha^3(u_2) \neq \emptyset \). Suppose for contradiction that \( \alpha^4(z_1) \cap \alpha^4(z_2) \neq \emptyset \). Let \( v \in \alpha^4(z_1) \cap \alpha^4(z_2) \). Then \( v \notin \alpha^3(u_1) \), because \( \alpha^3(u_1) \cap \alpha^3(u_2) \subseteq \alpha^3(u) \cap \alpha^3(u) = \emptyset \). Hence there is a hole \( h \) such that either \( tu_1 \in Y_t(h) \) and \( v \in P^h_t \) or \( tu_1 \in Y_r(h) \) and \( v \in P^h_r \). Say, \( tu_1 \in Y_t(h) \) and \( v \in V(P^h_t) \). Then \( V(P^h_t) \cap \alpha^3(u) \supseteq \emptyset \). This implies \( V(P^h_t) \subseteq \gamma^3(u_2) \), by Claim [7] and because \( P^h_t \) is connected. Let \( s \in \gamma^3(h) \) such that \( tu_1 \in Y(h,s) \). Then \( tu_2 \notin Y(h,s) \) and therefore \( V(P^h_t) \subseteq \sigma^4(z_2) \). This contradicts \( V(P^h_t) \cap \alpha^4(z_2) \neq \emptyset \). Thus \( \sigma^4(z_1) \cap \alpha^4(z_2) \neq \emptyset \), and by symmetry \( \alpha^4(z_1) \cap \alpha^4(z_2) = \emptyset \).
Finally, to prove [TL.5] let \( t \in V(D^3) \) such that \( \sigma^3(t) = \emptyset \) and \( \alpha^3(t) = V(H) \). Then \( \sigma^4(t) = \sigma^3(t) = \emptyset \) and thus \( \alpha^4(t) = \alpha^3(t) = V(H) \). 

\[ \text{Claim 10.} \]

1. For all \( t \in V(D^3) \) it holds that
   \[ \beta^3(t) \subseteq \beta^4(t) \subseteq \beta^3(t) \cup \bigcup_{i \text{ hole}} V(C_i). \] (16.2.3)

Furthermore, for every hole \( h \) we have \( \beta^4(t) \cap V(C_h) = \beta^3(t) \cap V(C_h) \) or \( \beta^4(t) \cap V(C_h) = (\beta^3(t) \cap V(C_h)) \cup V(P_h^r) \) or \( \beta^3(t) \cap V(C_h) = (\beta^3(t) \cap V(C_h)) \cup V(P_h^l) \).

2. For all \( tu \in E(D^3) \) it holds that
   \[ \sigma^3(u) \subseteq \beta^4(z_{tu}) = \sigma^4(u) \subseteq \sigma^3(u) \cup \bigcup_{i \text{ hole}} V(C_i). \] (16.2.4)

Furthermore, for every hole \( h \) we have \( \beta^4(z_{tu}) \cap V(C_h) = \sigma^3(u) \cap V(C_h) \) or \( \beta^4(z_{tu}) \cap V(C_h) = (\sigma^3(u) \cap V(C_h)) \cup V(P_h^r) \) or \( \beta^4(z_{tu}) \cap V(C_h) = (\sigma^3(u) \cap V(C_h)) \cup V(P_h^l) \).

3. For every hole \( h \) we have and every \( x \in V(D^4) \), either \( \beta^4(x) \cap V(C_h) = \{u_{h-1}, u_h\} \) or \( \beta^4(x) \cap V(C_h) = \{u_{h-1}, u_h\} \cup V(P_h^r) \) or \( \beta^4(x) \cap V(C_h) = \{u_{h-1}, u_h\} \cup V(P_h^l) \) or \( \beta^4(x) \supseteq V(C_h) \).

**Proof.** To prove (1), let \( t \in V(D^3) \). (16.2.3) follows immediately from the definitions of \( \sigma^4 \) and \( \alpha^4 \). As we always keep the sets \( V(P_h^r) \) and \( V(P_h^l) \) together, the only option for the set \( \beta^4(t) \cap V(C_h) \) other than those listed in (1) is that \( (\beta^3(t) \cap V(C_h)) \setminus \beta^3(t) = V(P_h^r) \cup V(P_h^l) \).

Suppose for contradiction that this is the case. Then there are \( u_r, u_l \in N_D^3(t) \) such that \( V(P_h^r) \subseteq \sigma^4(z_{tu_r}) \setminus \sigma^3(u_r) \) and \( V(P_h^l) \subseteq \sigma^4(z_{tu_l}) \setminus \sigma^3(u_l) \). Hence \( tu_r \in Y_r(h) \) and \( tu_l \in Y_l(h) \). Then \( t \in \zeta^3(h) \cup X_l(h) \) and \( t \in \gamma^3(h) \cup X_r(h) \). As \( \zeta^3(h) \cap \gamma^3(h) = \emptyset \) and \( X_l(h) \cap X_r(h) = \emptyset \), either \( t \in X_l(h) \cup \gamma^3(h) \) or \( t \in \zeta^3(h) \cap X_r(h) \). By symmetry, we may assume the former. Let \( s < D^3 t \) such that \( s \in \zeta^3(h) \). Then both \( s \) and \( t \) are \( h \)-critical in \( \Delta^3 \), but we have \( s < D^3 t \). This is a contradiction.

To prove (2), let \( tu \in V(D^3) \) and \( z := z_{tu} \). We have \( \gamma^4(z) = \gamma^4(u) \) by (16.2.2) and thus \( \beta^4(z) = \gamma^4(z) \setminus \alpha^4(u) = \sigma^4(u) \), because \( N_D^3(z) = \{u\} \). Now the inequalities in (16.2.4) follow from the definitions of \( \sigma^4 \). Suppose for contradiction that \( (\beta^4(z) \cap V(C_h)) \setminus \beta^3(u) = V(P_h^r) \cup V(P_h^l) \). Then \( V(P_h^r) \cup V(P_h^l) \subseteq \sigma^4(u) \setminus \sigma^3(u) \). Thus \( u \in X_l(h) \cap X_r(h) \), which is impossible.

Assertion (3) follows from (1), (2), and Claim 7.

To turn the treelike decomposition \((D^4, \sigma^4, \alpha^4)\) into an ordered treelike decomposition, we need to define the linear orders \( \leq^4 \) on the bags.

- For every \( t \in V(D^3) \) and \( v, w \in \beta^4(t) \), we let
  \[
  v \leq^4 w \iff \begin{cases} 
  v, w \in \beta^3(t) \text{ and } v \leq^3 w \\
  v \in \beta^3(t) \text{ and } w \in \beta^4(t) \setminus \beta^3(t) \\
  w, v \in \beta^4(t) \setminus \beta^3(t) \text{ and } v \in V(C_h) \text{ and } w \in V(C_i) \text{ for holes } h, i \text{ with } h < i \\
  v, w \in \beta^4(t) \setminus \beta^3(t) \text{ and } v, w \in V(P_h^r) \text{ for some hole } h \text{ and } v \leq^r_h w \\
  v, w \in \beta^4(t) \setminus \beta^3(t) \text{ and } v, w \in V(P_h^l) \text{ for some hole } h \text{ and } v \leq^l_h w.
  \end{cases}
  \]
• For every $tu \in E(D^3)$, we let $\leq_{ztu}^4$ be the restriction of $\leq_u^4$ to $\sigma^4(u) = \beta^4(z_{tu})$.

**Claim 11.** $\Delta^4 := (D^4, \sigma^4, \alpha^4, \leq^4)$ is an ordered treelike decomposition of $H$.

**Proof.** Follows immediately from Claims 9 and 10 and the definition of $\leq^4$. \hfill \blacksquare

**Claim 12.** There is a parametrised oδ-scheme $\Lambda^4(x^0, \ldots, x^4)$ that defines the ordered decomposition $\Delta^3$ of $H$ within $(G, u^0, \ldots, u^4)$.

**Proof.** Again, we need to be more specific about the nodes of the digraph $D^4$. Let $k \in \mathbb{N}$ such that $V(D^3) \subseteq V(G)^k$. We let

$$V(D^4) := \{\overline{v} \mid v \in V(D^3)\} \cup \{\overline{vw} \mid vw \in E(D^3)\}. $$

We identify each node $v \in V(D^3)$ with the tuple $\overline{v} \in V(D^4)$ and for each edge $vw \in E(D^3)$ we let $z_{vw} := \overline{vw}$. With this representation it is straightforward to define $\Delta^4$ in IFP. \hfill \blacksquare

The following claim shows what we have achieved in this step.

**Claim 13.** Let $h$ be a hole and $t \in V(D^4)$.

1. $t$ is $h$-critical in $\Delta^4$ if and only if $t \in V(D^3)$ and $t$ is $h$-critical in $\Delta^3$.

2. If $t$ is $h$-critical in $\Delta^4$ then either $\beta^4(t) \cap V(C_h) = \{u_{h-1}, u_h\}$ or $V(C_h) \subseteq \beta^4(t)$.

**Proof.** For the forward direction of (1), suppose that $t$ is $h$-critical in $\Delta^4$. Then $t \in V(D^3)$, because if $t = z_{tu}^r$ for some edge $tu \in E(D^3)$ then $\gamma^4(t) = \gamma^4(u)$. Furthermore, $t$ is $h$-critical in $\Delta^3$, because if $u \in D^3$ such that $V(C_h) \subseteq \gamma^3(u) = \gamma^4(u)$ then $t$ is not $h$-critical in $\Delta^4$.

For the backward direction, suppose that $t \in V(D^3)$ is $h$-critical in $\Delta^3$. Then $V(C_h) \subseteq \gamma^3(t) = \gamma^4(t)$. Suppose for contradiction that $t$ is not $h$-critical in $\Delta^4$. Then there is a $z \in N^D_+(t)$ such that $V(C_h) \subseteq \gamma^4(z)$. Let $u \in N^D_+(t)$ such that $z = z_{tu}$. Then $V(C_h) \subseteq \gamma^4(z) = \gamma^3(u)$. This contradicts $t$ being $h$-critical in $\Delta^3$.

To prove (2), suppose that $t$ is $h$-critical in $\Delta^4$ and thus in $\Delta^3$.

**Case 1:** $t \in \emptyset^3(h)$.

Let $u \in N^D_+(t)$. Then $tu \notin Y(s, h)$ for any $s \in \emptyset^3(h) \cup \emptyset^3(h)$ and thus $tu \notin Y_t(h) \cup Y_r(h)$.

It follows that for all $z = z_{tu} \in N^D_+(t)$ it holds that $\sigma^4(z) \cap V(C_h) = \sigma^3(u) \cap V(C_h)$ and thus $\alpha^4(z) \cap V(C_h) = \alpha^3(u) \cap V(C_h)$. This implies $\beta^4(t) \cap V(C_h) = \beta^3(t) \cap V(C_h) = \{u_{h-1}, u_h\}$.

**Case 2:** $t \in \emptyset^3(h)$.

Let $u \in N^D_+(t)$ such that $V(P_h^u) \subseteq \gamma^3(u)$. Then $tu \in Y(h, t)$ and thus $V(P_h^u) \subseteq \sigma^4(z_{tu}) \subseteq \beta^3(t)$. This implies $V(C_h) \subseteq \beta^4(t)$.

**Case 3:** $t \in \emptyset^3(h)$.

Symmetric to Case 2.

**Case 4:** $t \in \emptyset^3(h)$.

Then $V(C_h) \subseteq \beta^3(t) \subseteq \beta^4(t)$. \hfill \blacksquare
Step 4. Pulling the frames together.

In this step we shall define an ordered treelike decomposition $\Delta^5$ of $H$ where for all holes $h$ and all $h$-critical nodes $t$ either $h$ is framed in $t$ or $h$ is unframed in $t$ and there are children $u_r, u_l$ of $t$ such that $V(P_h^r) \subseteq \beta^5(u_l)$ and $V(P_h^l) \subseteq \beta^5(u_r)$. Only the existence of such children is new in $\Delta^5$, compared to $\Delta^4$. Technically, this step follows the previous step very closely.

We give all the definitions in full detail, but omit most of the proofs.

Let $h$ be a hole. By Claim 13 all $h$-critical nodes are in $\bigcup_1^4(h) \cup \bigcup^4(h)$. A node $z \in V(D^4)$ is $h$-left-subcritical if there is an $s \in \bigcup^4(h)$ such that $z \in N_+(s)$ and $P_h^s \subseteq \gamma^4(z)$. We denote the set of all $h$-left-subcritical nodes by $\bigcup^4(h)$. For every $z \in \bigcup^4(h)$ we let $X'(h, z)$ be the set of all $t \triangleright D^4$ such that $P_h^t \subseteq \gamma^4(t)$. Note that $X'(h, z)$ may be empty, as opposed to the sets $X(h, s)$ defined in the previous step. Let

$$X'_r(h) := \bigcup_{z \in \bigcup^4(h)} X'(h, z).$$

We define the set $\bigcup^4(h)$ of $h$-right-subcritical nodes and for each $z \in \bigcup^4(h)$ the set $X'(h, z)$ symmetrically, and we let

$$X'_l(h) := \bigcup_{z \in \bigcup^4(h)} X'(h, z).$$

Let me remark that there is no need to define sets $Y'_l(h)$ and $Y'_r(h)$ here, essentially because a node $z_{tu}$ belongs to $X'_l(h)$ if and only if the edge $tu \in E(D^3)$ would belong to $Y'_l(h)$. The reason we needed the sets $Y'_l(h)$ and $Y'_r(h)$ in the previous step is that we defined them in the graph $D^3$ and not the subdivided graph $D^4$.

Claim 14. Let $h$ be a hole. Then $X'_r(h) \cap X'_l(h) = \emptyset$.

Proof. Similar to the proof of Claim 8.

We define a decomposition $(D^5, \sigma^5, \alpha^5)$ as follows:

- We let $D^5 := D^4$.
- For all $t \in V(D^5)$, we let

$$\sigma^5(t) := \sigma^4(t) \cup \bigcup_{h \text{ hole with } t \in X'_l(h)} V(P_h^l) \cup \bigcup_{h \text{ hole with } t \in X'_r(h)} V(P_h^r).$$

- For all $t \in V(D^5)$, we let

$$\alpha^5(t) := \gamma^4(t) \setminus \sigma^5(t).$$

Then for all $t \in V(D^5)$ we have

$$\gamma^5(t) = \gamma^4(t) \quad \text{and} \quad \sigma^5(t) \supseteq \sigma^4(t) \quad \text{and} \quad \alpha^5(t) \subseteq \alpha^4(t). \quad (16.2.5)$$

Claim 15. $(D^5, \sigma^5, \alpha^5)$ is a treelike decomposition of $H$.

Proof. The proof is very similar to the proof of Claim 14. In particular, we have to distinguish between nodes of $D^5$ in $V(D^3)$ and nodes of the form $z_{tu}$. I leave the details to the reader. □
Claim 16. Let \( h \) be a hole. For all \( t \in V(D^5) \) it holds that

\[
\beta^4(t) \subseteq \beta^5(t) \subseteq \beta^4(t) \cup \bigcup_{\text{hole}i} V(C_i). \tag{16.2.6}
\]

Furthermore, \( \beta^5(t) \cap V(C_h) = \beta^4(t) \cap V(C_h) \) or \( \beta^5(t) \cap V(C_h) = (\beta^4(t) \cap V(C_h)) \cup V(P_h^r) \) or \( \beta^5(t) \cap V(C_h) = (\beta^4(t) \cap V(C_h)) \cup V(P_h^l) \).

Proof. Similar to the proof of Claim 10.

To define the linear orders \( \leq_5 \), for every \( t \in V(D^5) \) and \( v, w \in \beta^5(t) \), we let

\[
\begin{cases}
  v, w \in \beta^4(t) \text{ and } v \leq_4 w \\
  \text{or } v \in \beta^4(t) \text{ and } w \in \beta^5(t) \setminus \beta^4(t) \\
  v \leq_5 w \iff \begin{cases}
    \text{or } v, w \in \beta^5(t) \setminus \beta^4(t) \text{ and } v \in V(C_h) \text{ and } w \in V(C_i) \text{ for holes } h, i \text{ with } h < i \\
    \text{or } v, w \in \beta^5(t) \setminus \beta^4(t) \text{ and } v, w \in V(P_h^r) \text{ for some hole } h \text{ and } v \leq_h w \\
    \text{or } v, w \in \beta^5(t) \setminus \beta^4(t) \text{ and } v, w \in V(P_h^l) \text{ for some hole } h \text{ and } v \leq_h w.
  \end{cases}
\end{cases}
\]

Claim 17. \( \Delta^5 := (D^5, \sigma^5, \alpha^5, \leq^5) \) is an ordered treelike decomposition of \( H \).

Proof. Follows immediately from Claims 15 and 16 and the definition of \( \leq^5 \).

Claim 18. There is a parametrised od-scheme \( \Lambda^5(x^0, \ldots, x^4) \) that defines the ordered decomposition \( \Delta^5 \) of \( H \) within \( (G, u^0, \ldots, u^4) \).

Proof. Straightforward.

The following claim shows what we have achieved in this step.

Claim 19. Let \( h \) be a hole.

1. A node \( s \in V(D^5) \) is \( h \)-critical in \( \Delta^5 \) if and only if it is \( h \)-critical in \( \Delta^4 \).

2. \( \varnothing^4(h) = \varnothing^5(h) \).

3. \( \varphi^4(h) = \varphi^5(h) \). Furthermore, for all \( t \in \varnothing^4(h) \) there are nodes \( u_\ell, u_r \in N_{D^5}^+(t) \) such that \( V(P_h^\ell) \subseteq \beta^5(u_\ell) \) and \( V(P_h^r) \subseteq \beta^5(u_r) \).

4. For all \( t \in \varnothing^5(h) \) and all \( u \in N_{D^5}^+(t) \), if \( V(P_h^4) \subseteq \gamma^5(u) \) then \( V(P_h^r) \subseteq \beta^5(u) \), and if \( V(P_h^\ell) \subseteq \gamma^5(u) \) then \( V(P_h^\ell) \subseteq \beta^5(u) \).

Proof. (1) follows immediately from (16.2.5) and \( D^5 = D^4 \).

(2) follows from (16.2.6).

To prove (3), suppose that \( t \in \varnothing^4(h) \). Then \( t \notin X'(h, s) \) for any \( s \in \varnothing^4(h) \), because there can be no \( s \in D^4 \) that is \( h \)-critical. Thus \( \sigma^5(t) \cap V(C_h) = \sigma^4(t) \cap V(C_h) \). Similarly, for every \( z \in N_{D^5}^+(t) \) we have \( z \notin X'(h, s) \) for any \( s \in \varnothing^4(h) \), because \( t \) is the only parent of \( z \) and thus there can be no \( h \)-critical \( s \neq t \) such that \( s \in D^4 \). Hence \( \sigma^5(z) \cap V(C_h) = \sigma^4(z) \cap V(C_h) \). Combined with (16.2.5), this implies \( \beta^5(t) \cap V(C_h) = \beta^4(t) \cap V(C_h) = \{u_{h-1}, u_h\} \). Hence \( t \in \varnothing^5(h) \).

Let \( u_\ell \in \sqrt[4]{\varnothing(h)} \cap N_{D^5}^+(t) \). Then either \( V(P_h^\ell) \subseteq \beta^4(u_\ell) \subseteq \beta^5(u_\ell) \), and we are done, or there is a child \( x_\ell \in N_{D^5}^+(u_\ell) \) with \( V(P_h^\ell) \subseteq \gamma^4(x_\ell) \). Then \( x_\ell \in X'(h, u_\ell) \subseteq X^4(h) \) and thus \( V(P_h^\ell) \subseteq \beta^5(x_\ell) \subseteq \beta^5(u_\ell) \).
Finally, let \( u_r \in \mathcal{D}_\bigtriangleup(h) \cap N^D_\bigtriangleup(t) \). By symmetry, we have \( V(P^r_h) \subseteq \beta^5(u_r) \).

The proof of (3) also shows (4).

\[ \Box \]

**Step 5. Framing the unbalanced unframed holes.**

Let \( h \) be a hole and \( s \in \mathcal{D}^5(h) \). We let \( \mathcal{D}(h,s) \) be the set of all \( t \in N^D_\bigtriangleup(s) \) such that \( V(P^0_h) \subseteq \gamma^5(t) \), and we let \( \Sigma_r(h,s) := \{ \sigma^5(t) \mid t \in \mathcal{D}(h,s) \} \). We define \( \mathcal{D}_\bigtriangleup(h,s) \) and \( \Sigma_r(h,s) \) analogously. Remember that by Claim 19 for all \( t \in \mathcal{D}(h,s) \) it holds that \( V(P^0_h) \subseteq \beta^5(t) \) and for all \( t \in \mathcal{D}_\bigtriangleup(h,s) \) it holds that \( V(P^r_h) \subseteq \beta^5(t) \). The hole \( h \) is balanced in \( s \) if \( \Sigma_r(h,s) = \Sigma_r(h,s) \) and unbalanced otherwise.

We define a decomposition \( (D^6, \sigma^6, \alpha^6) \) as follows:

- We let \( D^6 := D^5 \).
- For all \( t \in V(D^6) \), we let
  \[
  \sigma^6(t) := \sigma^5(t) \cup \bigcup_{h \text{ hole}, s \in \mathcal{D}^5(h)} V(P^0_h) \cup \bigcup_{h \text{ hole}, s \in \mathcal{D}^5(h)} V(P^r_h).
  \]
  - For all \( t \in V(D^6) \), we let
    \[
    \alpha^6(t) := \gamma^5(t) \setminus \sigma^6(t).
    \]

Then for all \( t \in V(D^6) \) we have

\[
\gamma^6(t) = \gamma^5(t) \quad \text{and} \quad \sigma^6(t) \supseteq \sigma^5(t) \quad \text{and} \quad \alpha^6(t) \subseteq \alpha^5(t).
\] (16.2.7)

**Claim 20.** \( (D^6, \sigma^6, \alpha^6) \) is a treelike decomposition of \( H \).

**Proof.** Similar to the proofs of Claims 9 and 15. \( \Box \)

**Claim 21.** Let \( h \) be a hole. For all \( t \in V(D^6) \) it holds that

\[
\beta^5(t) \subseteq \beta^6(t) \subseteq \beta^5(t) \cup \bigcup_{i \text{ hole}} V(C_i).
\] (16.2.8)

Furthermore, if \( t \in \mathcal{D}^5(h) \) and \( h \) is unbalanced in \( t \) then \( V(C_h) \subseteq \beta^6(t) \), and otherwise \( \beta^6(t) \cap V(C_h) = \beta^5(t) \cap V(C_h) = \{ u_{h-1}, u_h \} \).

**Proof.** Straightforward. \( \Box \)

To define \( \leq^6 \), let \( t \in V(D^6) \). For subsets \( S_1, S_2 \subseteq \beta^5(t) \) we write \( S_1 \leq^6_t S_2 \) if \( S_1 \) is lexicographically smaller than or equal to \( S_2 \) with respect to the linear order \( \leq^5_t \) on \( \beta^5(t) \). This defines a linear order on the powerset \( 2^{\beta^5(t)} \). For subsets \( \Sigma_1, \Sigma_2 \subseteq 2^{\beta^5(t)} \) we write \( \Sigma_1 \leq^5_t \Sigma_2 \) if \( \Sigma_1 \) is lexicographically smaller than or equal to \( \Sigma_2 \) with respect to the linear order \( \leq^5_t \) on \( 2^{\beta^5(t)} \). To define \( \leq^6 \), for all \( v, w \in \beta^6(t) \), we let

\[
v \leq^6 w \iff \begin{cases} 
v, w \in \beta^5(t) \text{ and } v \leq^5_t w \\
v \in \beta^5(t) \text{ and } w \in \beta^6(t) \setminus \beta^5(t)
\end{cases}
\]

for some hole \( h \) with \( \Sigma_r(h,t) \not<^t \Sigma_r(h,t) \) and \( v \leq^r_t w \).

Preliminary Version
Claim 22. \( \Delta^6 := (D^6, \sigma^6, \alpha^6, \preceq^6) \) is an ordered treelike decomposition of \( H \).

Proof. Follows immediately from Claims 20 and 21 and the definition of \( \preceq^6 \).

Claim 23. There is a parametrised od-scheme \( \Lambda^6(x^0, \ldots, x^4) \) that defines the ordered decomposition \( \Delta^6 \) of \( H \) within \( (G, u^0, \ldots, u^4) \).

Proof. Straightforward.

Claim 24. Let \( h \) be a hole.

1. A node \( s \in V(D^6) \) is \( h \)-critical in \( \Delta^6 \) if and only if it is \( h \)-critical in \( \Delta^5 \).
2. \( \circ^6(h) = \circ^5(h) \cup \{ s \in \varnothing^5(h) \mid h \text{ unbalanced in } s \} \).
3. \( \varnothing^6(h) = \varnothing^5(h) \setminus \circ^6(h) \). Furthermore, for all \( s \in \varnothing^6(h) \) there are nodes \( t_1, t_r \in N^{D^6}_+(s) \) such that \( V(P^6_1) \subseteq \beta^6(t_1) \) and \( V(P^6_r) \subseteq \beta^6(t_r) \), and \( h \) is balanced in \( s \).
4. For all \( s \in \varnothing^6(h) \) and all \( t \in N^{D^6}_+(s) \), if \( V(P^6_1) \subseteq \gamma^6(t) \) then \( V(P^6_r) \subseteq \beta^6(t) \), and if \( V(P^6_r) \subseteq \gamma^6(t) \) then \( V(P^6_1) \subseteq \beta^6(t) \)

Proof. Straightforward.

Step 6. Framing the balanced unframed holes.

In this step, we will “frame” the remaining unframed holes. This is considerably harder than the previous steps. It is not hard to move both parts \( P^6_1 \) and \( P^6_r \) of the frame \( C_h \) to the bag of an \( h \)-critical node, the problem is that there is no generic way of ordering the resulting bag. What we do is create two new nodes with the same bag that contains \( C_h \). In one of these nodes we use the order \( \preceq^6 \) and in the other the order \( \preceq^6 \). Other than that, the nodes are indistinguishable, that is, they have the same parents and children, the same separator and the same component. This solves the problem for one hole, but there may be several unframed holes with the same critical node, and they may interact. This makes the construction technically complicated, though the idea remains the same.

The definition of the decomposition \( \Delta^7 \) requires some preparation. Let \( s \) be a node. For every hole \( h \) such that \( s \in \varnothing^6(h) \), let \( \varnothing(h, s) \) be the set of all \( t \in N^{D^6}_+(s) \) such that \( V(P^6_1) \subseteq \gamma^6(t) \), and let \( \varnothing(h, s) \) be the set of all \( t \in N^{D^6}_+(s) \) such that \( V(P^6_r) \subseteq \gamma^6(t) \). Remember that by Claim 24, for all \( t \in \varnothing(h, s) \) it holds that \( V(P^6_t) \subseteq \beta^6(t) \) and for all \( t \in \varnothing(h, s) \) it holds that \( V(P^6_r) \subseteq \beta^6(t) \). It also follows from Claim 24 that

\[
\{ \sigma^6(t) \mid t \in \varnothing(h, s) \} = \{ \sigma^6(t) \mid t \in \varnothing(h, s) \}.
\]

Let \( L_s \) be the graph defined as follows:

- \( V(L_s) \) is the set of all holes \( h \) such that \( s \in \varnothing^6(h) \).
- \( E(L_s) := \{ hi \mid h, i \in V(L_s), (\varnothing(h, s) \cup \varnothing(h, s)) \cap (\varnothing(i, s) \cup \varnothing(i, s)) \neq \emptyset \} \).

Note that the graph \( L_s \) may be empty. For every connected component \( A \) of \( L_s \) we create two fresh nodes \( v_\ell(s, A), v_r(s, A) \). Here “fresh” means that \( v_\ell(s, A), v_r(s, A) \not\in V(D^6) \) and that all nodes \( v_\ell(s, A), v_r(s, A) \) for \( s \in V(D^6) \) and connected components \( A \) of \( L_s \) are mutually
distinct. We let \( V(s) := \{v_L(s, A), v_r(s, A) \mid A \text{ connected component of } L_s \} \). Note that if \( L_s \) is empty then \( V(s) \) is empty as well. We let
\[
W(s) := N^D_+ (s) \setminus \bigcup_{h \in V(L_s)} (\mathcal{O}(h, s) \cup \mathcal{O}_\mathcal{N}(h, s)).
\]
We are ready to define the directed graph \( D^7 \):

- \( V(D^7) := V(D^6) \cup \bigcup_{s \in V(D^6)} V(s) \).
- To define \( E(D^7) \), for every \( s \in V(D^6) \), we let
  \[
  N^D_+ (s) := V(s) \cup W(s),
  \]
  and for every connected component \( A \) of \( L_s \) we let
  \[
  N^D_+ (v_L(s, A)) := N^D_+ (v_r(s, A)) := \bigcup_{h \in V(A)} (\mathcal{O}(h, s) \cup \mathcal{O}_\mathcal{N}(h, s)).
  \]

Before we define \( \sigma^7 \) and \( \alpha^7 \), we make a few observations about \( D^7 \). Remember that \( D^6 = D^4 \) is obtained from \( D^3 \) by subdividing all edges and that all nodes \( s \) that are critical for some hole \( h \) are in \( V(D^3) \). Thus the children of these nodes have in-degree 1. Now let \( s \in V(D^6) \) such that \( L_s \neq \emptyset \) and let \( t \in N^D_+ (s) \). Then \( s \) is \( h \)-critical for all \( h \in V(L_s) \) and thus \( s \) is the only parent of \( t \) in \( D^6 \). In \( D^7 \), either \( s \) remains the only parent of \( t \) (if \( t \in W(s) \)), or there is exactly one connected component \( A \) of \( L_s \) such that the two parents of \( t \) are \( v_L(s, A) \) and \( v_r(s, A) \). To see this, note that for distinct components \( A, A' \) of \( L_s \) we have
\[
\bigcup_{h \in V(A)} (\mathcal{O}(h, s) \cup \mathcal{O}_\mathcal{N}(h, s)) \cap \bigcup_{h' \in V(A')} (\mathcal{O}(h', s) \cup \mathcal{O}_\mathcal{N}(h', s)) = \emptyset
\]
by the definition of \( L_s \). Let us turn to the definition of \( \sigma^7 \) and \( \alpha^7 \):

- For all \( t \in V(D^6) \), we let
  \[
  \sigma^7 (t) := \sigma^6 (t) \cup \bigcup_{h \text{ hole, } s \in \mathcal{O}_6(h)} V(P^6_h) \cup \bigcup_{h \text{ hole, } s \in \mathcal{O}_6(h)} V(P^6_h).
  \]

For all \( s \in V(D^6) \) and all connected components \( A \) of \( L_s \) we let
\[
\sigma^7 (v_L(s, A)) := \sigma^7 (v_r(s, A)) := \bigcup_{h \in V(A)} \sigma^6 (t).
\]

- For all \( t \in V(D^6) \) we let \( \alpha^7 (t) := \alpha^6 (t) \setminus \sigma^7 (t) \).

For all \( s \in V(D^6) \) and all connected components \( A \) of \( L_s \) we let
\[
\alpha^7 (v_L(s, A)) := \alpha^7 (v_r(s, A)) := \bigcup_{h \in V(A)} \alpha^6 (t).
\]
Claim 25. Let $t \in V(D^6)$.

(1) $\sigma^7(t) \supseteq \sigma^6(t)$ and $\alpha^7(t) \subseteq \alpha^6(t)$ and $\gamma^7(t) = \gamma^6(t)$.

(2) If $t \in W(s)$ for some $s \in V(D^6)$, then $\sigma^7(t) = \sigma^6(t)$ and $\alpha^7(t) = \alpha^6(t)$.

Proof. (1) is obvious; for the equality $\gamma^7(t) = \gamma^6(t)$ just note that if $V(P^6_i) \subseteq \sigma^7(t) \setminus \sigma^6(t)$ then $V(P^6_h) \subseteq \gamma^6(t)$ and similarly for $V(P^6_h)$.

To prove (2), suppose that $t \in W(s)$ for some $s \in V(D^6)$. Then $t \not\in \mathcal{O}(h,s) \cup \mathcal{S}(h,s)$ for any hole $h$. Suppose for contradiction that $t \in \mathcal{O}(h,s') \cup \mathcal{S}(h,s')$ for some hole $h$ and node $s' \in V(D^6)$. Then $s \neq s'$, and $s'$ is $h$-critical. By Claims 24, 19, and 13 we have $s' \in V(D^3)$. As $t$ is a child of $s'$ in $D^6 = D^4$, it is of the form $z_{s'\alpha}$ for some edge $s' \in E(D^3)$. Thus $s'$ is the only parent of $t$ in $D^6$. Hence $s = s'$, which is a contradiction.

Claim 26. $(D^7, \sigma^7, \alpha^7)$ is a treelike decomposition of $H$.

Proof. (TL.1) is immediate from the construction of $D^7$ from the acyclic digraph $D^6$.

(TL.2) follows from Claim 25 for nodes $t \in V(D^6)$. For nodes $t$ of the form $v_t(s,A)$ and $v_t(s,A)$ it follows from (TL.2) and (TL.4) for $D^6$ that $\sigma^7(t)$ and $\alpha^7(t)$ are disjoint and from (TL.2) for $\Delta^6$ that $N_h^+(\alpha^7(t)) \subseteq \sigma^7(t)$.

To prove (TL.3) let us first consider a node $t \in V(D^6)$. Let $u \in N_{D^7}^+(t)$. If $u \in W(t)$, then $\sigma^7(u) = \sigma^6(u)$ and $\alpha^7(u) = \alpha^6(u)$. Thus $\gamma^7(u) = \gamma^6(u) \subseteq \gamma^6(t) = \gamma^7(t)$. Suppose for contradiction that $\sigma^7(u) \nsubseteq \sigma^7(t)$. Let $v \in \sigma^7(u) \setminus \sigma^7(t)$. Then $v \in V(P^6_h) \cup V(P^6_h)$ for some hole $h$ such that the parent $s$ of $t$ is in $\mathcal{S}(h,s)$. Then $v \in \mathcal{S}(h,t)$ by Claim 24. This contradicts $v$ being in $\sigma^6(u) = \alpha^6(u)$. Suppose next that $u = v_t(t,A)$ or $u = v_t(t,A)$ for some connected component $A$ of the graph $L_t$. Then

$$\gamma^7(u) = \bigcup_{x \in \mathcal{O}(h,t) \cup \mathcal{S}(h,t)} \gamma^6(x) \subseteq \bigcup_{x \in N_{D^6}^+(t)} \gamma^6(x) \subseteq \gamma^6(t) = \gamma^7(t).$$

Suppose for contradiction that $\sigma^7(u) \nsubseteq \sigma^7(t)$. Let $v \in \sigma^7(u) \setminus \sigma^7(t)$. Then there is a hole $h \in V(A)$ and a node $x \in \mathcal{O}(h,t) \cup \mathcal{S}(h,t)$ such that $v \in \sigma^6(x) \subseteq \sigma^6(t)$. Now we argue as above. Since $v \not\in \alpha^7(t)$, we have $v \in V(P^6_h) \cup V(P^6_h)$ for some hole $h$ such that the parent $s$ of $t$ is in $\mathcal{S}(h,t)$. Then $v \in \mathcal{S}(h,t)$ by Claim 24, which contradicts $v$ being in $\sigma^6(x)$.

Now consider a node $t \in \{v_t(s,s), v_t(s,s)\}$ for some $s \in V(D^6)$ and some connected component $A$ of $L_s$. Let $u \in N_{D^7}^+(t)$. Then $u \in \mathcal{O}(h,s) \cup \mathcal{S}(h,s)$ for some hole $h \in V(A)$ and thus $\sigma^6(u) \subseteq \sigma^7(t)$ and $\sigma^6(u) \subseteq \sigma^6(t)$. As $\alpha^7(u) \subseteq \sigma^6(u)$ and $\gamma^7(u) = \gamma^6(u)$, this implies (TL.3).

To prove (TL.4) again we first consider a node $t \in V(D^6)$. Let $u_1, u_2 \in N_{D^7}^+(t)$.

Case 1: $u_1, u_2 \in \{v_t(t,A), v_t(t,A)\}$ for some connected component $A$ of $L_t$.

Then $\sigma^7(u_1) = \sigma^7(u_2)$ and $\alpha^7(u_1) = \alpha^7(u_2)$.

Case 2: $u_1 \in \{v_t(t,A), v_t(t,A)\}$ and $u_2 \in \{v_t(t,A), v_t(t,A)\}$ for distinct connected components $A_1, A_2$ of $L_t$.

We shall prove that $\alpha^7(u_1) \cap \gamma^7(u_2) = \emptyset$. By symmetry, this implies $\gamma^7(u_1) \cap \alpha^7(u_2) = \emptyset$ and thus $u_1 \perp \Delta^7 u_2$.

So let $v \in \alpha^7(u_1)$. Then there is a hole $h_1 \in V(A_1)$ and an $x_1 \in \mathcal{O}(h_1,t) \cup \mathcal{S}(h_1,t)$ such that $v \in \sigma^6(x_1)$. First of all, this implies that $v \not\in \sigma^6(x)$ for any $x \in N_{D^6}^+(t)$.
and thus \( v \not\in \sigma^7(u_2) \). Suppose for contradiction that \( v \in \alpha^7(u_2) \). Then there is a hole \( h_2 \in V(A_2) \) and an \( x_2 \in \mathcal{O}(h_2, t) \cup \mathcal{O}_\gamma(h_2, t) \) such that \( v \in \alpha^6(x_2) \). From (TL.4) for \( \Delta^6 \) it follows that \( \sigma^6(x_1) = \sigma^6(x_2) \) and \( \alpha^6(x_1) = \alpha^6(x_2) \). Let us assume without loss of generality that \( x_1 \in \mathcal{O}(h_1, t) \) and \( x_2 \in \mathcal{O}_\gamma(h_2, t) \). Then \( V(P_{h_1}^t) \subseteq \gamma^6(x_1) = \gamma^6(x_2) \) and thus \( x_2 \in \mathcal{O}(h_1, t) \). Hence \( \mathcal{O}(h_1, t) \cap \mathcal{O}_\gamma(h_2, t) \neq \emptyset \), and thus \( h_1h_2 \in E(L_t) \). This contradicts \( h_1 \in V(A_1) \) and \( h_2 \in V(A_2) \) for distinct connected components \( A_1, A_2 \) of \( L_t \).

**Case 3:** \( u_1 \in \{v_l(t, A), v_r(t, A)\} \) for some connected component \( A \) of \( L_t \) and \( u_2 \in W(t) \).

We shall prove that \( \alpha^7(u_1) \cap \gamma^7(u_2) = \emptyset \) and \( \gamma^7(u_1) \cap \alpha^7(u_2) = \emptyset \). This implies \( u_1 \perp \Delta^7 u_2 \).

Remember that we have \( u_2 \in N^6_+(t) \) and \( \sigma^7(u_2) = \sigma^6(u_2) \) and \( \alpha^7(u_2) = \alpha^6(u_2) \). To prove that \( \alpha^7(u_1) \cap \gamma^7(u_2) = \emptyset \), let \( v \in \alpha^7(u_1) \). Then there is a hole \( h \in V(A) \) and an \( x \in \mathcal{O}(h, t) \cup \mathcal{O}_\gamma(h, t) \) such that \( v \in \alpha^6(x) \). This implies \( v \not\in \sigma^6(u_2) = \sigma^7(u_2) \).

Suppose for contradiction that \( v \in \alpha^7(u_2) = \alpha^6(u_2) \). Then by (TL.4) for \( \Delta^6 \) we have \( \sigma^6(x) = \sigma^6(u_2) \) and \( \alpha^6(x) = \alpha^6(u_2) \). Let us assume without loss of generality that \( x \in \mathcal{O}(h, t) \). Then \( V(P_{h_1}^t) \subseteq \gamma^6(x) = \gamma^6(u_2) \) and thus \( u_2 \in \mathcal{O}(h, t) \subseteq V(t) \). This contradicts \( u_2 \in W(t) \).

To prove \( \gamma^7(u_1) \cap \alpha^7(u_2) = \emptyset \), let \( v \in \alpha^7(u_1) \). Then \( \sigma^6(x) \) for any \( x \in N^6_+(t) \) and thus \( v \not\in \gamma^7(u_2) \). We have already seen that \( \alpha^7(u_1) \cap \alpha^7(u_2) \subseteq \alpha^7(u_1) \cap \gamma^7(u_2) = \emptyset \). Hence \( v \not\in \alpha^7(u_1) \) as well.

**Case 4:** \( u_1 \in W(t) \) and \( u_2 \in \{v_l(t, A), v_r(t, A)\} \) for some connected component \( A \) of \( L_t \).

Symmetric to Case 3.

**Case 5:** \( u_1, u_2 \in W(t) \).

Then for \( i = 1, 2 \) we have \( u_i \in N^6_+(t) \) and \( \sigma^7(u_i) = \sigma^6(u_i) \) and \( \alpha^7(u_i) = \alpha^6(u_i) \). Thus the assertion follows immediately from (TL.4) for \( \Delta^6 \).

It remains to prove (TL.4) for nodes of the form \( v_l(s, A) \) or \( v_r(s, A) \). Let \( t \in \{v_l(s, A), v_r(s, A)\} \) for some \( s \in V(D^6) \) and some connected component \( A \) of \( L_t \). Let \( u_1, u_2 \in N^6_+(t) \). Then for \( i = 1, 2 \) we have \( u_i \in N^6_+(t) \) and \( \sigma^7(u_i) \geq \sigma^6(u_i) \) and \( \alpha^7(u_i) \subseteq \alpha^6(u_i) \) and \( \gamma^7(u_i) = \gamma^6(u_i) \).

**Case 1:** \( u_1 \perp \Delta^6 u_2 \).

Then \( \gamma^7(u_1) = \gamma^7(u_2) \). We shall prove that \( \sigma^7(u_1) = \sigma^7(u_2) \), then \( \alpha^7(u_1) = \alpha^7(u_2) \) follows. Suppose for contradiction that \( \sigma^7(u_1) \neq \sigma^7(u_2) \). Say, \( \sigma^7(u_1) \setminus \sigma^7(u_2) \neq \emptyset \). Then there is a hole \( h \in V(A) \) such that \( u_1 \in \mathcal{O}(h, t) \) and \( u_2 \not\in \mathcal{O}(h, t) \) or \( u_1 \in \mathcal{O}_\gamma(h, t) \) and \( u_2 \not\in \mathcal{O}_\gamma(h, t) \). Say, \( u_1 \in \mathcal{O}(h, t) \) and \( u_2 \not\in \mathcal{O}(h, t) \). Then \( V(P_h^t) \subseteq \gamma^6(u_1) \) and \( V(P_h^t) \not\subseteq \gamma^6(u_1) \), which contradicts \( \gamma^6(u_1) = \gamma^6(u_2) \).

**Case 2:** \( u_1 \perp \Delta^6 u_2 \).

Then we have \( \gamma^7(u_1) \cap \alpha^7(u_2) \subseteq \gamma^6(u_1) \cap \alpha^6(u_2) = \emptyset \), because \( \gamma^7(u_1) = \gamma^6(u_1) \) and \( \alpha^7(u_2) \subseteq \alpha^6(u_2) \). By symmetry, we have \( \alpha^7(u_1) \cap \gamma^7(u_2) = \emptyset \). Thus \( u_1 \perp \Delta^7 u_2 \).

To prove (TL.5) let \( t \in V(D^6) \) such that \( \sigma^6(t) = \emptyset \) and \( \alpha^6(t) = V(H) \). Without loss of generality we may assume that \( t \) is \( \leq D^6 \)-minimal. Then \( \sigma^7(t) = \sigma^6(t) = \emptyset \) and thus \( \alpha^7(t) = \alpha^6(t) = V(H) \).

**Claim 27.**
(1) For all \( t \in V(D^6) \) it holds that \( \beta^7(t) = \beta^6(t) \).

(2) For all \( s \in V(D^6) \) and all connected components \( A \) of \( L_s \) it holds that

\[
\beta^7(v_r(s, A)) = \beta^7(v_r(s, A)) = \bigcup_{h \in V(A)} \sigma^6(t) \cup \bigcup_{h \in V(A)} V(C_h) \subseteq \beta^6(s) \cup \bigcup_{h \in V(A)} V(C_h).
\]

Proof. To prove (1), let \( t \in V(D^6) \). We have \( \gamma^7(t) = \gamma^6(t) \) and

\[
\bigcup_{u \in N^D_+(t)} \alpha^7(t) = \bigcup_{u \in V(t)} \alpha^7(u) \cup \bigcup_{u \in W(t)} \alpha^7(u)
= \bigcup_{h \in V(L_t)} \alpha^6(x) \cup \bigcup_{u \in W(t)} \alpha^6(u)
= \bigcup_{u \in N^{D'}_+(t)} \alpha^6(u).
\]

Thus \( \beta^7(t) = \beta^6(t) \).

To prove (2), let \( s \in V(D^6) \), and let \( A \) be a connected component of \( L_s \). Clearly, we have \( \beta^7(v_l(s, A)) = \beta^7(v_r(s, A)) \). Let \( t := v_r(s, A) \). We first prove that

\[
\beta^7(t) \subseteq \bigcup_{h \in V(A)} \sigma^6(u) \cup \bigcup_{h \in V(A)} V(C_h) \tag{16.2.9}
\]

Recall that \( N^D_+(t) = \bigcup_{h \in V(A)} (\emptyset(h, s) \cup \mathcal{Q}(h, s)) \). Hence the first union on the right-hand side of (16.2.9) ranges precisely over all \( u \in N^D_+(t) \). Let \( v \in \beta^7(t) \). Suppose that \( v \not\in \sigma^6(u) \) for any \( u \in N^D_+(t) \). As \( v \in \gamma^7(t) = \bigcup_{u \in N^D_+(t)} (\sigma^6(u) \cup \alpha^6(u)) \) and \( v \not\in \alpha^7(u) \) for any \( u \in N^D_+(t) \), there is a \( u \in N^D_+(t) \) such that \( v \in \alpha^6(u) \setminus \alpha^7(u) \), or equivalently \( v \in \sigma^7(u) \setminus \sigma^6(u) \). Thus \( v \in V(P_h^\emptyset) \cap V(P_h^\emptyset) \subseteq V(C_h) \) for some hole \( h \) such that \( s \in \emptyset(h) \). Moreover, since \( v \in \gamma^6(u) \) and thus either \( V(P_h^\emptyset) \subseteq \gamma^6(u) \) or \( V(P_h^\emptyset) \subseteq \gamma^6(u) \), we have \( u \in \emptyset(h, s) \cup \mathcal{Q}(h, s) \). Since \( u \in N^{D'}_+(t) = \bigcup_{h \in V(A)} (\emptyset(h, s) \cup \mathcal{Q}(h, s)) \) and

\[
(\emptyset(h, s) \cup \mathcal{Q}(h, s)) \cap (\emptyset(h', s) \cup \mathcal{Q}(h', s)) = \emptyset
\]

for \( h \not\in V(A), h' \in V(A) \) we have \( h \in V(A) \).

To prove the converse inclusion, note first that

\[
\bigcup_{h \in V(A)} \sigma^6(u) = \sigma^7(t) \subseteq \beta^7(t).
\]

Let \( h \in V(A) \). As \( s \in \emptyset(h) \), there is a \( u_t \in \emptyset(h, s) \) and a \( u_r \in \mathcal{Q}(h, s) \), and we have \( u_t, u_r \in N^{D'}_+(t) \). Moreover, we have \( V(P_h^\emptyset) \subseteq \sigma^7(u_t) \subseteq \beta^7(t) \) and \( V(P_h^\emptyset) \subseteq \sigma^7(u_r) \subseteq \beta^7(t) \).
and thus $V(C_h) \subseteq \beta^7(t)$. This completes the proof that

$$
\beta^7(v_L(s, A)) = \beta^7(v_r(s, A)) = \bigcup_{h \in V(A)} \sigma^6(t) \cup \bigcup_{h \in V(A)} V(C_h).
$$

It remains to prove that

$$
\bigcup_{h \in V(A)} \sigma^6(t) \cup \bigcup_{h \in V(A)} V(C_h) \subseteq \beta^6(s) \cup \bigcup_{h \in V(A)} V(C_h).
$$

However, this is obvious, because $\sigma^6(t) \subseteq \beta^6(s)$ for all $t \in \sqrt{\partial}(h, s) \cup \sqrt{\partial}(s, h) \subseteq N^{D^6}(s)$. 

It remains to define $\preceq^7$. For every $t \in V(D^6)$ we let $\preceq^7_{v_L(s, A)} := t$. Let $s \in V(D^6)$, and let $A$ be a connected component of $L_s$. To define $\preceq^7_{v_r(s, A)}$ and $\preceq^7_{v_L(s, A)}$, we need some preparation.

We call an edge $hi$ $E(L_s)$ straight if $\sqrt{\partial}(s, h) \cap \sqrt{\partial}(s, i) \neq \emptyset$ or $\sqrt{\partial}(s, h) \cap \sqrt{\partial}(s, i) \neq \emptyset$. We call $hi \in E(L_s)$ twisted if $\sqrt{\partial}(s, h) \cap \sqrt{\partial}(s, i) \neq \emptyset$ or $\sqrt{\partial}(s, h) \cap \sqrt{\partial}(s, i) \neq \emptyset$.

**Claim 28.** No edge of $L_s$ is both straight and twisted.

**Proof.** Suppose for contradiction that $hi \in E(L_s)$ is straight and twisted. Say, $\sqrt{\partial}(s, h) \cap \sqrt{\partial}(s, i) \neq \emptyset$ and $\sqrt{\partial}(s, h) \cap \sqrt{\partial}(s, i) \neq \emptyset$; all other cases are symmetric. Then there is a $u \in N^*_v(s, i)$ such that $V(P^i_h), V(P^i_s) \subseteq \gamma^6(u)$ and an $x \in N^*_v(s, h)$ such that $V(P^i_h), V(P^i_s) \subseteq \gamma^6(u)$. If $\gamma^6(u) = \gamma^6(x)$ then $V(C_i) \subseteq \gamma^6(u)$, which contradicts $s$ being $i$-critical. Hence $\gamma^6(u) \cap \gamma^6(x) = \sigma^6(u) \cap \sigma^6(x)$ and thus $V(P^i_h) \subseteq \sigma^6(u) \subseteq \beta^6(s)$. This contradicts $s \in \sqrt{\partial}(h)$. 

Recall that the holes are integers in $[n]$ and hence ordered by the natural linear order. We define an enumeration

$$
h_1, h_2, \ldots, h_m
$$

of $V(A)$ as follows: we let $h_1 := \min (V(A))$, and for every $j \in [m - 1]$, we let

$$
h_{j+1} := \min \{ h \in V(A) \setminus \{h_1, \ldots, h_j\} \mid ih \in E(L_s) \text{ for some } i \in \{h_1, \ldots, h_j\} \}.
$$

Note that $h_{j+1}$ is well defined, because $A \subseteq L_s$ is connected. We declare $h_1, \ldots, h_m$ to be **straight** or **twisted** inductively as follows. $h_1$ is straight. For $j \in [m - 1]$, let $i \in [j]$ be minimum such that $h_i h_j \in E(L_i)$. If $h_i$ is straight and the edge $h_i h_{j+1}$ is straight then $h_{j+1}$ is straight. If $h_i$ is straight and the edge $h_i h_{j+1}$ is twisted then $h_{j+1}$ is twisted. If $h_i$ is twisted and the edge $h_i h_{j+1}$ is straight then $h_{j+1}$ is twisted. If $h_i$ is twisted and the edge $h_i h_{j+1}$ is twisted then $h_{j+1}$ is straight.

Now we are ready to define $\preceq_{v_L(s, A)}^7$ and $\preceq_{v_r(s, A)}^7$. For all $v, w \in \beta^7(v_L(s, A)) = \beta^7(v_r(s, A)) \subseteq \beta^6(s) \cup \bigcup_{h \in V(A)} V(C_h)$ we let

$$
v \preceq_{v_L(s, A)}^7 w \iff \begin{cases} 
& v, w \in \beta^6(s) \text{ and } v \preceq^6_s w \\
& \text{or } v \in \beta^6(s) \text{ and } w \notin \beta^6(s) \\
& \text{or } v, w \notin \beta^6(s) \text{ and } v \in V(C_h) \text{ and } w \in V(C_i) \\
& \text{for holes } h, i \text{ with } h < i \\
& \text{or } v, w \in V(C_h) \setminus \beta^6(s) \text{ for some straight } h \in V(A) \text{ and } v \preceq^t_h w \\
& \text{or } v, w \in V(C_h) \setminus \beta^6(s) \text{ for some twisted } h \in V(A) \text{ and } v \preceq^t_h w.
\end{cases}
$$
and

\[ v \leq_{\nu_v(s,A)} w \iff \begin{cases} 
\forall, w \in \beta^6(s) \text{ and } v \prec_\nu w \\
\quad \text{or } v \in \beta^6(s) \text{ and } w \not\in \beta^6(s) \\
\quad \text{or } v, w \not\in \beta^6(s) \text{ and } v \in V(C_h) \text{ and } w \in V(C_i) \\
\quad \text{for holes } h, i \text{ with } h < i \\
\quad \text{or } v, w \in V(C_h) \setminus \beta^6(s) \text{ for some straight } h \in V(A) \text{ and } v \leq_h^t w \\
\quad \text{or } v, w \in V(C_h) \setminus \beta^6(s) \text{ for some twisted } h \in V(A) \text{ and } v \leq_h^t w.
\]

Claim 29. \( \Delta^7 = (D^7, \sigma^7, \alpha^7, \leq^7) \) is an ordered treelike decomposition of \( H \).

**Proof.** Straightforward.

Claim 30. There is a parametrised od-scheme \( \Lambda^7(x^0, \ldots, x^4) \) that defines the ordered decomposition \( \Delta^7 \) of \( H \) within \( (G, u^0, \ldots, u^4) \).

**Proof.** The proof of this claim is not as straightforward as the proofs of the previous definability claims, because it is neither clear how to define the new nodes of \( D^7 \) in IFP nor how to define \( \leq^7 \).

Suppose that \( V(D^6) \subseteq V(G)^k \). Then we let \( V(D^7) \subseteq V(G)^k+1 \). We represent each node \( \nu \in V(D^6) \) by the \((k+1)\)-tuple \( \nu u^0 \). Furthermore, for each \( \nu \in V(D^6) \) and each connected component \( A \), let \( h := \min(V(A)) \). We represent \( \nu_t(\nu, A) \) by the \((k+1)\)-tuple \( \nu u_h^t \) and \( \nu_t(\nu, A) \) by the \((k+1)\)-tuple \( \nu u_i^t \). Note that we have \( u_h^t \neq u_i^t \), because \( h \) is a hole, which implies that the patch \( Q_h \) is nontrivial. Using the facts the set \( V(D^6) \) is definable in IFP with parameters \( u^0, \ldots, u^4 \) and that for every node \( \nu \in V(D^6) \) the graph \( L_\nu \) is IFP-definable, we can construct an IFP-formula that defines \( V(D^7) \) with parameters \( u^0, \ldots, u^4 \). Once we have defined \( V(D^7) \) in IFP, it is not hard to define \( E(D^7), \sigma^7 \), and \( \alpha^7 \).

It is not clear how to define the linear order \( \leq^7_i \) for a node \( t = \nu_t(s,A) = \nu u_h^t \) or \( t = \nu_t(s,A) = \nu u_h^t \), because there is no definable relation between \( u_h^t \) and \( u_i^t \) for holes \( h \neq i \). That is, even if we use \( u_h^t \) as a parameter we cannot distinguish between \( u_i^t \) and \( u_i^t \) in IFP. This means that we cannot tell if an edge of \( L_s \) is straight or twisted. However, we can define in IFP for every hole \( i \in V(A) \) a vertex \( u_{t,i} \in \{ u_i^t, u_i^t \} \) such that for \( t = \nu_t(s,A) = \nu u_h^t \) we have

\[ u_{t,i} = \begin{cases} 
\nu_i^t & \text{if } i \text{ is straight}, \\
\nu_i^t & \text{if } i \text{ is twisted},
\end{cases} \]

and for \( t = \nu_t(s,A) = \nu u_h^t \) we have

\[ u_{t,i} = \begin{cases} 
\nu_i^t & \text{if } i \text{ is straight}, \\
\nu_i^t & \text{if } i \text{ is twisted},
\end{cases} \]

This suffices to define the order \( \leq^7_i \), because for the elements in \( V(C_i) \setminus \beta^6(s) \) we can use the order \( \text{int-order}[G, u_{i-1}, u_i, u_{i+1}, y_1, y_2] \).

Claim 31. Let \( h \) be a hole and \( t \in V(D^7) \) such that \( t \) is \( h \)-critical in \( \Delta^7 \). Then \( h \) is framed in \( t \), that is, \( V(C_h) \subseteq \beta^7(t) \).

**Proof.** This follows easily from Claim 27, either \( t \in \partial^6(h) \), then \( t \in \partial^7(h) \), because \( \beta^7(t) = \beta^6(t) \). Or \( t \in \{ \nu_l(s,A), \nu_r(s,A) \} \) for some \( s \in \partial(h) \) and \( A \) with \( h \in V(A) \). Then \( V(C_h) \subseteq \beta^7(t) \) by Claim 27.

M. Grohe, Definable Graph Structure Theory
Step 7. Filling the holes.
In this final step, we extend the decomposition to $G$. We define a decomposition $(D^8, \sigma^8, \alpha^8)$ of $G$ as follows:

- We let $D^8 := D^7$.
- For each $t \in V(D^8)$, we let
  \[
  \sigma^8(t) := \sigma^7(t) \cup \bigcup_{h \text{ hole}, V(C_h) \subseteq \sigma^7(t)} V(J_h).
  \]
- For each $t \in V(D^8)$, we let
  \[
  \gamma^8(t) := \gamma^7(t) \cup \bigcup_{h \text{ hole}, V(C_h) \subseteq \gamma^7(t)} V(J_h),
  \]
  and we let $\alpha^8(t) := \gamma^8(t) \setminus \sigma^8(t)$.

Note that for every $t \in V(D^8)$ we have $\sigma^7(t) = \sigma^8(t) \cap V(H)$ and $\gamma^7(t) = \gamma^8(t) \cap V(H)$ and $\alpha^7(t) = \alpha^8(t) \cap V(H)$.

Claim 32. $(D^8, \sigma^8, \alpha^8)$ is a treelike decomposition of $G$.
Proof. Straightforward. 

Claim 33. For every $t \in V(D^8)$ we have
  \[
  \beta^8(t) := \beta^7(t) \cup \bigcup_{h \text{ hole}, V(C_h) \subseteq \beta^7(t)} V(J_h).
  \]

Proof. Let $t \in V(D^8)$. We have $\beta^8(t) \cap V(H) = \beta^7(t)$. Let $h$ be a hole. If $V(C_h) \subseteq \beta^7(t) \subseteq \gamma^7(t)$ then $V(J_h) \subseteq \gamma^8(t)$. Moreover, for every $u \in N_{D^8}^+(t)$ we have $V(C_h) \cap \alpha^7(u) = \emptyset$. Thus either $V(C_h) \not\subseteq \gamma^7(u)$ or $V(C_h) \subseteq \sigma^7(u)$. In both cases, $V(J_h) \cap \alpha^8(u) = \emptyset$. It follows that
  \[
  V(J_h) \subseteq \gamma^8(t) \setminus \bigcup_{u \in N_{D^8}^+(t)} \alpha^8(u) = \beta^8(u).
  \]

Now suppose that $V(C_h) \not\subseteq \beta^7(t)$. Then either $V(C_h) \not\subseteq \gamma^7(t)$ or $V(C_h) \cap \alpha^7(u) \neq \emptyset$ for some $u \in N_{D^8}^+(t)$.

Case 1: $V(C_h) \not\subseteq \gamma^7(t)$.
Then $V(J_h) \cap \beta^8(t) \subseteq V(J_h) \cap \gamma^8(t) = \emptyset$.

Case 2: $V(C_h) \subseteq \gamma^7(t)$ and $V(C_h) \cap \alpha^7(u) \neq \emptyset$ for some $u \in N_{D^8}^+(t)$.

Case 2a: $V(C_h) \subseteq \gamma^7(u)$.
Then $V(J_h) \subseteq \gamma^8(u)$. Moreover, since $V(C_h) \not\subseteq \sigma^7(u)$ we have $V(J_h) \cap \sigma^8(u) = \emptyset$ and thus $V(J_h) \subseteq \alpha^8(u)$, which implies $V(J_h) \cap \beta^8(t) = \emptyset$.
Case 2b: $V(C_h) \nsubseteq \gamma^7(u)$.

As $V(C_h) \subseteq \gamma^7(t)$ and $V(C_h) \nsubseteq \beta^7(t)$, the node $t$ is not $h$-critical, and thus there is a $u' \in N^D_{\leq 8}(t)$ such that $V(C_h) \subseteq \gamma^7(u')$. But then $\alpha^7(u) \cap \gamma^7(u') \neq \emptyset$, which contradicts \([TL.4]\).

For every $t \in V(D^8)$, we define the order $\leq^8_t$ as follows: for $v, w \in \beta^8(t)$, we let

\[
v \leq^8_t w :\iff \begin{cases} 
\quad v, w \in \beta^7(t) \text{ and } v \leq^7_t w \\
\quad \text{or } v \in \beta^7(t) \text{ and } w \in \beta^8(t) \setminus \beta^7(t) \\
\quad \text{or } v \in V(J_h) \text{ and } w \in V(J_i) \text{ for holes } h, i \text{ with } h < i \\
\quad \text{or } v, w \in V(J_h) \text{ for some hole } h \text{ and } u^h \leq^7_t u^i \text{ and } v \leq^7_t w \\
\quad \text{or } v, w \in V(J_h) \text{ for some hole } h \text{ and } u^h \leq^7_t u^i \text{ and } v \leq^7_t w.
\end{cases}
\]

Claim 34. $\Delta^8 := (D^8, \sigma^8, \alpha^8, \leq^8)$ is an ordered treelike decomposition of $G$.

Proof. Straightforward.

Claim 35. There is a parametrised od-scheme $\Lambda^8(x^0, \ldots, x^4)$ such that $\Lambda^8[G, u^0, \ldots, u^4] = \Delta^8$.

Proof. Straightforward.

We complete the proof with an application of the Parameter Elimination Lemma 7.2.3 to $\Lambda^8(x^0, \ldots, x^4)$ to obtain an unparametrised od-scheme $\Lambda$ that defines an ordered treelike decomposition on $G$.

16.3 Decomposing Almost Embeddable Graphs and Their Minors

Theorem 16.3.1 (Definable Structure Theorem for Almost Embeddable Graphs).

For all $p, q, r \in \mathbb{N}$, the class $\mathcal{M}(\mathcal{AE}_{p,q,r})$ of all minors of $(p, q, r)$-almost embeddable graphs admits $\text{IFP}$-definable ordered treelike decompositions.

With the results of the previous sections, it is fairly straightforward to prove the following lemma. Recall that the partial order $<^*$ on $\mathbb{N}^2$ is a restriction of the lexicographical order that we introduced in Section 15.1.2 to order surfaces with respect to simplicity.

Lemma 16.3.2. Let $p, q, r \in \mathbb{N}$, and suppose that for all $(r', q') <^* (r, q)$ the class $\mathcal{AE}_{p,q,r'}$ admits $\text{IFP}$-definable ordered treelike decompositions. Then the class $\mathcal{Z}_3 \cap \mathcal{AE}_{p,q,r}$ of all 3-connected $(p, q, r)$-almost embeddable graphs admits $\text{IFP}$-definable ordered treelike decompositions.

Proof. If $(r, q) \leq^* (0, 1)$, the assertion follows from the Definable Structure Theorem for Almost Planar Graphs 13.5.1. Let us assume that $(r, q) >^* (0, 1)$. Let

\[
\mathcal{R} := \mathcal{N}_{4p} \left( \bigcup_{(r', q') <^* (r, q)} \mathcal{U}(\mathcal{AE}_{p,q',r'}) \right).
\]

It follows from the definition of $<^*$ that the union is finite. Hence by the assumption of this lemma, the Finite Extension Lemma 7.3.1 the Union Lemma for Definable Ordered

M. Grohe, Definable Graph Structure Theory
Decompositions 13.11, and the Component Lifting Lemma 7.3.4, the class $\mathcal{R}$ admits $\text{IFP}$-definable ordered treelike decompositions. By Lemma 15.1.6, the reduction of every graph in $\mathcal{AE}_{p,q,r}$ with respect to a simplifying curve is in $\mathcal{R}$.

Let $G = (Z_3 \cap \mathcal{AE}_{p,q,r}) \setminus \mathcal{R}$, and let $(G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q)$ be a $p$-arrangement of $G$ in a surface $S$ of Euler genus $r$ with $q$ cuffs. Observe that $G$ together with the arrangement $(G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q)$ and the surface $S$ satisfy Assumption 15.1.7(3). To see that Assumption 15.1.7(5) is satisfied, suppose for contradiction that for some $j \in [q]$ we have $n_j := |R^j| \leq 2$. This implies $V(R^j) \setminus \pi(R^j) = \emptyset$, because $G$ is 3-connected and $\pi(R^j)$ separates $R^j$ from $G \setminus R^j$. But then we can find a $p$-arrangement of $G$ in the surface $S'$ obtained from $S$ by gluing a disk on the cuff $c^j$. Thus $G \in \mathcal{AE}_{p,q,r}$, which is a contradiction.

Case 1: The representativity of $G_0$ in $S$ is at most 4.

Let $g \subseteq S$ be a $G_0$-normal simplifying curve with $|g \cap V(G_0)| \leq 4$. By Lemma 15.1.6, we have $G \setminus \pi(g \cap V(G_0)) \in \mathcal{R}$ and thus $G \in N_4(\mathcal{R})$. As $\mathcal{R}$ admits $\text{IFP}$-definable ordered treelike decompositions, by the Finite Extension Lemma 7.3.1, $N_4(\mathcal{R})$ admits $\text{IFP}$-definable ordered treelike decompositions as well.

In the following, suppose that the representativity of $G_0$ in $S$ is at least 5. Then Assumption 15.1.7(4) is satisfied as well. We can choose a surface $S$ satisfying Assumption 15.1.7(6). Then Assumption 15.1.7 is satisfied, and by Lemma 15.1.10, there are disks $D^1, \ldots, D^q \subseteq S$ satisfying Assumption 15.3.1(1). Finally, since we have $\mathcal{P} = \mathcal{AE}_{0,0,0} \subseteq \mathcal{R}$, by Lemma 15.1.6, Assumption 15.3.1(2) is satisfied. Hence Assumption 15.3.1 is fully satisfied.

Case 2: The representativity of $G_0$ in $S$ is at least 5 and $G$ has a canonical simplifying patch.

Note that in this case, all assumptions of Lemma 16.1.2 are satisfied, and thus we can define an ordered treelike decomposition on $G$.

Case 3: The representativity of $G_0$ in $S$ is at least 5 and $G$ has no canonical simplifying patch.

Then $G$ satisfies Assumption 15.5.1. In this case, all assumptions of the Last Extension Lemma 16.2.1 are satisfied, and again we can define an ordered treelike decomposition on $G$.

The lemma seems to give us the key argument for an inductive proof of the Definable Structure Theorem 16.3.1 if not for the full class $\mathcal{M}(\mathcal{AE}_{p,q,r})$ then at least for the class $\mathcal{AE}_{p,q,r}$. However, we need a reduction to 3-connected graphs, and the decomposition of a graph into its 3-connected components has torsos that are minors, but not necessarily subgraphs of the original graphs. Like the classes of almost planar graphs, where the same problem occurred, the classes of almost embeddable graphs do not seem to be closed under taking minors. Therefore, we first need to understand the structure of minors of almost embeddable graphs. Fortunately, we can use the structure of minors of almost planar graphs, as described in Section 13.5.2, as a blueprint.

Note that $\mathcal{P} = \mathcal{AE}_{p,0,0} \subseteq \mathcal{AE}_{p,1,0} = \mathcal{AP}_p$ for all $p \in \mathbb{N}$, thus for the case $r = 0$ and $q \leq 1$, Theorem 16.3.1 follows from the Definable Structure Theorem for Almost Planar Graphs 13.5.1. For the rest of this section, we make the following assumption.

Assumption 16.3.3. $p, q, r \in \mathbb{N}$ such that either $r > 0$ or $q > 1$ and

$$
\mathcal{M}R := N_{4p} \bigg( \bigcup_{(r', q') \prec^* (r, q)} U(\mathcal{M}(\mathcal{AE}_{3p,q', r'})) \bigg).
$$

(16.3.1)
Let \( G \in \mathcal{AE}_{p,q,r} \), and let \((G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q)\) be a \( p \)-arrangement of \( G \) in a surface \( S \) of Euler genus \( r \) with \( q \) cuffs. Then by Lemma 15.1.6, for every \( G_0 \)-normal simplifying curve \( g \subseteq S \) it holds that \( G \setminus \pi(V(G_0) \cap g) \in \mathcal{MR} \).

### 16.3.1 The Structure of Minors of Almost Embeddable Graphs

The following definition generalises Definition 13.5.5 from the almost planar to the almost embeddable setting.

**Definition 16.3.4.** An \( m \)-arrangement of a graph \( G^* \) in a surface \( S \) with \( q \) cuffs is a tuple

\[
(G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q, F)
\]

such that:

1. \( (G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q) \) is an arrangement of the graph \( G := \pi(G_0) \cup \bigcup_{i=1}^{q} R^i \) in \( S \).
2. \( F \subseteq E(\pi(G_0)) \), and all edges in \( F \) have both endvertices in \( \bigcup_{i=1}^{q} \pi(\pi^i) \).
3. \( G^* = G/F \).

\((G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q, F)\) is a \( p \)-\( m \)-arrangement, for some \( p \in \mathbb{N} \), if for all \( i \in [q] \) the pair \((R^i, \pi(\pi^i))\) is a \( p \)-ring, and it is a local \( p \)-\( m \)-arrangement if \((R^i, \pi(\pi^i))\) is a \( 2p \)-vortex.

The class of all graphs that have a \( p \)-\( m \)-arrangement in a surface of Euler genus \( r \) with \( q \) cuffs is denoted by \( \mathcal{MAE}_{p,q,r} \).

**Lemma 16.3.5.** \( \mathcal{MAE}_{p,q,r} = \mathcal{M}(\mathcal{AE}_{p,q,r}) \).

**Proof.** Analogous to the proof of Lemma 13.5.6 \( \square \)

**Definition 16.3.6.** An \( m \)-arrangement \((G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q, F)\) is reduced if for all \( e \in F \) and all \( e_0 \in E(G_0) \) with \( e = \pi(e_0) \) the curve \( e_0 \subseteq S \) is a contractible loop in \( S \).

The class of all graphs that have a reduced \( p \)-\( m \)-arrangement in a surface of Euler genus \( r \) with \( q \) cuffs is denoted by \( \mathcal{MAE}_{p,q,r}^{\text{red}} \).

**Lemma 16.3.7.** \( \mathcal{MAE}_{p,q,r}^{\text{red}} \subseteq \mathcal{MAE}_{p,q,r} \subseteq \mathcal{N}_1(\mathcal{MR}) \cup \mathcal{MAE}_{p,q,r}^{\text{red}} \).

**Proof.** The first inclusion is trivial. To prove the second, let \((G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q, F)\) be a \( p \)-\( m \)-arrangement of a graph \( G^* \) in a surface \( S \) of Euler genus \( r \) with \( q \) cuffs, and let \( G := \pi(G_0) \cup \bigcup_{i=1}^{q} R^i \).

Let \( e \in F \) and \( e_0 \in E(G_0) \) such that \( e = \pi(e_0) \). Then by Definition 16.3.4(ii), the curve \( e_0 \) is either an open loop or a link in \( S \). If \( e_0 \) is a noncontractible loop or a link and thus a simplifying curve, then \( G \setminus \{v, w\} \in \mathcal{MR} \). Let \( v_e \) be the vertex of \( G^* \) corresponding to \( e \). Then \( G^* \in \mathcal{N}_1(\mathcal{MR}) \).

If for all \( e \in F \) and \( e_0 \in E(G_0) \) such that \( e = \pi(e_0) \) the curve \( e_0 \) is a contractible loop, then \( G^* \in \mathcal{MAE}_{p,q,r}^{\text{red}} \). \( \square \)

The following generalisation of Lemma 13.5.8 describes the structure of the graphs in the class \( \mathcal{MAE}_{p,q,r}^{\text{red}} \).

**Lemma 16.3.8.** Let \((G_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q, F)\) be a reduced \( p \)-\( m \)-arrangement of a graph \( G^* \) in a surface \( S \), and let \( G := \pi(G_0) \cup \bigcup_{i=1}^{q} R^i \). Then there is a tree decomposition \((T^*, \beta^*)\) of \( G^* \) in a surface \( S_r \) for the root \( r \) of \( T^* \), and for every \( t \in V(T^*) \setminus \{r\} \) a closed disk \( D_t \) such that the following conditions are satisfied:
(i) $S_r$ is homeomorphic to $S$.

(ii) $S_r \cup \bigcup_{t \in V(T^*) \setminus \{r\}} D_t = S$.

(iii) $\text{int}(S_r) \cap \text{int}(D_t) = \emptyset$ for all $t \in V(T^*) \setminus \{r\}$ and $\text{int}(D_t) \cap \text{int}(D_u) = \emptyset$ for all distinct $t, u \in V(T^*) \setminus \{r\}$.

(iv) $\text{bd}(S_r) \cup \bigcup_{t \in V(T^*) \setminus \{r\}} \text{bd}(D_t) = \text{bd}(S) \cup \bigcup_{f \in \mathcal{F}} f$.

(v) The torso $\tau^*(r)$ has a 3p-arrangement $(G_{t0}, \pi_t, R_t^1, \tau_t^1, \ldots, R_t^q, \tau_t^q)$ in $S_r$ such that:

a. For all $u \in N^T_+(r)$ it holds that $K[\sigma^*(u)] \subseteq R_t$ for some $t \in [q]$.

b. $V(R_t^i) \subseteq V((R_t^i + F)/F)$ for all $i \in [q]$.

c. $G_{t0} \subseteq G_0/F$, and the embeddings of $G_{t0}$ in $S_r$ and of $G_0$ in $S$ strongly coincide on $G_0 \cap \bigcup F$.

(vi) For all $t \in V(T^*) \setminus \{r\}$ the torso $\tau^*(t)$ has a 3p-arrangement $(G_{t0}, \pi_t, R_t, r_t)$ in $D_t$ such that:

a. For all $u \in \{t\} \cup N^T_+(t)$ it holds that $K[\sigma^*(u)] \subseteq R_t$.

b. There is an $i \in [q]$ such that $V(R_t^i) \subseteq V((R_t^i + F)/F)$.

c. $G_{t0} \subseteq G_0/F$, and the embeddings of $G_{t0}$ in $D_t$ and of $G_0$ in $S$ strongly coincide on $G_0 \cap \bigcup F$.

Proof. Let $\overline{S}$ be a surface without boundary and $D^1, \ldots, D^q \subseteq \overline{S}$ closed disks such that $S = \overline{S} \setminus \bigcup_{i=1}^q \text{int}(D^i)$. For every $i \in [q]$, let $F^i \subseteq F$ be the set of all edges in $F$ with both endvertices in $\overline{S}$. Then $F = \bigcup_{i=1}^q F^i$. Since for every $e \in F^i$ the curve $e$ is a contractible loop, there are mutually disjoint closed disks $\overline{D}^1, \ldots, \overline{D}^q \subseteq \overline{S}$ such that $D^i \cup \bigcup_{e \in F^i} e \subseteq \text{int}(\overline{D}^i)$ for every $i \in [q]$. This follows from Fact 9.1.9.

Let $i \in [q]$. Let $s^i$ be a sphere with $\overline{D}^i \subseteq s^i$, and let $d^i := s^i \setminus \text{int}(D^i)$. Let $G^i_0 := G_0[V(G_0) \cap \overline{D}^i]$. Then $G^i_0$ is embedded in $d^i$. Let $\pi^i$ be the restriction of $\pi$ to $V(G^i_0)$. Then $(G^i_0, \pi^i, R^i, \tau^i)$ is a p-arrangement of $G^i := \pi(G^i_0) \cup R^i$ in the disk $d^i$, and $(G^i_0, \pi^i, R^i, \tau^i)$ is a p-m-arrangement of $G^i/F^i$ in $d^i$. We apply Lemma 3.5.8 to this m-arrangement and obtain a tree decomposition $(T^i, \beta^i)$ of $G^i/F^i$ and closed disks $D^i_{r^i}$, for all $t \in V(T^i)$, satisfying the conditions of that lemma. As the disks $D^i_{r^i}$ have mutually disjoint interiors and boundaries in $\text{bd}(D^i) \cup \bigcup_{e \in F^i} e \subseteq \text{int}(\overline{D}^i)$, there is a node $r^i \in V(T^i)$ such that $\text{bd}(\overline{D}^i) \subseteq D^i_{r^i}$. Without loss of generality we may assume that $r^i$ is the root of $T^i$.

Now we combine and extend the tree decompositions $(T^i, \beta^i)$ into a decomposition $(T^*, \beta^*)$ of $G$. Without loss of generality we assume that the trees $T^1, \ldots, T^q$ are mutually disjoint. We let $T^*$ be the tree obtained from the union of $T^1, \ldots, T^q$ by identifying the roots $r^1, \ldots, r^q$. Let $r$ denote the new root of $T^*$. Then $V(T^*) = \{r\} \cup \bigcup_{i=1}^q V(T^i) \setminus \{r^i\}$. For every $i \in [q]$ and $t \in V(T^i) \setminus \{r^i\}$ we let $\beta^i(t) := \beta^i(t)$ and $D_t := D^i_t$. Then clearly the disks $D_t$, for $t \in V(T^*) \setminus \{r\}$ are mutually disjoint, and the assertions in (vi) follow from Lemma 3.5.8 (iv).

We let

$$S_r := S \setminus \text{int} \left( \bigcup_{t \in V(T^*)} D_t \right) = (S \setminus \bigcup_{i=1}^q \text{int}(D^i)) \cup \bigcup_{i=1}^q (D^i_{r^i} \cap D^i).$$
To see that $S_r$ is homeomorphic to $S$, note first that $(S \setminus \bigcup_{i=1}^q \text{int}(D^i))$ is homeomorphic to $S$, because the disks $\overline{D}^i$ are mutually disjoint and contain $D^i$. For every $i \in [q]$, the disk $D^i_{r_i} \subseteq S^i$ contains $\text{bd}(\overline{D}^i)$ and is disjoint from $\text{int}(D^i)$. Thus $D^i_{r_i} \cap \overline{D}^i$ is a cylinder homeomorphic to the cylinder $\overline{D}^i \setminus \text{int}(D^i)$. This implies that $S_r$ is homeomorphic to $S$, that is, assertion (i). Assertions (ii) and (iii) are immediate from the construction, and (iv) follows from Lemma 13.5.8(iii). We let

$$\beta^*(r) := \left( V(G) \setminus \bigcup_{i=1}^q V(G^i) \right) \cup \bigcup_{i=1}^q \beta^i(r^i).$$

To see that $(T^*, \beta^*)$ is a tree decomposition of $G$, note that every vertex of $G^i$ that is adjacent to a vertex in $V(G) \setminus V(G^i)$ is contained in $\beta^i(r^i)$. To complete the proof of the lemma, note that the assertions in (v) follow from Lemma 13.5.8(iv).

Lemma 16.3.9. Let $(G_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q, F)$ be a p-m-arrangement of a graph $G^*$ in a surface $S$ of Euler genus $\tau$ with $q$ cuffs. Let $(T^*, \beta^*)$ be the tree decomposition of $G^*$ obtained by Lemma 16.3.8. Let $r$ be the root of $T^*$ and $S_r$ obtained by Lemma 16.3.8. Let $(G_{r_0}, \pi_r, R^1_r, \tau^1_r, \ldots, R^q_r, \tau^q_r)$ be the $q$-p-arrangement of $\tau^*(r)$ in $S_r$ satisfying the conditions in Lemma 16.3.8(v). Let $g \subseteq S_r$ be a $G_{r_0}$-normal simplifying curve, and let

$$G^* \setminus g := G^* \setminus \pi_r(\tau_r \cap g).$$

Then $G^* \setminus g \in \mathcal{MR}$.

Proof. Let $G := \pi(G_0) \cup \bigcup_{i=1}^q R^i$. Note that if $g$ is a proper noncontractible curve or cuff-separating curve in $S_r$, then it is a noncontractible simple closed curve or cuff-separating curve in $S$ as well. Since $\text{bd}(S_r) \subseteq \text{bd}(S) \cup \bigcup_{f \in F} f$ and the endvertices of all edges $f \in F$ are in $\text{bd}(S)$, we have $V(G^*) \setminus \text{bd}(S_r) \subseteq \text{bd}(S)$. Thus if $g$ is a noncontractible loop or a link in $S_r$, then its endpoints are in $\text{bd}(S)$, because $g$ is $G_{r_0}$-normal. It follows that $g$ is a noncontractible loop or a link, in $S$ as well. Hence, in all cases $g$ is a simplifying curve in $S$.

Possibly sliding its endpoints along $\text{bd}(S)$, we may further assume that $g$ is $G_{r_0}$-normal. Thus by Lemma 15.1.6 we have

$$G \setminus g := G \setminus \pi(V(G_0) \cap g) \in \mathcal{M}_4 \left( \bigcup_{(r',q') \prec (r,q)} \mathcal{U}(\mathcal{A}\mathcal{E}_{r,q,r'}) \right) \subseteq \mathcal{MR}$$

Since $G^* \setminus g \cong G \setminus g$, the assertion of the lemma follows.

16.3.2 Proof of the Definable Structure Theorem 16.3.1

The key to the proof of the Definable Structure Theorem for Almost Embeddable Graphs is the following variant of Lemma 16.3.2, which will take care of the inductive step in the proof of the Definable Structure Theorem.

Lemma 16.3.10. Let $p, q, r \in \mathbb{N}$, and suppose that the class $\mathcal{MR}$ admits $\text{IFP}$-definable ordered treelike decompositions. Then the class $\mathcal{Z}_3 \cap \mathcal{MAE}_{r,q,r}^\text{red}$ admits $\text{IFP}$-definable ordered treelike decompositions.
Proof. The proof closely follows the proof of Lemma 16.3.2. If \((r, q) \leq^* (0, 1)\), the assertion follows from the Definable Structure Theorem for Almost Planar Graphs [3.5.1]. So we assume that \((r, q) >^* (0, 1)\). Let \(G^* \in \mathcal{MA}^{\text{red}}_{p,q,r} \setminus \mathcal{MR}\), and let \((G_0, \pi, R_1, \bar{T}_1, \ldots, R_q, \bar{T}_q, F)\) be a \(p\)-\(m\)-arrangement of \(G^*\) in a surface \(S\) of Euler genus \(r\) with \(q\) cuffs. Let \(G := \pi(G_0) \cup R\).

Let \((T^*, \beta^*)\) be the tree decomposition of \(G^*\) obtained by Lemma 16.3.8. Let \(r\) be the root of \(T^*\) and \(S_r\) the surface of the lemma. Let \((G_{r_0}, \pi_r, R_1, \bar{T}_1, \ldots, R_q, \bar{T}_q)\) be the \(3p\)-arrangement of \(\tau^*(r)\) in \(S_r\) satisfying the conditions in Lemma 16.3.11. Then \((G_{r_0}, \pi_r, \bar{T}_1, \ldots, \bar{T}_q)\) is a local \(3p\)-arrangement of \(G^*\) in the surface \(S_r\). Observe that \(G^*\) together with this arrangement and the surface \(S_r \simeq S\) satisfy Assumption 15.1.7(1) with \(3p\) instead of \(p\). By a similar argument as in the proof of Lemma 16.3.2 it can be shown that Assumption 15.1.7(5) is satisfied as well.

**Case 1:** The representativity of \(G_{r_0}\) in \(S_r\) is at most 4.

Let \(g \subseteq S_r\) be a \(G_{r_0}\)-normal simplifying curve with \(|g \cap V(G_{r_0})| \leq 4\). By Lemma 16.3.9 we have \(G^* \setminus g \in \mathcal{MR}\) and thus \(G^* \in \mathcal{N}_4(\mathcal{MR})\). As \(\mathcal{MR}\) admits IFP-definable ordered treelike decompositions, by the Finite Extension Lemma 7.3.1 \(\mathcal{N}_4(\mathcal{MR})\) admits IFP-definable ordered treelike decompositions as well.

Suppose that the representativity of \(G_{r_0}\) in \(S_r\) is at least 5. Then Assumption 15.1.7(4) is satisfied as well, and we can choose a surface \(\bar{S}\) satisfying Assumption 15.3.1(6) Then Assumption 15.1.7 is satisfied. By Lemma 15.4.10 there are disks \(D_1, \ldots, D_4 \subseteq S\) satisfying Assumption 15.3.1(1). Finally, since we have \(\mathcal{P} = \mathcal{AE}_{0,0,0} \subseteq \mathcal{MR}\), by Lemma 16.3.9 Assumption 15.3.1(2) (with \(\mathcal{MR}\) instead of \(\mathcal{R}\)) is satisfied as well.

**Case 2:** The representativity of \(G_{r_0}\) in \(S_r\) is at least 5 and \(G^*\) has a canonical simplifying patch.

Note that in this case, all assumptions of Lemma 16.1.2 are satisfied, and thus we can define an ordered treelike decomposition on \(G^*\).

**Case 3:** The representativity of \(G_{r_0}\) in \(S_r\) is at least 5 and \(G^*\) has no canonical simplifying patch.

Then \(G^*\) satisfies Assumption 15.5.1 as well. In this case, all assumptions of the Last Extension Lemma 16.2.1 are satisfied, and again we can define an ordered treelike decomposition on \(G^*\).

**Proof of the Definable Structure Theorem for Almost Embeddable Graphs 16.3.1.** We prove the theorem by induction on the order \(<^*\) of pairs \((r, q)\). Note that in each step of the induction we prove the theorem for all \(p\). For the base step, suppose that \((r, q) \leq^* (0, 1)\). Then the assertion of the Theorem follows from the Definable Structure Theorem for Almost Planar Graphs [3.5.1].

For the inductive step, suppose that \((r, q) >^* (0, 1)\), and let \(p \in \mathbb{N}\). We define \(\mathcal{MR}\) as in 16.3.1. By the induction hypothesis, for all \((r', q') <^* (r, q)\) the class \(\mathcal{M}(\mathcal{AE}_{3p,q',r'})\) admits

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Preliminary Version
Chapter 16. Decompositions of Almost Embeddable Graphs

IFP-definable ordered treelike decompositions. Thus by the Union Lemma for Definable Ordered Decompositions 7.1.10 and the Finite Extension Lemma 7.3.1, the class MR admits IFP-definable ordered treelike decompositions. By Lemma 16.3.7 and again the Union Lemma 7.1.10 and the Finite Extension Lemma 7.3.1, \( \mathcal{M}(\mathcal{AE}_{p,q,r}) \) admits IFP-definable ordered treelike decompositions. Thus by Lemma 16.3.7 and again the Union Lemma 7.1.10 and the Finite Extension Lemma 7.3.1, \( \mathcal{M}(\mathcal{AE}_{p,q,r}) \) admits IFP-definable ordered treelike decompositions.

16.4 Almost Embeddable Completions

Definition 16.4.1. Let \( p, q, r \in \mathbb{N} \), and let \( G \) be a graph.

1. An \( \mathcal{AE}_{p,q,r} \)-star decomposition of \( G \) is a tuple

\[
(S, \beta, s, S, H_0, \pi, R^1, \bar{r}^1, \ldots, R^q, \bar{r}^q)
\]

such that:

(i) \((S, \beta)\) is a star decomposition of \( G \), and \( s \) is the centre of \( S \).

(ii) \( S \) is a surface of Euler genus \( r \) with \( q \) cuffs, and \((H_0, \pi, R^1, \bar{r}^1, \ldots, R^q, \bar{r}^q)\) is a \( p \)-arrangement of the torso \( \tau(s) \) in \( S \).

(iii) For all tips \( t \in V(S) \setminus \{s\} \),

a. either \( K[\sigma(t)] \subseteq R \),

b. or there is a subgraph \( C_t \subseteq H_0 \) such that \( \pi(C_t) = K[\sigma(t)] \) and \( C_t \cong K_i \) for some \( i \in [3] \), and if \( C_t \cong K_3 \) then \( C_t \) is a facial cycle of \( H_0 \).

If (iii-a) holds for all \( t \in V(S) \setminus \{s\} \), then \((S, \beta, s, S, H_0, \pi, R^1, \bar{r}^1, \ldots, R^q, \bar{r}^q)\) is a simple \( \mathcal{AE}_{p,q,r} \)-star decomposition of \( G \).

2. Let \( \Phi \) be a pre-decomposition of \( G \). Then an \( \mathcal{AE}_{p,q,r} \)-star completion of \( \Phi \) is an \( \mathcal{AE}_{p,q,r} \)-star decomposition \((S, \beta, s, S, H_0, \pi, R^1, \bar{r}^1, \ldots, R^q, \bar{r}^q)\) of \( G \) such that for all tips \( t \) of \( S \) there is a \( t' \in V(\Phi) \) such that \( t \parallel^{(S, \beta), \Phi} t' \).

Our goal in this section is to prove the following generalisation of the Almost Planar Completion Theorem 14.1.3.

Theorem 16.4.2 (Almost Embeddable Completion Theorem). Let \( p, q, r \in \mathbb{N} \). Then there exists \( d, d' \in \mathbb{N} \) such that for all pd-schemes \( \Psi \) there exists an od-scheme \( \Lambda \) such that for every graph \( G \) the following holds. If \( \Psi[G] \) is a tight pre-decomposition that has an \( \mathcal{AE}_{p,q,r} \)-star completion, then \( \Lambda[G] \) is an ordered completion of the \((d, d')\)-derivation of \( \Psi[G] \).

The proof of this theorem follows the final steps of the proof of the Almost Planar Completion Theorem 14.1.3 in Section 14.6. The only real difference is in the very last step of the proof, where we use an inductive argument similar to the proof of the Definable Structure Theorem for Almost Embeddable Graphs 16.3.1.

16.4.1 \( \mathcal{MAE}_{p,q,r} \)-Star Completions

Definition 16.4.3. Let \( p, q, r \in \mathbb{N} \), and let \( G \) be a graph.
16.4. Almost Embeddable Completions

(1) An $\mathcal{MAE}_{p,q,r}$-star decomposition of $G$ is a tuple

$$(S, \beta, s, S, H_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q, F)$$

satisfying the following conditions.

- (i) $(S, \beta)$ is a star decomposition of $G$ and $s$ is the centre of $S$.
- (ii) $S$ is a surface of Euler genus $r$ with $q$ cuffs, and $(H_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q, F)$ is a $p$-m-arrangement of the torso $\tau(s)$ in $S$.
- (iii) For all tips $t \in V(S) \setminus \{s\}$,
  - a. either there is an $i \in [q]$ such that $K[\sigma(t)] \subseteq (R^i + F)/F$,
  - b. or there is a subgraph $C_t \subseteq H_0$ such that $K[\sigma(t)] = \pi(C_t)/F$ and $C_t \cong K_i$ for some $i \in [3]$, and if $C_t \cong K_3$ then $C_t$ is a facial cycle of $H_0$.

If (iii-a) holds for all $t \in V(S) \setminus \{s\}$, then the $\mathcal{MAE}_{p,q,r}$-star decomposition is simple.

If the $p$-m-arrangement $(H_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q, F)$ in (ii) is reduced then the $\mathcal{MAE}_{p,q,r}$-star decomposition is reduced.

(2) Let $\Phi$ be a pre-decomposition of $G$. An $\mathcal{MAE}_{p,q,r}$-star completion of $\Phi$ in $G$ is an $\mathcal{MAE}_{p,q,r}$-star decomposition $(S, \beta, s, S, H_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q, F)$ of $G$ such that for all tips $t$ of $S$ there is a $t' \in V(\Phi)$ with $t \parallel (S, \beta), \Phi t'$.

Next, we generalise the lemmas of Section [14.5] Fortunately, the proofs of these lemmas go through almost literally.

**Lemma 16.4.4.** Let $p, q, r \in \mathbb{N}$ such that $p \geq 3$. Let $G$ be a graph and $\Phi$ a tight pre-decomposition of $G$ that has an $\mathcal{MAE}_{p,q,r}$-star completion. Let $J^*$ be a quasi-4-connected component of $G$ and $\Phi^*$ the pre-decomposition of $J^*$ induced by $\Phi$ on $J^*$.

Then $(\Phi^*)^{(p,p)}$ has a simple $\mathcal{MAE}_{p,q,r}$-star completion in $J^*$.

**Proof.** Similar to the proof of Lemma [14.5.2] $\square$

The following generalisation of Lemma [14.5.3] only applies to reduced $\mathcal{MAE}_{p,q,r}$-star completions. We will turn to non-reduced $\mathcal{MAE}_{p,q,r}$-star completions in Lemma [16.4.6].

**Lemma 16.4.5.** Let $p, q, r \in \mathbb{N}$ such that $p \geq 3$. Let $G$ be a graph and $\Phi$ a pre-decomposition of $G$ that has a reduced $\mathcal{MAE}_{p,q,r}$-star completion. Let $J^*$ be a quasi-4-connected component of $G$ and $\Phi^*$ the pre-decomposition induced by $\Phi$ on $J^*$.

Then $J^*$ has a tree decomposition $(T^*, \beta^*)$ such that for all $t \in V(T^*)$ one of the following three conditions is satisfied.

- (i) $t$ is a leaf of $T^*$, and there is a $t' \in V((\Phi^*)^{(p,p)})$ such that $t \parallel (T^*, \beta^*), (\Phi^*)^{(p,p)} t'$.
- (ii) $\tau^*(t)$ has a 3$p$-arrangement $(H_0, \pi_t, R_t, \tau_t)$ in a disk $D_t$ such that for all $u \in \{t\} \cup N^+_t(t)$ it holds that $K[\sigma^*(u)] \subseteq R_t$.
- (iii) $t$ is the root of $T^*$, and $\tau^*(t)$ has a 3$p$-arrangement $(H_0, \pi_t, R^1_t, \tau^1_t, \ldots, R^q_t, \tau^q_t)$ in a surface $S$ of Euler genus $r$ with $q$ cuffs such that for all $u \in N^+_t(t)$ there is an $i \in [q]$ such that $K[\sigma^*(u)] \subseteq R^i_t$.  

Preliminary Version
Lemma 16.4.6. Let $p, q, r \in \mathbb{N}$. Let $G^*$ be a graph and $\Phi$ be a pre-decomposition of $G^*$ that has a non-reduced $\mathcal{MAE}_{p,q,r}$-star completion.

Then there are $q', r' \in \mathbb{N}$ with $(r', q') <^\ast (r, q)$ and a set $W \subseteq V(G^*)$ of size $|W| \leq 4p + 1$ such that the following holds. Let $G'$ be a connected component of $G^* \setminus W$ and $\Phi'$ the pre-decomposition induced by $\Phi$ on $G'$. Then $(\Phi')^{(p,0)}$ has an $\mathcal{MAE}_{p,q,r}$-star completion.

Proof. Let $(S, \beta, s, S_0, \pi, R_1, \pi_1, \ldots, R_0, \pi_0)$ be a non-reduced $\mathcal{MAE}_{p,q,r}$-star completion of $G^*$. Let $H^* := \tau(s)$, and let $H := \pi(S) \cup \bigcup_{i=1}^p R_i$. Then $H^* = H/F$. Let $f_0 \in E(H_0)$ such that $f := \pi(f_0) \in F$ and $f_0$ is a noncontractible loop or a link in $S$. Let $w$ be the vertex of $G^*$ corresponding to $f$.

By Lemma [15.1.7] we can find $(r', q') <^\ast (r, q)$ and a set $X \subseteq V(H)$ of size $|X| \leq 4p$ such that every connected component $H'$ of $H \setminus f_0 \setminus X$ is contained in $\mathcal{AE}_{p,q,r'}$. As a matter of fact, we need to recall the construction described in Section [15.1.3] that we used to prove Lemma [15.1.1] to understand how the arrangement $(H_0, \pi, R_1, \pi_1, \ldots, R_0, \pi_0)$ of $H$ in $S$ yields an arrangement of $H'$ in a surface $S'$ with $eg(S') = r'$ and $cf(S') = q'$.

We let $X^* \subseteq V(H^*)$ be the set corresponding to $X$ (after contracting the edges in $f$) and $W := X^* \cup \{w\}$. Now let $G'$ be a connected component of $G^* \setminus W$, and let $H'^* := H^*[V(G') \cap V(H')]$. Then either $H'$ is empty, or it is a connected component of $H^* \setminus W$, corresponding to a connected component $H''$ of $H \setminus f_0 \setminus X$. The arrangement of $H''$ obtained from Lemma [15.1.7] as described above yields a $\mathcal{MAE}_{p,q',r'}$-star decomposition of $G'$, which can easily be turned into a $\mathcal{MAE}_{p,q',r'}$-star completion of $(\Phi')^{(p,0)}$.

16.4.2 Completion Versions of the Combination Lemma and the Last Extension Lemma

Lemma 16.4.7. Let $d, d' \in \mathbb{N}$. Let $\Lambda^1, \Lambda^2(\overline{x}, y)$ be od-schemes and $\Psi(\overline{x})$ a pd-scheme, and let $\varphi(\overline{x}, y)$ and $\psi(\overline{x}, y_1, y_2)$ be $\text{IFP}$-formulae. Then there exists an od-scheme $\Lambda$ such that the following holds.

Let $G$ be a graph and $\overline{v} \in V(G)$. Let $W := \varphi[G, \overline{v}, y]$, and let $A_1, \ldots, A_m$ be the connected components of $G \setminus W$. Let $\Phi := \Psi[G, \overline{v}]$, and for every $i \in [m]$, let $\Phi_i$ be the restriction of $\Phi^{(d,d')}$ to $V(A_i)$. Suppose that the following conditions are satisfied:

(i) The od-scheme $\Lambda^1$ defines an ordered treelike decomposition on $G/A_1/ \cdots /A_m$.

(ii) For every $i \in [m]$ and every $a \in V(A_i)$, the od-scheme $\Lambda^2$ defines an ordered completion of $\Phi_i$ in $A_i$ within $(G, \overline{v}, a)$.

(iii) For every $i \in [m]$, the restriction of the binary relation $\psi[G, \overline{v}, y_1, y_2]$ to the vertices of attachment of $A_i$ is a linear order.

Then $\Lambda[G]$ is an ordered completion of $\Phi^{(d,d')}$ in $G$.

Proof. The proof is an easy adaptation of the proof of the Combination Lemma [16.1.1]. We use the notation of that proof and carry out Step 1 completely analogously. We need to modify the definition of $\Delta'$ at the beginning of Step 2 slightly. For every $i \in [m]$ and $a \in V(A_i)$ we let $\Delta_{i,a} := (D_{i,a}, \sigma_{i,a}, \alpha_{i,a}) := \Lambda^2[\overline{v}, a]$. By assumption (ii), $\Delta_{i,a}$ is an ordered completion of $\Phi_i$ in $A_i$. Without loss of generality we assume that $\Delta_{i,a}$ is normal. We further assume that the node sets $V(D_{i,a})$ are mutually disjoint, and that they are all disjoint from $V(D')$. We define a decomposition $(D'', \sigma'', \alpha'')$ of $G$ as follows:

M. Grohe, Definable Graph Structure Theory
16.4. Almost Embeddable Completions

\[
\begin{align*}
\bullet \ V(D'') & := V(D') \cup \bigcup_{i=1}^{m} \bigcup_{a \in V(A_i)} V(D_{i,a}) ; \\
\bullet \ E(D'') & := E(D') \cup \bigcup_{i=1}^{m} \bigcup_{a \in V(A_i)} \left( E(D_{i,a}) \cup \{ (s,t) \mid s \in M_i, t \in V(D_{i,a}) \leq \{ D_{i,a} \} \} \right) ; \\
\bullet \ \sigma''(t) & := \begin{cases} 
\sigma'(t) \setminus \{ a_1, \ldots, a_m \} & \text{if } t \in V(D') , \\
\sigma_i(t) & \text{if } t \text{ is an extension node of } \Delta_{i,a} , \\
\sigma_i(t) & \text{for some } i \in [m], a \in V(A_i) , \\
\sigma(t) & \text{if } t \text{ is a ground node of } \Delta_{i,a} , \\
\sigma(t) & \text{for some } i \in [m], a \in V(A_i) ; \\
\alpha_i(t) & \text{if } t \in V(D_{i,a}) \\
\alpha_i(t) & \text{for some } i \in [m], a \in V(A_i) .
\end{cases}
\end{align*}
\]

We can prove that \((D'', \sigma'', \alpha'')\) is a treelike decomposition of \(G\) as in the proof of Claim 3 in the proof of the Combination Lemma; there is only one additional difficulty: to verify \((\text{TL.2})\) we need to handle ground nodes of the decompositions \(\Delta_{i,a}\) differently than extension nodes (for extension nodes, we proceed as in the proof of the Combination Lemma). So let \(i \in [m]\) and \(a \in V(A_i)\), and let \(t\) be a ground node of \(\Delta_{i,a}\). Let \(t' \in V(\Phi_i)\) such that \(t \parallel \Delta_{i,a} \Phi_i t'\). As \(\Phi_i\) is the restriction of \(\Phi^{(d,d')}\) to \(A_i\), we have \(t' \in V(\Phi^{(d,d')}))\) with \(\gamma^{\Phi^{(d,d')}}(t') \subseteq V(A_i)\) and \(\sigma^{\Phi^{(d,d')}}(t') = \sigma_i(t') = \sigma_{i,a}(t) = \sigma''(t)\). Furthermore, \(\alpha^{\Phi^{(d,d')}}(t') = \alpha_{i,a}(t) = \alpha''(t)\). Moreover, as the derivation \(\Phi^{(d,d')}\) is a proper pre-decomposition of \(G\), we have \(\sigma^{\Phi^{(d,d')}}(t') \cap \alpha^{\Phi^{(d,d')}}(t') = \emptyset\) and \(N^{G}(\alpha^{\Phi^{(d,d')}}(t')) \subseteq \sigma^{\Phi^{(d,d')}}(t')\).

Once we have proved that \((D'', \sigma'', \alpha'')\) is a treelike decomposition, we see that it is a completion of \(\Phi^{(d,d')}\) whose ground nodes are the ground nodes of the \(\Delta_{i,a}\). We can define a linear order on all extension nodes as we did for all nodes of \(D''\) in the proof of the Combination Lemma and turn it into an ordered completion.

\[\square\]

The next lemma is a “completion version” of the Last Extension Lemma 16.2.1. For a graph \(G\) satisfying Assumption 15.1.7 for some surface \(S\), some arrangement of \(G\) in \(S\), and cetera, let \(R'(G)\) be the set of all subgraphs of \(g\)-reductions with respect to the arrangement for all simplifying curves \(g \subseteq S\), and let \(R(G) := R'(G) \cup P\). For simplicity, we just write \(R'(G)\) and \(R(G)\) even though the classes depend not only on \(G\) but also on the surface \(S\) and the arrangement. Note that all graphs \(G' \in R'(G)\) are subgraphs of \(G\). Hence if \(\Phi\) is a pre-decomposition of \(G\), then \(\Phi\) induces a pre-decomposition \(\Phi'\) on \(G'\).

\textbf{Lemma 16.4.8.} Let \(d,d', p, q, r \in \mathbb{N}\) with \((r, q) >^* (0, 1)\). Let \(A\) an od-scheme and \(\Psi\) a pd-scheme. Then there is an od-scheme \(\Lambda\) such that for all graphs \(G\) the following holds. Suppose that

\[\text{(i) } G \text{ and } R(G) \text{ satisfy Assumptions 15.1.7, 15.3.1 and 15.5.1;}\]

\[\text{(ii) for all } G' \in R'(G) \text{ the od-scheme } \Lambda'[G'] \text{ is an ordered completion of } (\Phi')^{(d,d')}, \text{ where } \Phi' \text{ is the pre-decomposition induced by } \Psi[G] \text{ on } G'.\]

Preliminary Version
Then \( \Lambda[G] \) is an ordered completion of \( (\Psi[G])^{(d,d')}. \)

**Proof.** Note that by taking the “union” of \( \Lambda' \) with an od-scheme that defines an ordered treelike decomposition on all planar graphs, we obtain an od-scheme \( \Lambda'' \) that satisfies the following condition, provided \( G \) satisfies (ii):

(iii) for all \( G' \in \mathcal{R}(G) \), if \( G' \in \mathcal{P} \) then \( \Lambda''[G'] \) is an ordered treelike decomposition of \( G' \), and if \( G' \in \mathcal{R}(G) \setminus \mathcal{P} \) then \( \Lambda''[G'] \) is an ordered completion of \( (\Phi')^{(d,d')} \), where \( \Phi' \) is the pre-decomposition induced by \( \Psi[G] \) on \( G' \).

With \( \Lambda'' \) and (ii'), the proof of the Last Extension Lemma \[16.2.1\] goes through with only minor adaptations. \( \square \)

### 16.4.3 Proof of the Almost Embeddable Completion Theorem \[16.4.2\]

We shall prove the following lemma, which immediately implies the Almost Embeddable Completion Theorem \[16.4.2\].

**Lemma 16.4.9.** Let \( p, q, r \in \mathbb{N} \). Then there exists \( d, d' \in \mathbb{N} \) such that for all pd-schemes \( \Psi \) there exists an od-scheme \( \Lambda \) such that for every graph \( G \) the following holds. If \( \Psi[G] \) is a tight pre-decomposition that has an \( \mathcal{MAE}_{p,q,r} \)-star completion, then \( \Lambda[G] \) is an ordered completion of the \( (d,d') \)-derivation of \( \Psi[G] \).

**Proof.** The proof is by induction on the order \( <^* \).

The base step is Lemma \[14.6.2\] (the strengthening of the Almost Planar Completion Theorem \[14.1.3\] where we only assumed \( \Psi[G] \) to have a \( \mathcal{MAP}_p \)-star completion).

For the inductive step, let \( (r, q) >^* (0, 1) \) and suppose that the lemma is proved for all \( (r', q') <^* (r, q) \). Without loss of generality we may assume that \( p \geq 3 \).

The strategy of the inductive step is to first carry it out for the quasi-4-connected components of \( G \), which we will do in Claim \[2\] and then apply the QC4 Completion Lemma \[12.6.2\].

To prove Claim \[2\] we will repeatedly use the following claim.

**Claim 1.** Let \( p', q', r' \in \mathbb{N} \) such that \( (r', q') <^* (r, q) \), and let \( \Psi \) be a pd-scheme. Then there exist \( d, d' \in \mathbb{N} \) and an od-scheme \( \Lambda' \) such that the following holds. Let \( G \) be a graph such that \( \Phi := \Psi[G] \) is a tight pre-decomposition of \( G \). Suppose that there is a tree decomposition \( (T, \beta) \) of \( G \) such that for all \( t \in V(T) \) one of the following three conditions is satisfied.

(i) \( t \) is a leaf of \( T \), and there is a \( t' \in V(\Phi) \) such that \( t >^{(T,\beta)} \Phi t' \).

(ii) \( \tau(t) \) has a \( p' \)-arrangement \( (H_{10}, \pi_t, R_t, \tau_t) \) in a disk \( D_t \) such that for all \( u \in \{t\} \cup N_T(t) \) it holds that \( K[\sigma(u)] \subseteq R_t \).

(iii) \( t \) is the root of \( T \) and \( \tau(t) \) has a \( p' \)-arrangement \( (H_{10}, \pi_t, R_t^1, \tau_t^1, \ldots, R_t^q, \tau_t^q) \) in a surface \( S \) with \( eg(S) = r' \) and \( cf(S) = q' \) such that for all \( u \in N_T(t) \) there is an \( i \in [q] \) with \( K[\sigma(u)] \subseteq R_t^i \).

Then \( \Lambda'[G] \) is an ordered completion of \( \Phi^{(d,d')} \).

**Proof.** We proceed very similarly to the proof of Lemma \[14.6.1\] Let \( G \) be a graph such that \( \Phi := \Psi[G] \) is a tight pre-decomposition of \( G \). Without loss of generality we may assume that \( G \) is connected. Let \( (T, \beta) \) be a tree decomposition of \( G \) satisfying (i)–(iii). By Corollary \[12.1.4\] we may assume without loss of generality that \( (T, \beta) \) is tight. We may further assume that

M. Grohe, *Definable Graph Structure Theory*
$V(\Phi)$ is the set of all leaves $t \in V(T)$ satisfying (i), and that for all $t \in V(\Phi)$ we have $\sigma(t) = \sigma^\Phi(t)$ and $\alpha(t) = \alpha^\Phi(t)$. It follows from (ii) and (iii) that the adhesion of $(T, \beta)$ and hence the adhesion of $\Phi$ is at most $p'$.

Let $t \in V(T)$ and $G^t := G[\gamma(t)] \cup K[\sigma(t)]$. Let $\Phi^t$ be the pre-decomposition of $G^t$ defined by $V(\Phi^t) := N^G_2(t)$ and $\sigma^\Phi^t(u) := \sigma(u)$, $\alpha^\Phi^t(u) := \alpha(u)$ for all $u \in V(\Phi^t)$. By our assumption that $(T, \beta)$ is tight, $\Phi^t$ is a tight pre-decomposition of $G^t$.

Let $t_0$ be the root of $T$ and note that $G^{t_0} = G$. It follows from (ii) that for all $t \in V(T) \setminus (V(\Phi) \cup \{t_0\})$

(A) $\Phi^t$ has an $\mathcal{AP}_{p'}$-star completion in $G^t$.

It follows from (iii) that

(B) $\Phi^{t_0}$ has an $\mathcal{AE}_{p', q, p'}$-star completion in $G$.

For every tuple $\overline{v} := (v_1, \ldots, v_{p'+1}) \in V(G)^{p'+1}$, let

$$S_\overline{v} := \begin{cases} \{v_1, \ldots, v_{p'}\} & \text{if } v_{p'+1} \notin \{v_1, \ldots, v_{p'}\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $A_{\overline{v}}$ be the connected component of $G \setminus S_{\overline{v}}$ that contains $v_{p'+1}$, and let

$$H_{\overline{v}} := G[V(A_{\overline{v}}) \cup S_{\overline{v}}] \cup K[S_{\overline{v}}].$$

Note that $A_{\overline{v}} = H_{\overline{v}} = G$ if $v_{p'+1} \in \{v_1, \ldots, v_{p'}\}$ by our assumption that $G$ is connected. Let $P$ be the set of all $\overline{v} \in V(G)^{p'+1}$ such that $S_{\overline{v}} = N^G(A_{\overline{v}}) = \partial^G(V(H_{\overline{v}}))$. For every subset $Q \subseteq P$, we define a pre-decomposition $\Phi_Q$ of $G$ by letting $V(\Phi_Q) := Q$ and $\sigma^{\Phi_Q}(\overline{w}) := S_{\overline{w}}$, $\alpha^{\Phi_Q}(\overline{w}) := V(A_{\overline{w}})$ for all $\overline{w} \in Q$. For every $\overline{v} \in P$, we let $\Phi_{Q, \overline{v}}$ be the “restriction” of $\Phi_Q$ to $H_{\overline{v}}$, that is, we let $V(\Phi_{Q, \overline{v}}) := \{\overline{w} \in Q \mid V(H_{\overline{v}}) \subseteq V(H_{\overline{v}})\}$ and $\sigma^{\Phi_{Q, \overline{v}}}(\overline{w}) := S_{\overline{w}}$, $\alpha^{\Phi_{Q, \overline{v}}}(\overline{w}) := V(A_{\overline{w}})$ for all $\overline{w} \in V(\Phi_{Q, \overline{v}})$. It follows from the definition of $P$ that $\Phi_Q$ and $\Phi_{Q, \overline{v}}$ are tight pre-decompositions of $G$ and $H_{\overline{v}}$, respectively. Note that there is a pd-scheme $\Psi'(X, \overline{v})$, where $X$ is a $(p'+1)$-ary relation variable, that defines $\Phi_{Q, \overline{v}}$ within $(G, Q, \overline{v})$.

By the induction hypothesis of the global induction in the proof of the lemma, there is an od-scheme $\Lambda'(X, \overline{v})$ such that for all $Q \subseteq P$ and $\overline{v} \in P$, if $\Psi'(Q, \overline{v})$ has an $\mathcal{AP}_{p'}$-star completion or an $\mathcal{AE}_{p', q, p'}$-star completion in $H_{\overline{v}}$ then $\Lambda'(X, \overline{v})$ defines an ordered completion of $\Phi_{Q, \overline{v}}$ in $H_{\overline{v}}$ within $(G, Q, \overline{v})$. For all $Q \subseteq P$ and $\overline{v} \in P$, let $\Delta_{Q, \overline{v}} := \Lambda''(G, Q, \overline{v})$.

For a tuple $\overline{v} \in P$ and a node $t \in V(T)$, we write $\overline{v} \parallel t$ if $\sigma(t) = S_{\overline{v}}$ and $\alpha(t) = V(A_{\overline{v}})$. We inductively define sequences $(P_i)_{i \in \mathbb{N}}$ of subsets of $P$ as follows. We let $P_0$ be the set of all $\overline{v} \in P$ such that $\overline{v} \parallel t$ for some $t \in V(\Phi)$. Then $\Phi$ and $\Phi_{P_0}$ are equivalent pre-decompositions of $G$ (in the sense of Definition 12.1.7). For all $i \in \mathbb{N}$, we let $P_{i+1}$ be the set of all $\overline{v} \in P_i$ such that either $\overline{v} \in P_i$ or $\Delta_{P_{i+1}, \overline{v}}$ is an ordered completion of $\Phi_{P_i, \overline{v}}$. We let $P_{\infty} := \bigcup_{i \in \mathbb{N}} P_i$.

A straightforward induction that uses (A), (B), and the facts that $(T, \beta)$ is tight and has adhesion at most $p'$ shows that for all $t \in V(T)$ there is a $\overline{v} \in P_{\infty}$ such that $\overline{v} \parallel t$. In particular, there is a $\overline{v}_0 \in P_{\infty}$ such that $\overline{v}_0 \parallel t_0$.

By induction on $i \in \mathbb{N}$, for every $\overline{v} \in P_i$, we can define an ordered completion $\Delta_{\overline{v}} := (D_{\overline{v}}, \sigma_{\overline{v}}, \alpha_{\overline{v}}, \leq_{\overline{v}})$ of the restriction $\Phi_{P_i, \overline{v}}$ of $\Phi_{P_i} = \Phi$ to $H_{\overline{v}}$ by combining the o-decompositions $\Delta_{P_{i+1}, \overline{v}}$ appropriately. Then $\Delta_{\overline{v}_0}$ is an ordered completion of $\Phi$ in $H_{\overline{v}_0} = G$.

It is important to note that in our definition of the decompositions $\Delta_{\overline{v}}$ we make no assumption on the definability of the tree decomposition $(T, \beta)$; we only assume that such a
decomposition exists. As all steps of the construction can easily be formalised in \( \text{IFP} \), there is an od-scheme \( \Lambda'(\overline{x}) \) that defines \( \Delta_{\overline{x}_0} \) within \( (G, \overline{x}_0) \). The claim follows.

The next claim is a generalisation of Lemma [14.6.1]

\textbf{Claim 2.} There are \( d, d' \in \mathbb{N} \) such that for every pd-scheme \( \Psi \) there is a parametrised od-scheme \( \Lambda(\overline{x}) \) such that the following holds. Let \( G \) be a graph such that \( \Phi := \Psi(G) \) is a tight pre-decomposition of \( G \) that has an \( \mathcal{MAE}_{p,q,r} \)-star completion. Let \( J^* \) be a quasi-4-connected component of \( G \) with index \( \overline{v} \in V(G) \), and let \( \Phi^* \) be the pre-decomposition induced by \( \Phi \) on \( J^* \). Then the scheme \( \Lambda(\overline{x}) \) defines an ordered completion of \( (\Phi^*);d,d' \) on \( J^* \) within \( (G, \overline{v}) \).

\textbf{Proof.} Let \( \Psi \) be a pd-scheme and \( G \) a graph such that \( \Phi := \Psi(G) \) is tight and has a \( \mathcal{MAE}_{p,q,r} \)-star completion in \( G \). Let \( J^* \) be a quasi-4-connected component of \( G \) with index \( \overline{v} \in V(G) \), and let \( J \) be the torso and matching corresponding to \( J^* \). Let \( \Phi^* \) be the pre-decomposition induced by \( \Phi \) on \( J^* \) and \( \Phi_1 := (\Phi^*)^{(p,p)} \). By Lemma 16.4.4, \( \Phi_1 \) has a simple \( \mathcal{MAE}_{p,q,r} \)-star completion in \( J^* \).

Suppose first that \( \Phi_1 \) has a non-reduced \( \mathcal{MAE}_{p,q,r} \)-star completion in \( J^* \). Then by Lemma 16.4.6 there is a pair \( (r', q') <^* (r, q) \) and a set \( W^* \subseteq V(J^*) \) of size \( |W^*| \leq 4p + 1 \) such that for every connected component \( A \) of \( J^* \setminus W^* \) and the pre-decomposition \( \Phi_1, A \) induced by \( \Phi_1 \) on \( A \), the derived pre-decomposition \( \Phi_1^{A} \) has a \( \mathcal{MAE}_{p,q',r'} \)-star completion.

Let \( W \subseteq V(J) \subseteq V(G) \) be the subset that is contracted to \( W^* \) in the transition from \( J \) to the minor \( J^* = J/M \). Then \( |W| \leq 8p + 2 \).

By the induction hypothesis (of the global induction in the proof of the lemma), there are \( d_1, d'_1 \in \mathbb{N} \) and a parametrised od-scheme \( \Lambda_1(\overline{x}, y, Z) \) such that for every connected component \( A \) of \( J^* \setminus W^* \) and every \( v \in V(A) \), the od-scheme \( \Lambda_1(\overline{x}, y_1, y_2, Z) \) defines an ordered completion of \( (\Phi_1^{A});(d_1, d'_1) \) within \( (G, \overline{v}, v_1, v_2, W) \), where \( v_1 = v_2 = v \) if \( v \in V(J) \), and otherwise \( v_1 \) and \( v_2 \) are the endvertices of the edge \( e \in M \) that is contracted to \( v \). To construct \( \Lambda_1 \), we use that there are only finitely many \( (r', q') <^* (r, q) \). From \( \Lambda_1(\overline{x}, y_1, y_2, Z) \), we can construct an od-scheme \( \Lambda(\overline{x}, Z) \) that defines an ordered completion of \( (\Phi_1^{(p,0)});(d_1, d'_1) \) within \( (G, \overline{v}, W) \), where \( \Phi_2 \) is the pre-decomposition induced by \( \Phi_1 \) on \( J^* \setminus W \). By the Finite Extension Lemma for Ordered Completions 12.5.4 there is an od-scheme \( \Lambda \) that defines an ordered completion of \( (\Phi_1^{(p,0)});(d_1 + 8p + 2, d'_1 + 8p + 2) \) within \( (G, \overline{v}) \). As

\[
(\Phi_1^{(p,0)});(d_1 + 8p + 2, d'_1 + 8p + 2) \lesssim (\Phi^*);(d_1 + 8p + 2, d'_1 + 8p + 2) \lesssim (\Phi^*);(d_1 + 8p + 2, d'_1 + 9p + 2)
\]

by Lemma 12.4.3 this proves the claim.

In the following, we assume that all \( \mathcal{MAE}_{p,q,r} \)-star completions of \( \Phi_1 \) in \( J^* \) are reduced. We apply Lemma 16.4.5 and obtain a tree decomposition \( (T^*, \beta^*) \) of \( J^* \) that satisfies conditions (i)–(iii) of the lemma. We say that a node \( t' \in V(\Phi_1) \) \textit{appears} in \( (T^*, \beta^*) \) if there is a leaf \( t \) of \( T^* \) such that \( t \parallel t' \).

Let \( t_0 \) be the root of \( T^* \) and \( H := \tau^*(t_0) \). Let \( (H_0, \pi, R^1, \tau^1, \ldots, R^i, \tau^i) \) be a 3p-arrangement of \( H \) in a surface \( S \) of Euler genus \( r \) with \( q \) cuffs such that for all \( u \in N^1_x(t_0) \) there is an \( i \in [q] \) such that \( K[\omega^i(u)] \subseteq R^i \). Let \( c^1, \ldots, c^q \) be the cuffs of \( S \). Without loss of generality we can make the following assumptions:

\[ (C) \pi \text{ is the identity on } H_0 \setminus \bigcup_{j=1}^q \overline{c}_j. \]
(D) For all \( j \in [q] \) we have \( |\mathcal{P}^j| \geq 3 \).

To see that we can assume (D), just note that if \( |\mathcal{P}^j| < 3 \) then it follows from the 3-connectedness of \( J^* \) that \( V(R^j) \setminus \pi(\mathcal{P}^j) = \emptyset \) and there is no \( u \in N^R_+(t_0) \) with \( \sigma^*(u) \subseteq V(R^j) \) and \( \alpha^*(u) \neq \emptyset \). So we can just omit the vortex \( R^j \) and obtain an arrangement of \( H \) in a simpler surface. Now we can apply Claim \( 1 \) to complete the proof.

Furthermore, we may assume:

(E) The representativity of the embedding of \( H_0 \) in \( S \) is at least 5.

If the representativity of the embedding is less than 5, we can delete 4 vertices to obtain an arrangement in a simpler surface, and complete the proof by an application of Claim \( 1 \) and the Finite Extension Lemma for Ordered Completions \( 12.5.4 \).

For every \( j \in [q] \), let \( \mathcal{R}_j \) be the subgraph of \( J^* \) defined as follows:

\[
V(\mathcal{R}_j) := V(R^j) \cup \bigcup_{u \in N^R_+(t_0)} \alpha^*(u),
\]

\[
E(\mathcal{R}_j) := \{ uv \in E(J^*) \mid u, w \in V(\mathcal{R}_j), vw \notin \pi(E(H_0)) \}.
\]

Then \( (H_0, \pi, \mathcal{R}_1, \mathcal{R}_1, \ldots, \mathcal{R}_q, \mathcal{R}_q) \) is a local \( 3p \)-arrangement of \( J^* \) in \( S \). Note that the graph \( J^* \), the surface \( S \), and the arrangement \( (H_0, \pi, \mathcal{R}_1, \mathcal{R}_1, \ldots, \mathcal{R}_q, \mathcal{R}_q) \) satisfy Assumption \( 15.1.7(1) \).

Of course we can choose \( \mathcal{S} \) and \( D_1, \ldots, D_q \) to satisfy Assumption \( 15.1.7(6) \). Then by Lemma \( 15.1.10 \) there are disks \( \overline{D}_1, \ldots, \overline{D}_q \subseteq \mathcal{S} \) satisfying Assumption \( 15.3.1(1) \).

We let \( \mathcal{R} \) be the class consisting of all \( g \)-reductions of \( J^* \) by \( H_0 \)-normal simplifying curves \( g \subseteq \mathcal{S} \), all their subgraphs, and all planar graphs. Note that this is precisely the class \( \mathcal{R}(J^*) \) as defined before the statement of Lemma \( 16.4.8 \) With this \( \mathcal{R} \), assumption Assumption \( 15.3.1(2) \) is satisfied as well.

**Case 1:** \( J^* \) has a canonical simplifying patch.

Let \( k \) be the length of the tuple \( \mathcal{P} \) of variables of the formula \( \text{bridge-ord}(x, x', \mathcal{P}, z_1, z_2) \) of Lemma \( 15.4.23 \) and \( \ell := 2 + k + 4p \). Let \( Q = Q[v_1, v_2] \) be a minimal canonical simplifying patch of \( J^* \). Let \( (v_3, \ldots, v_{k+2}) \) be such that for every external \( Q \)-bridge \( B \) the binary relation \( \text{bridge-ord}[J^*, \mathcal{P}, z_1, z_2] \) is a linear order on the vertices of attachment of \( B \). Let \( W := \text{int-vert}[J^*, v_1, v_2, y] = V(I(Q)) \). Then \( J^* \setminus W \in \mathcal{R} \), and by the definition of \( \mathcal{R} \), this means that \( J^* \setminus W \) is either planar or a subgraph of the reduction of \( J^* \) by an \( H_0 \)-normal simplifying curve \( g \subseteq \mathcal{S} \). If \( J^* \setminus W \) is planar, we can construct an od-scheme \( \Lambda(\mathcal{P}) \) that defines an ordered treelike decomposition of \( J^* \) within \( (G, \mathcal{P}) \) similarly to the proof of Lemma \( 16.1.2 \).

In the following, we assume that \( J^* \setminus W \) is a subgraph of the reduction of \( J^* \) by an \( H_0 \)-normal simplifying curve \( g \subseteq \mathcal{S} \). We choose a set \( S \subseteq V(J^* \setminus W) \) of size \( |S| \leq 12p \) according to Lemma \( 15.1.6 \) applied to the local \( 3p \)-arrangement \( (H_0, \pi, \mathcal{R}_1, \mathcal{R}_1, \ldots, \mathcal{R}_q, \mathcal{R}_q) \) of \( J^* \), and let \( J' := J^* \setminus (W \cup S) \). Let \( A \) be a connected component of \( J' \), and let \( \Phi_A \) be the pre-decomposition induced by \( \Phi_1 \) on \( A \). We apply the construction described in Section \( 15.1.3 \) (used to prove Lemma \( 15.1.6 \) there) to \( (H_0, \pi, \mathcal{R}_1, \mathcal{R}_1, \ldots, \mathcal{R}_q, \mathcal{R}_q) \) and obtain a local \( 3p \)-arrangement of \( A \) into a simpler surface, and from this we obtain \( (r_A, q_A) \) a tree decomposition \( (T_A, \beta_A) \) of \( A \) with the same tree \( T_A := T^* \) such that for every node \( t \in V(T_A) \) one of the following three conditions is satisfied:
(F) \( t \) is a leaf of \( T_A \), and there is a \( t' \in V(\Phi_A) \) such that \( t \sim (T_A, \beta_A, \Phi_A) t' \).

(G) \( \tau_A(t) \) has a 3\( p \)-arrangement \((H_{t0}, \pi_t, R_t, r_t)\) in a disk \( D_t \) such that for all \( u \in \{t\} \cup N_T^+(t) \) it holds that \( K[\sigma_A(u)] \subseteq R_t \).

(H) \( t = t_0 \) is the root of \( T_A \) and \( \tau_A(t) \) has a 3\( p \)-arrangement \((H_{t0}, \pi_t, R^1_t, \pi^1_t, \ldots, R^q_t, \pi^q_t)\) in a surface \( S_A \) with \( \text{eg}(S_A) = r_A \) and \( \text{cf}(S_A) = q_A \) such that for all \( u \in N_T^+(t) \) there is an \( i \in [q] \) with \( K[\sigma_A(u)] \subseteq R_i^t \).

By Claim 1, we obtain constants \( d, d' \) and a parametrised od-scheme \( \Lambda'(\pi, y) \) such that for every connected component \( A \) of \( J' \) and every \( a \in V(A) \) the scheme \( \Lambda'(\pi, y) \) that defines an ordered completion of \( \Phi_A^{(d,d')} \) in \( A \) within \((J^*, \pi, a)\). (To obtain one od-scheme that works for all connected components, we use the fact that there are only finitely many \((r'^{r'}, q'^q) <^* (r, q)\).) Then, using the Finite Extension Lemma for Ordered Completions 12.5.4, we can construct a parametrised od-scheme \( \Lambda^2(\pi, y) \) such that for every connected component \( A \) of \( J^* \setminus W \) and every \( a \in V(A) \) the scheme \( \Lambda^2(\pi, y) \) defines an ordered completion of \( \Phi_A^{(d+12p,d'+12p)} \) in \( A \) within \((J^*, \pi, a)\), where again \( \Phi_A \) is the pre-decomposition induced by \( \Phi_1 \) on \( A \).

Let \( \Lambda^1 \) be an od-scheme that defines an ordered treelike decomposition on all graphs embeddable in a surface of Euler genus at most \( r \). Such an od-scheme exists by Theorem 9.4.1. Let \( \varphi(\pi, y) := \text{int-vert}(x_1, x_2, y) \) (of Lemma 15.4.20) and \( \psi(\pi, y_1, y_2) := \text{bridge-ord}(x_1, \ldots, x_{k+2}, y_1, y_2) \). By Lemma 16.4.7 obtain an od-scheme \( \Lambda''(\pi) \) that defines an ordered completion of \( (\Phi_1)^{(d+12p,d'+12p)} \) in \( J^* \) within \((G, \pi)\).

**Case 2:** \( J^* \) has no canonical simplifying patch.

Then Assumption 15.5.1 is satisfied. Arguing similarly to Case 1, we can show that \( J^* \) and \( \mathcal{R} = \mathcal{R}(J^*) \) satisfy condition (ii) of Lemma 16.4.8. Then an application of the lemma completes the proof.

Now the lemma follows by an application of the Q4C Completion Lemma 12.6.2.

M. Grohe, *Definable Graph Structure Theory*
Finally, we are ready to prove the main theorem of this book, the Definable Structure Theorem for Graphs with Excluded Minors.

Every class of graphs with excluded minors admits IFP-definable ordered treelike decompositions.

The proof heavily builds on Robertson and Seymour’s structure theory for graphs with excluded minors. We give an an outline of this theory in Section 17.1. The technical result that we need from this theory is the Local Structure Theorem 17.1.3. Intuitively, it states that all “highly connected regions” of a graph excluding some fixed graph as a minor are almost embeddable in some surface after removing a bounded number of vertices. One obstacle in understanding the precise technical statement is the definition of highly connected regions in terms of so-called “tangles”. However, once we have mastered the technical obstacles, we will see that the Local Structure Theorem is extremely powerful. In particular, it implies the “Global” Structure Theorem [17.1.1] already mentioned in the introduction to this book, stating that a graph excluding some fixed graph as a minor has a tree decomposition into torsos that are almost embeddable in some surface after removing a bounded number of vertices.

The construction used to prove the Global from the Local Structure Theorem is simple and generic (in the sense that it does not depend on the specific graph classes with excluded minors we are interested in here, but only on the description of the local structure in terms of tangles). We will see that our theory of completions of pre-decomposition nicely fits into this framework, and this will allow us to make the construction IFP-definable. Then we will only have to plug in our Almost Embeddable Completion Theorem [16.4.2] to deal with the local structure of graphs with excluded minors, and this will allow us to complete the proof of the Definable Structure Theorem.

17.1 The Structure of Graphs with Excluded Minors

Robertson and Seymour proved the following structure theorem for graphs with excluded minors.

Theorem 17.1.1 (Structure Theorem). Let $\mathcal{C}$ be a class of graphs with excluded minors. Then there are $\ell, p, q, r \in \mathbb{N}$ such that every graph $G \in \mathcal{C}$ has a tree decomposition $(T, \beta)$ with the following properties. For every node $t \in V(T)$ there is a $\zeta_t \subseteq \beta(t)$ of size $|\zeta_t| \leq \ell$ such...
that $\tau(t) \setminus \zeta_t \in \mathcal{A}\mathcal{E}_{p,q,r}$, that is, the graph $\tau(t) \setminus \zeta_t$ is $p$-almost embeddable in a surface of Euler genus $r$ with $q$ cuffs.

Intuitively, this is exactly what we need, because we know how to handle almost embeddable graphs, and we can lift ordered treelike decompositions from the torsos of a decomposition of a graph to the whole graph. The only thing that remains to do is define a treelike decomposition of our graphs whose torsos are almost embeddable. (Robertson and Seymour’s Structure Theorem tells us that such a decomposition exists.) Unfortunately, this approach turns out to be a dead end, because we do not know how to define such decompositions in \textsc{ifp}. To prove our Definable Structure Theorem, we take a different (though related) route through an even stronger theorem due to Robertson and Seymour, the Local Structure Theorem described next.

17.1.1 The Local Structure Theorem

Let $G$ be a graph. A separation of $G$ is a pair $(A, B)$ of subgraphs of $G$ such that $A \cup B = G$ and $E(A \cap B) = \emptyset$. The order of a separation $(A, B)$ is $|V(A) \cap V(B)|$. Let $k \in \mathbb{N}^+$. A tangle of order $k$ in a graph $G$ is a set $\mathcal{T}$ of separations of $G$ of order $< k$ such that the following axioms are satisfied:

\begin{enumerate}[(TA.1)]
    \item For every separation $(A, B)$ of $G$ of order $< k$, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$. \hfill (TA.1)

    \item For all $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ it holds that $A_1 \cup A_2 \cup A_3 \neq G$. \hfill (TA.2)

    \item For all $(A, B) \in \mathcal{T}$ it holds that $V(A) \neq V(G)$. \hfill (TA.3)
\end{enumerate}

Example 17.1.2. Let $k \in \mathbb{N}^+$. Let $G$ be a graph and $X \subseteq V(G)$ a clique in $G$ of size $|X| \geq 3k - 2$. Let $\mathcal{T}$ be the set of all separations $(A, B)$ of $G$ of order $< k$ with $X \subseteq V(B)$. Then $\mathcal{T}$ is a tangle of order $k$ in $G$.

To see this, note first that if $(A, B)$ is a separation of $G$ then either $X \subseteq V(A)$ or $X \subseteq V(B)$, because $X$ is a clique. This implies (TA.1). To verify (TA.2) let $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$. Then for $i = 1, 2, 3$ it holds that $X \subseteq V(B)$ and thus $|X \cap V(A)| \leq |V(A \cap B)| < k$. Thus $|X \cap V(A_1 \cup A_2 \cup A_3)| \leq 3k - 3$, and this implies that $V(G) \setminus V(A_1 \cup A_2 \cup A_3) \supseteq X \setminus V(A_1 \cup A_2 \cup A_3) \neq \emptyset$. This last inequality also implies (TA.3) if we let $(A_1, B_1) = (A_2, B_2) = (A_3, B_3)$.

We may view a tangle $\mathcal{T}$ as describing a “highly connected region” in a graph in the following sense. By (TA.1) the tangle gives a “direction” to each separation of $G$ of low order. Let us a view the graph $B$ in a separation $(A, B) \in \mathcal{T}$ as the “big part” of the separation. Then the “region” described by the tangle consists of all vertices that are in the big part of most tangle elements. It is “highly connected”, because the small parts of all low order separations do not belong to the region. Unfortunately, there is no easy way to make this intution precise. For example, the intersection of all big parts $B$ for $(A, B) \in \mathcal{T}$ may well be empty. Actually, the notion of a tangle is an attempt to make the intutive notion of highly connected region in a graph precise. For more background on tangles, I refer the reader to the survey [51].

In the following, let $\mathcal{T}$ be a tangle of order $k$ in $G$. For a set $\zeta \subseteq V(G)$ with $\ell := |\zeta| < k$ we let $\mathcal{T} \setminus \zeta$ be the set of all separations $(A', B')$ of $G \setminus \zeta$ of order $k - \ell$ such that there exists a separation $(A, B) \in \mathcal{T}$ with $\zeta \subseteq V(A \cap B)$ and $A \setminus \zeta = A'$ and $B \setminus \zeta = B'$. Robertson and Seymour [109] proved that $\mathcal{T} \setminus \zeta$ is a tangle of order $k - \ell$ in $G \setminus \zeta$.

M. Grohe, Definable Graph Structure Theory
Let \((S, \beta)\) be a star decomposition of \(G\). Then \((S, \beta)\) is \(\Sigma\)-central if there is no tip \(t\) of \(S\) and separation \((A, B) \in \Sigma\) such that \(V(B) \subseteq \beta(t)\). An \(\mathcal{AE}_{p,q,r}\)-star decomposition
\[
(S, \beta, s, S, H_0, \pi, R^1, \tau^1, \ldots, R^q, \tau^q)
\]
of \(G\) is \(\Sigma\)-central if \((S, \beta)\) is \(\Sigma\)-central.

The following theorem is essentially Theorem 3.1 of [111] rephrased in our terminology. In Appendix A we give the exact statement of Robertson and Seymour’s theorem and show how our version can be derived from it.

**Theorem 17.1.3 (Local Structure Theorem).** Let \(C\) be a class of graphs with excluded minors. Then there are integers \(k, \ell, p, q, r \in \mathbb{N}\) such that \(k > \ell\) and for every graph \(G \in C\) and every tangle \(T\) of order at least \(k\) in \(G\) there exists a set \(\zeta \subseteq V(G)\) of size \(|\zeta| \leq \ell\) and an \(\mathcal{AE}_{p,q,r}\)-star decomposition of \(G \setminus \zeta\) that is \(T \setminus \zeta\)-central.

At first sight, the “Global” Structure Theorem [17.1.1] looks like a much stronger and more useful statement; after all it describes the structure of the whole graph in a transparent way, whereas the Local Structure Theorem only describes the structure of certain regions of the graph determined by its tangles. However, it turns out that the Local Structure Theorem is the more powerful result. We shall derive the Structure Theorem from the Local Structure Theorem in Section [17.1.2] below; the construction will also be the basis of the proof of our Definable Structure Theorem [17.2.1] in Section [17.2].

Before we do this, let me make a few intuitive remarks on why we should expect the Local (and hence the Global) Structure Theorem to hold. Let \(C\) be a class of graphs with excluded minors. Let \(m\) be sufficiently large such that \(K_m\) is not a minor of any graph in \(C\). Let \(G \in C\). With some handwaving, we may read the Local Structure Theorem as saying that every highly connected region \(R\) in \(G\), determined by a tangle of sufficiently high order (that is, order at least \(k\) where the parameter \(k\) depends on the class \(C\)), is \(p\)-almost embeddable in a surface of Euler genus \(r\) with \(q\) cuffs after removing at most \(\ell\) vertices (for suitable parameters \(p, q, r, \ell\), again depending on \(C\)). So let us consider the region \(R\). For the sake of the argument, we may just think of \(R\) as a subgraph of \(G\). As it is highly connected, it must have high tree width, because low tree width implies the existence of a low order separation, which contradicts the high connectivity. Then by the Excluded Grid Theorem (Fact [14.2.1], \(R\) contains a large grid. Let us choose a grid \(H\) of maximum size in \(R\). We think of the grid \(H\) as being embedded in the plane in the natural way and the rest of the graph \(R\) as being attached to \(H\). The key observation is that the components of \(R \setminus H\) cannot be attached to \(H\) arbitrarily, but must “almost” respect the planar structure, because otherwise we can find a large complete graph as a minor. For example, if there are many mutually disjoint paths that are all internally disjoint from \(H\) such that all endvertices of these paths are far apart in the grid, then we can construct a \(K_m\) minor, which contradicts \(G\) being in \(C\). Figure 17.1 shows an example.

For a similar reason, there cannot be many mutually disjoint “crosses” on the grid that are mutually far apart. Here a “cross” consist of two disjoint paths that are internally disjoint from the grid and both have their endvertices in one hexagon of the grid, in such a way that they are interleaving (see Figure 17.2).

Arguments like this lead us to a structure where there are bounded number of vertices that are attached to the grid arbitrarily, these will go into the set \(\zeta\). Moreover, there may be a bounded number of places where we have many external paths with their endpoints
Chapter 17. Graphs with Excluded Minors

Figure 17.1. A grid with three external paths whose endvertices are far apart, and a $K_6$-minor in this grid

Figure 17.2. A grid with a cross

M. Grohe, *Definable Graph Structure Theory*
“close” together and many crosses close together and close to the endpoints of the paths. These places will be the vortices of the arrangement of $R$ in a surface. Note that so far the arrangement is planar, but we have not yet considered the components of $R \setminus H$ that are attached only to the perimeter of the grid. These components cannot be attached completely arbitrarly, because then we can construct a $K_m$-minor again. The components may create additional vortices. However, the most interesting case is that the components form many parallel paths connecting different segments of the boundary of the grid, possibly in a twisted way. Then the idea is to glue the corresponding boundary segments of the grid together and obtain an embedding in a surface of higher genus. (Recall that all surfaces can be constructed by gluing together boundary segments of polygons.) Thus we end up with an arrangement of $R$ in some surface with a bounded number of vortices. The surface must have bounded genus and the vortices must have bounded width, because otherwise we will obtain a $K_m$-minor again.

17.1.2 From Local to Global

Note that a each node $s$ of a tree decomposition $(T, \beta)$ of the form described in the Structure Theorem yields an $\A_\epsilon p,q,r$-star decomposition of $G \setminus \zeta_s$. The centre of this star decomposition is $s$ and the tips correspond to the connected components of $T \setminus \{s\}$. (There are some minor issues on how the bags of the centre and the tips of the star decomposition may intersect. Let us ignore these for the sake of this discussion.) Hence the Structure Theorem says that we can decompose a graph into pieces that have exactly the structure described in the Local Structure Theorem. What makes the Local Structure Theorem so useful, and in some sense stronger than the global theorem, is that it says that this structure is “pervasive” in the graph, that is, it appears in every highly connected region (determined by a tangle), and not only in the pieces of a fixed decomposition.

Let us now show how to derive the Structure Theorem from the Local Structure Theorem. Our proof is a variant of Robertson and Seymour’s proof from [109][111].

Let $G$ be a graph and $X \subseteq V(G)$. Recall that a set $Y \subseteq V(G)$ cracks $X$ if for every connected component $A$ of $G \setminus Y$ it holds that $|(V(A) \cap X) \cup Y| < |X|$. We call $X$ $k$-crackable, for some $k \in \mathbb{N}^+$, if there is a $Y \subseteq V(G)$ of size $|Y| < k$ that cracks $X$; otherwise, $X$ is $k$-uncrackable.

**Lemma 17.1.4.** Let $G$ be a graph and $X \subseteq V(G)$ a $k$-uncrackable set of size at least $3k - 2$. Let $\mathfrak{T}_X$ be the set of all separations $(A, B)$ of $G$ of order $< k$ such that

$$|(V(B) \cap X) \cup V(A \cap B)| \geq |X|.$$ 

Then $\mathfrak{T}_X$ is a tangle of order $k$ in $G$.

**Proof.** Let $(A, B)$ be a separation of $G$ of order $< k$ and $Y := V(A \cap B)$. Observe that if $(A, B) \in \mathfrak{T}_X$, that is, $|(V(B) \cap X) \cup Y| \geq |X|$, then

$$|V(A) \cap X| \leq k - 1,$$

(17.1.1)

because $|X \setminus V(A)| = |(V(B) \cap X) \setminus Y| = |(V(B) \cap X) \cup Y| - |Y| \geq |X| - (k - 1)$.

To prove (17.1.1) let $C$ be a connected component of $G \setminus Y$ such that $|(V(C) \cap X) \cup Y| \geq |X|$. Such a component exists because $X$ is $k$-uncrackable. If $C \subseteq B$ then $(A, B) \in \mathfrak{T}_X$. Otherwise, $C \subseteq A$ and thus $(B, A) \in \mathfrak{T}_X$.

Preliminary Version
To prove \([\text{TA.2]}\) let \((A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}_X\). Then by \([\text{17.1.1]}\)
\[|(V(A_1) \cup V(A_2) \cup V(A_3)) \cap X| \leq 3k - 3 < |X|
\]
and thus \(X \not\subseteq V(A_1 \cup A_2 \cup A_3)\), which implies \(V(A_1 \cup A_2 \cup A_3) \neq V(G)\).

To prove \([\text{TA.3]}\) let \((A, B) \in \mathcal{T}_X\). Then \(|V(A) \cap X| \leq k - 1\) and thus \(X \not\subseteq V(A)\), which implies \(V(A) \neq V(G)\).

\[\square\]

Proof of the Structure Theorem \([\text{17.1.1]}\) (assuming the Local Structure Theorem \([\text{17.1.3]}\)). Let \(\mathcal{C}\) be a class of graphs with excluded minors. We choose parameters \(k', \ell', p', q', r'\) according to the local structure theorem.

We let \(k := \max\{k', p' + \ell' + 1, 1 + \ell' + 1\}, \ell := 4k - 3, p := p', q := q', r := r'\). By induction on \(|G|\), we prove that for every graph \(G \in \mathcal{C}\) and every set \(X \subseteq V(G)\) of size \(|X| \leq 3k - 2\) there is a tree decomposition \((T, \beta)\) of \(G\) with the following two properties.

(i) \(X \subseteq \beta(s)\) for the root \(s\) of \(T\).

(ii) For every node \(t \in V(T)\), there is a set \(\zeta_t \subseteq \beta(t)\) of size \(|\zeta_t| \leq \ell\) such that \(\tau(t) \setminus \zeta_t \in \mathcal{AE}_{p', q', r'}\).

For the base step, just note that if \(|G| \leq \ell\) then the trivial one-node tree decomposition satisfies (i) and (ii) for every set \(X \subseteq V(G)\), because we can let \(\zeta_t := V(G)\) for the unique node \(t\) of the tree.

For the inductive step, let \(G \in \mathcal{C}\) such that \(|G| > \ell\) and \(X \subseteq V(G)\) such that \(|X| \leq 3k - 2\). As \(|G| \geq 3k - 2\), without loss of generality we may assume that \(|X| = 3k - 2\).

**Case 1:** \(X\) is \(k\)-crackable.

Let \(Y \subseteq V(G)\) be a set of size \(|Y| < k\) that cracks \(X\). Let \(A_1, \ldots, A_m\) be the connected components of \(G \setminus Y\). For every \(i \in [m]\), let \(G_i := G[V(A_i) \cup Y]\) and \(X_i := (X \cap V(A_i)) \cup Y\). Then \(|X_i| < |X| = 3k - 2\). This implies \(X \not\subseteq V(G_i)\) and thus \(|G_i| < |G|\). We apply the inductive hypothesis to \(G_i, X_i\) and obtain a tree decomposition \((T_i, \beta_i)\) of \(G_i\) that satisfies (i) and (ii).

We form a tree \(T\) by taking the disjoint union of \(T_1, \ldots, T_m\) and adding a fresh root node \(s\) and edges from \(s\) to the roots of \(T_1, \ldots, T_m\). We define \(\beta : V(T) \to 2^{|V(G)|}\) by \(\beta(s) := Y \cup X\) and \(\beta(t) := \beta_i(t)\) for all \(t \in V(T_i), i \in [m]\). It is easy to see that \((T, \beta)\) is a tree decomposition of \(G\) that satisfies (i) and (ii). (For (ii), we let \(\zeta_s := \beta(s) = Y \cup X\).)

**Case 2:** \(X\) is \(k\)-uncrackable.

Let \(\mathcal{T}_X\) be the tangle defined in Lemma \([\text{17.1.4]}\). As the order of \(\mathcal{T}_X\) is \(k \geq k'\), by the Local Structure Theorem \([\text{17.1.3]}\) we find a set \(\zeta \subseteq V(G)\) of size \(|\zeta| \leq \ell'\) and a \(\mathcal{T}_X \setminus \zeta\)-central \(\mathcal{AE}_{p', q', r'}\)-star decomposition \((S', \beta', s', \mathcal{S}, H_0, \pi, R^1, \tau^1, \ldots, R^d, \tau^d)\) of \(G' := G \setminus \zeta\).

Let \(t_1, \ldots, t_m\) be the tips of the star \(S'\). For every \(i \in [m]\), let \(G'_i := G'[\gamma(t_i)]\) and \(B_i := (V(G') \setminus \alpha'(t_i), E(G') \setminus E(G'_i))\). Then \((G'_i, B_i)\) is a separation of \(G'\) of order \(|\sigma'(t_i)| \leq \max\{p', 3\} < k - \ell'\). As our star decomposition is \(\mathcal{T}_X \setminus \zeta\)-central, we have \((G'_i, B_i) \in \mathcal{T}_X \setminus \zeta\), and thus \(|G'_i| = |G'|\) by \([\text{TA.3]}\). Let \(G_i := G[V(G'_i) \cup \zeta]\) and note that \(|G_i| < |G|\). Let \(X_i := (X \cap \alpha'(t_i)) \cup \sigma'(t_i) \cup \zeta\). As \((G'_i, B_i) \in \mathcal{T}_X\), we have
\[|(X \cap \alpha'(t_i))| + |\sigma'(t_i)| \geq |(V(B_i) \cap X) \cup V(G'_i \cap B_i)| \geq |X|,
\]
which implies \(|X \cap \alpha'(t_i)| = |X| - |X \setminus \alpha'(t_i)| \leq |\sigma'(t_i)|\) and thus
\[|X_i| \leq 2|\sigma'(t_i)| + \ell \leq 2 \max\{p', 3\} + \ell' \leq 3k - 2.
\]

M. Grohe, *Definable Graph Structure Theory*
We apply the inductive hypothesis to \( G_i, X_i \) and obtain a tree decomposition \((T_i, \beta_i)\) of \( G_i \) satisfying (i) and (ii).

We form a tree \( T \) by taking the disjoint union of the trees \( T_i \) for \( i \leq [m] \) and adding a fresh root node \( s \) and edges from \( s \) to the roots of all trees \( t_i \). We define \( \beta : V(T) \to 2^{V(G)} \) by letting \( \beta(s) := \beta'(s') \cup X \cup \zeta \) and \( \beta(t) := \beta(t) \) for all \( t \in V(T_i), i \leq [m] \). It is easy to see that \((T, \beta)\) satisfies (i) and (ii); the only thing that is not completely obvious is (ii) for the root \( s \). We let \( \zeta_s := X \cup \zeta \). Note that \(|\zeta_s| \leq 4k - 3 = \ell', \) because \(|\zeta| \leq \ell' \leq k - 1 \).

We have \( \tau(s) \setminus \zeta_s = \tau'(s) \in \mathcal{AE}_{p,q,r} \).

\[ \square \]

### 17.2 The Main Theorem

**Theorem 17.2.1 (Definable Structure Theorem).** Let \( C \) be a class of graphs with excluded minors. Then \( C \) admits IFP-definable ordered treelike decompositions.

**Corollary 17.2.2.** Let \( C \) be a class of graphs with excluded minors. Then \( C \) admits IFP+C-definable canonisation.

**Corollary 17.2.3.** Let \( C \) be a class of graphs with excluded minors. Then \( \text{IFP+C captures PTIME on} \ C \).

**Proof of the Definable Structure Theorem [17.2.1].** Without loss of generality we may assume that \( C \) is a minor ideal. By the Component Lifting Lemma [7.3.4] it suffices to show that the class \( C \cap \mathbb{Z} \) of connected graphs in \( C \) admits IFP-definable ordered treelike decompositions. Choose \( k', \ell', p', q', r' \in \mathbb{N} \) according to the Local Structure Theorem [17.1.3] and let \( p := \max\{p', 3\}, \ q := q', \ r := r' \). Choose \( d = d(p, q, r), \ d' = d'(p, q, r) \in \mathbb{N} \) according to the Almost Embeddable Completion Theorem [16.4.2]. Let \( k := \max\{k', p + \ell' + d' + 1, d\} \).

To explain the definition of the decomposition, we fix a connected graph \( G \in C \). Without loss of generality we may assume that \(|G| > 4k - 3 \), because on graphs of order at most \( 4k - 3 \) we can easily define an ordered treelike decomposition. In the first part of the proof, we define a decomposition \( \Delta^1 = (D^1, \sigma^1, \alpha^1) \) of \( G \) such that for all nodes \( t \in V(D^1) \) we will be able to define an ordered treelike decomposition on \( \tau^1(t) \) by using the Almost Embeddable Completion Theorem [16.4.2]. The construction of \( \Delta^1 \) is based on the proof of the Structure Theorem [17.1.1] from the Local Structure Theorem [17.1.3] that we gave in Section [17.1.2]. In the second part of the proof, we use a product construction (similar to the Decomposition Lifting Lemma [5.6.2]) to lift the ordered treelike decompositions of the torsos of \( \Delta^1 \) to an ordered treelike decomposition of \( G \).

**Part 1. Construction of \( \Delta^1 \).**
Let \( n := 7k - 4 \). The nodes of \( D^1 \) will be elements of \( V(G)^n \). For every tuple \( \overline{u} = (u_1, \ldots, u_n) \in V(G)^n \) we let

\[
\overline{u}_S := (u_1, \ldots, u_{3k-1}) \in V(G)^{3k-2}, \\
\overline{u}_X := (u_{3k}, \ldots, u_{6k-3}) \in V(G)^{3k-2}, \\
\overline{u}_Y := (u_{6k-2}, \ldots, u_{7k-4}) \in V(G)^{k-1}, \\
\overline{u}_Z := (u_{6k-2}, \ldots, u_{6k+\ell'-3}) \in V(G)^{\ell'}.
\]

Note that \( \overline{u}_Z \) is well-defined because \( \ell' \leq k - 1 \). We let

\[
S_{\overline{u}} := \begin{cases} 
\emptyset & \text{if } u_1 \in \overline{u}_S, \\
\overline{u}_S & \text{otherwise}
\end{cases}
\]
and \( X_\pi := \overline{u}_X \) and \( Y_\bar{u} := \overline{u}_Y \) and \( Z_\bar{u} := \overline{u}_Z \). Furthermore, we let \( A_\pi \) be the connected component of \( G \setminus S_\pi \) that contains \( u_1 \), and we let \( G_\pi := G[V(A_\pi) \cup S_\pi] \). We let \( U \) be the set of all \( \overline{u} \in V(G)^n \) satisfying the following conditions:

(A) \( S_\pi = N(A_\pi) \);

(B) \( S_\pi \subseteq X_\pi \subseteq V(G_\pi) \) and \( Y_\pi, Z_\pi \subseteq V(G_\pi) \);

(C) either \( X_\pi = V(G_\pi) \) or \(|X_\pi| = 3k - 2 \).

Note that, by (A), \( G_\pi \) is connected for all \( \overline{u} \in U \).

There will be four kinds of nodes in \( V(D^1) \), r-nodes (“root nodes”), b-nodes (“bounded nodes”), c-nodes (“crackable nodes”), and uc-nodes (“uncrackable nodes”).

(D) An \( r \)-node is a tuple \( \overline{u} = (u_1, \ldots, u_n) \in U \) with \( u_1 \in \overline{u}_S \) and \( \overline{u}_S = X_\pi \).

Let \( U_r \) be the set of all r-nodes. Observe that for every \( \overline{u} \in U_r \) we have \( A_\pi = G_\pi = G \), because \( G \) is connected.

(E) A \( b \)-node is a tuple \( \overline{u} \in U \) such that \( X_\pi = V(G_\pi) \).

Let \( U_b \) be the set of all b-nodes. Observe that \( U_r \cap U_b = \emptyset \), because \( |G| > 3k - 2 \).

(F) A \( c \)-node is a tuple \( \overline{u} \in U \setminus (U_r \cup U_b) \) such that \( Y_\pi \) cracks \( X_\pi \) in \( G_\pi \).

(G) A \( uc \)-node is a tuple \( \overline{u} \in U \setminus (U_r \cup U_b) \) such that \( X_\pi \) is \( k \)-uncrackable in \( G_\pi \).

Let \( U_c \) and \( U_{uc} \) be the sets of all c-nodes and uc-nodes, respectively.

For every subset \( V \subseteq U \) we define a pre-decomposition \( \Phi_V \) by letting \( V(\Phi) := V \) and \( \sigma^{\Phi_V}(\overline{u}) := S_\pi \) and \( \alpha^{\Phi_V}(\overline{u}) := V(A_\pi) \) for all \( \overline{u} \in V \). Furthermore, for every \( \overline{u} \in U \) we let \( \Phi_{V,\pi} \) be the restriction of \( \Phi_V \) to \( V(G_\pi) \), that is, the subdecomposition of \( \Phi_V \) with node set \( V(\Phi_{V,\pi}) := \{ \overline{u} \in V \mid \gamma^{\Phi_V}(\overline{u}) \subseteq V(G_\pi) \} \).

We need a second, more complicated pre-decomposition \( \Phi'_{V,\pi} \) for all \( V \subseteq U \) and \( \overline{u} \in U_{uc} \). To define it, we fix \( V \subseteq U \) and \( \overline{u} \in U_{uc} \). Let

\[ G'_\pi := G_\pi \setminus Z_\pi. \]

\( \Phi'_{V,\pi} \) will be a pre-decomposition of \( G'_\pi \). For every tuple \( \overline{w} = (w_1, \ldots, w_{p+1}) \in V(G'_\pi)^{p+1} \), let \( S_{\pi,\overline{w}} = \emptyset \) if \( w_1 \in \{w_2, \ldots, w_{p+1}\} \) and \( S_{\pi,\overline{w}} := \{w_2, \ldots, w_{p+1}\} \) otherwise, and let \( A_{\pi,\overline{w}} \) be the connected component of \( G'_\pi \setminus S_{\pi,\overline{w}} \) that contains \( w_1 \). Furthermore, we let \( G_{\pi,\overline{w}} := G'_\pi[V(A_{\pi,\overline{w}}) \cup S_{\pi,\overline{w}}] \). (Note that \( S_{\pi,\overline{w}} \) does not depend on \( \pi \), but \( A_{\pi,\overline{w}} \) and \( G_{\pi,\overline{w}} \) do.) We let \( W_\pi \) be the set of all \( \overline{w} = (w_1, \ldots, w_{p+1}) \in V(G'_\pi)^{p+1} \) with \( S_{\pi,\overline{w}} = N_{G'_\pi}(A_{\pi,\overline{w}}) = \partial G'_\pi(G_{\pi,\overline{w}}) \).

Letting \( \sigma_{\pi,\overline{w}}(\overline{w}) := S_{\pi,\overline{w}} \) and \( \alpha_{\pi,\overline{w}}(\overline{w}) := V(A_{\pi,\overline{w}}) \) for all \( \overline{w} \in W_\pi \), we obtain a tight pre-decomposition \( (W_\pi, \sigma_{\pi,\overline{w}}, \alpha_{\pi,\overline{w}}) \). We shall define \( \Phi'_{V,\pi} \) to be the maximal subdecomposition of \( (W_\pi, \sigma_{\pi,\overline{w}}, \alpha_{\pi,\overline{w}}) \) whose \((d, d')\)-derivation is “contained” in \( \Phi_{V,\pi} \), albeit for a technical notion of containment, specifically tailored to the situation at hand and different from Definition 12.1.7. We let \( V(\Phi'_{V,\pi}) \) be the set of all \( \overline{w} \in W_\pi \) such that for all pairs \((S, A)\), where \( S \subseteq V(G'_\pi) \) and \( A \) is a connected component of \( G'_\pi \setminus S \) such that \( |S| \leq d \) and \( S = N_{G'_\pi}(A) = \partial G'_\pi(V(A) \cup S) \) and

\[ |(V(A) \cup S) \setminus (V(A_{\pi,\overline{w}}) \cup S_{\pi,\overline{w}})| \leq d'. \]
there is a \( \pi \in V(\Phi'_{V,\pi}) \) with \( A_{\pi} = A \setminus X_{\pi} \). For every \( \varpi \in V(\Phi'_{V,\pi}) \) we let \( \sigma^{\Phi'_{V,\pi}}(\varpi) := S_{\pi,\varpi} \) and \( \alpha^{\Phi'_{V,\pi}}(\varpi) := V(A_{\pi,\varpi}) \).

**Claim 1.** There are pd-schemes \( \Psi(X,\pi) \) and \( \Psi'(X,\pi) \) (not depending on \( G \)) such that 
\[ \Psi[G, V, \pi] = \Phi_{V, \pi} \] for all \( V \subseteq U \) and \( \pi \in U \) and \( \Psi'[G, V, \pi] = \Phi'_{V, \pi} \) for all \( V \subseteq U \) and \( \pi \in U_{uc} \).

**Proof.** Straightforward.

**Claim 2.** There is an od-scheme \( \Lambda(X, \pi) \) with the following property: for all \( V \subseteq U \) and \( \pi \in U_{uc} \), if \( \Phi'_{V, \pi} \) has an \( \mathcal{A}\mathcal{E}_{p,q,r} \)-star completion in \( G'_{\pi} \), then
\[ \Delta_{V, \pi} := (D_{V, \pi}, \sigma_{V, \pi}, \alpha_{V, \pi}, \leq_{V, \pi}) := \Lambda[V, \pi] \]
is an ordered completion of \( \Phi_{V, \pi} \) in \( G_{\pi} \) that has a unique \( \leq_{D_{V, \pi}} \)-minimal node \( s \), which is a completion node with \( \beta_{V, \pi}(s) = X_{\pi} \cup Z_{\pi} \).

**Proof.** We apply (a parametrised version of) the Almost Embeddable Completion Theorem \( \text{[16.4.2]} \) to \( \Psi(X, \pi) \) and obtain an od-scheme \( \Lambda'(X, \pi) \) with the following property: for all \( V \subseteq U \) and \( \pi \in U_{uc} \), if \( \Phi'_{V, \pi} \) has an \( \mathcal{A}\mathcal{E}_{p,q,r} \)-star completion in \( G_{\pi} \), then \( \Delta_{V, \pi} := (D'_{V, \pi}, \sigma_{V, \pi}^{t'}, \alpha_{V, \pi}^{t'}, \leq_{V, \pi}^{t'}) := \Lambda'[V, \pi] \) is an ordered completion of \( (\Phi'_{V, \pi})^{(d', d)} \) in \( G_{\pi} \). Without loss of generality we may assume that \( \Delta_{V, \pi} \) is normal.

We define the o-decomposition \( \Delta_{V, \pi} := (D_{V, \pi}, \sigma_{V, \pi}, \alpha_{V, \pi}, \leq_{V, \pi}) \) of \( G_{\pi} \) as follows. We let \( D_{V, \pi} \) be the digraph obtained from \( D'_{V, \pi} \) by adding a new root node \( s \) and edges from \( s \) to all \( \leq_{D_{V, \pi}} \)-minimal nodes. We let \( \sigma_{V, \pi}(s) := \emptyset \) and \( \alpha_{V, \pi}(s) := V(G_{\pi}) \). For all completion nodes \( t \) of \( \Delta_{V, \pi} \) we let \( \sigma_{V, \pi}(t) := \sigma_{V, \pi}^{t'}(t) \cup X_{\pi} \cup Z_{\pi} \) and \( \alpha_{V, \pi}(t) := \alpha_{V, \pi}^{t'}(t) \setminus X_{\pi} \). The definition is slightly more complicated for ground nodes \( \Delta_{V, \pi} \). Let \( t \) be such a ground node and \( S := \sigma_{V, \pi}(t) \) and \( A := G_{\pi}'[\sigma_{V, \pi}(t)] \). As \( t \) is parallel to a node of \( (\Phi'_{V, \pi})^{(d', d)} \), we have \( S = N_{G_{\pi}'}(A) = \partial_{G_{\pi}'}(V(A) \cup S) \) and \( |S| \leq d \) and
\[ |(V(A) \cup S) \setminus (V(A_{\pi, \varpi}) \cup S_{\pi, \varpi})| \leq d' \]
for some \( \varpi \in V(\Phi'_{V, \pi}) \). Thus, by the definition of \( \Phi'_{V, \pi} \), there is a \( \overline{\pi} \in V \) such that \( A_{\overline{\pi}} = A \setminus X_{\overline{\pi}} \). Then \( S_{\overline{\pi}} = N_{G_{\pi}'}(A_{\overline{\pi}}) \subseteq S \cup X_{\pi} \cup Z_{\pi} \). We let \( \alpha_{V, \pi}(t) := \alpha_{V, \pi}^{t'}(t) \setminus X_{\pi} = V(A_{\pi}) = \alpha_{V, \pi}^{t'}(\overline{\pi}) \) and \( \sigma_{V, \pi}(t) := N_{G_{\pi}'}(\alpha_{V, \pi}(t)) = S_{\overline{\pi}} = \sigma_{V, \pi}^{t'}(\overline{\pi}) \). Note that the definition of \( \alpha_{V, \pi}(t) \) and \( \sigma_{V, \pi}(t) \) only depends on \( \sigma_{V, \pi}(t) \), but not on the choice of \( \overline{\pi} \in V \).

It is straightforward to verify that \( (D_{V, \pi}, \sigma_{V, \pi}, \alpha_{V, \pi}, \leq_{V, \pi}) \) is a treelike decomposition. Furthermore, it is a completion of the pre-decomposition \( \Phi_{V, \pi} \) such that all ground nodes of \( \Delta_{V, \pi} \) are also ground nodes of \( (D_{V, \pi}, \sigma_{V, \pi}, \alpha_{V, \pi}, \leq_{V, \pi}) \).

To turn it into an ordered completion, observe that \( \beta_{V, \pi}(s) = X_{\pi} \cup Z_{\pi} \) and \( \beta_{V, \pi}(t) \subseteq \beta_{V, \pi}^{t'}(t) \cup X_{\pi} \cup Z_{\pi} \) for all completion nodes \( t \). Let \( \leq^{XZ} \) be the linear order on \( X_{\pi} \cup Z_{\pi} \subseteq \overline{u} \) induced by the indices of the first appearance of the elements in the tuple \( \overline{u} \). We let \( (\leq^{XZ})_{s} := \leq^{XZ} \), and for each completion node \( t \) we define \( (\leq^{XZ})_{t} \) by \( (\leq^{XZ})_{t} \) \( w \) if and only if either \( v, w \in X_{\pi} \cup Z_{\pi} \) and \( v \leq^{XZ} w \) or \( v \in X_{\pi} \cup Z_{\pi} \) and \( w \in \beta_{V, \pi}^{t'}(t) \setminus (X_{\pi} \cup Z_{\pi}) \) or \( v, w \in \beta_{V, \pi}^{t'}(t) \setminus (X_{\pi} \cup Z_{\pi}) \).

It is straightforward to define \( \Delta_{V, \pi} := (D_{V, \pi}, \sigma_{V, \pi}, \alpha_{V, \pi}, \leq_{V, \pi}) \) in \( \text{IFP} \).

By induction on \( i \in \mathbb{N} \), we define sets \( U^{i} \subseteq U \). We let \( U^{0} := \emptyset \). For the inductive step, let \( i \in \mathbb{N} \) and \( \pi \in U \).

- If \( \pi \in U^{i} \), then \( \pi \in U^{i+1} \) if for every connected component \( A \) of \( G \setminus X_{\pi} \) there is a \( \sigma_{\pi} \in U^{i} \) such that \( A_{\pi} = A \).
• If \( \overline{u} \in U_b \), then \( \overline{u} \in U^{i+1} \).
• If \( \overline{u} \in U_c \), then \( \overline{u} \in U^{i+1} \) if for every connected component \( A \) of \( G_{\overline{u}} \) there is \( \overline{u} \in U^i \) with \( A_{\overline{u}} = A \).
• If \( \overline{u} \in U_{uc} \), then \( \overline{u} \in U^{i+1} \) if the o-decomposition \( \Delta_{U^i,\overline{u}} := A[G, U^i, \overline{u}] \) of Claim 2 is an ordered completion of \( \Phi_{U^i,\overline{u}} \) in \( G_{\overline{u}} \).
• Otherwise, that is, if \( \overline{u} \in U \setminus (U_r \cup U_b \cup U_c \cup U_{uc}) \), then \( \overline{u} \notin U^{i+1} \).

Observe that

\[
U^0 \subseteq U^1 \subseteq U^2 \subseteq \ldots \subseteq U_r \cup U_b \cup U_c \cup U_{uc}.
\]

We let \( U^\infty := \bigcup_{i \in \mathbb{N}} U^i \). Note that for all \( i \geq |G|^n \geq |U| \) we have \( U^i = U^{i+1} \) and thus \( U^i = U^\infty \). For every \( \overline{u} \in U^\infty \), we define the rank \( \text{rk}(\overline{u}) \) to be the unique \( i \in \mathbb{N}^+ \) such that \( \overline{u} \in U^i \setminus U^{i-1} \).

**Claim 3.** For every connected subgraph \( A \subseteq G \) with \( |N^G(A)| \leq 3k - 2 \) there is a \( \overline{u} \in U^\infty \) such that \( A_{\overline{u}} = A \).

**Proof.** The proof is by induction on \( |A| + |N^G(A)| \). For the base step, note that if \( |A| + |N^G(A)| \leq 3k - 2 \) then there is a node \( \overline{u} \in U_b \subseteq U^\infty \) such that \( A_{\overline{u}} = A \) and \( X_{\overline{u}} = V(A) \cup N^G(A) \). For the inductive step, let \( A \) be a connected subgraph of \( G \). Let \( S := N^G(A) \) and \( H := G[V(A) \cup S] \). Assume that \( |S| \leq 3k - 2 \) and \( |H| > 3k - 2 \), and let \( X \subseteq V(H) \) such that \( |X| = 3k - 2 \) and \( S \subseteq X \).

**Case 1:** \( X \) is \( k \)-crackable in \( H \).

Let \( Y \subseteq V(H) \) a set of size \( < k \) that cracks \( X \). There is a node \( \overline{u} \in U_c \) such that \( S_{\overline{u}} = S \) and \( A_{\overline{u}} = A = X \times Y \). To see that \( \overline{u} \in U^\infty \), let \( A' \) be a connected component of \( H \setminus (X \cup Y) \). We have to prove that there is a node \( \overline{v} \in U^\infty \) such that \( A_{\overline{v}} = A' \). Let \( A'' \) be the connected component of \( H \setminus Y \) with \( A' \subseteq A'' \). Then \( N^H(A') \subseteq (X \cap A'') \cup Y \) and thus \( |N^H(A')| < |X| = 3k - 2 \), because \( Y \) cracks \( X \). Note that \( N^G(A') = N^H(A') \), because \( \partial^G(H) \subseteq X \) and \( X \cap V(A') = \emptyset \). Thus \( |N^G(A')| < 3k - 2 \). Moreover, \( |A'| + |N^G(A')| < |A| + |N^G(A)| \), because \( V(A') \cap N^G(A') = V(A') \cup N^H(A') \subseteq V(H) = V(A) \cup N^G(A) \) and \( X \not\subseteq N^G(A') \). Thus we can apply the induction hypothesis to \( A' \) and obtain a node \( \overline{u} \in U^\infty \) with \( A_{\overline{u}} = A' \).

**Case 2:** \( X \) is \( k \)-un crackable in \( H \).

Let \( \Sigma_X \) be the tangle of \( H \) defined in Lemma 17.1. As by the Local Structure Theorem, there is a set \( Z \subseteq V(G) \) of size \( |Z| \leq p' \) and a \( \Sigma_X \setminus Z \)-central \( \mathcal{A}_\mathcal{E}_{p',q',r'} \)-star decomposition of \( H \setminus Z \). Moreover, there is an \( u \)-node \( \overline{u} \) such that \( A_{\overline{u}} = A \) (and hence \( S_{\overline{u}} = S \) and \( G_{\overline{u}} = H \)) and \( X_{\overline{u}} = X \) and \( Z_{\overline{u}} = Z \). Let \( H' := H \setminus Z \). Let \( (S^*, \beta^*, s^*, S, H_0, \pi, R^1, \tau^1, \ldots, R^n, \tau^n) \) be a \( \Sigma_X \setminus Z \)-central \( \mathcal{A}_\mathcal{E}_{p',q',r'} \)-star decomposition of \( H' \). As \( p' \leq p \) and \( q' = q \), \( r' = r \), it is also a \( \mathcal{A}_\mathcal{E}_{p,q,r} \)-star decomposition.

Without loss of generality, we may assume that the decomposition is tight. To see this, suppose that \( t \) is a tip of \( S^* \) such that either \( \alpha^*(t) \) is not connected or \( \sigma^*(t) \neq N^H(t) \) or \( \sigma^*(t) \neq \partial^H(\gamma^*(t)) \). Then we modify the decomposition as follows. Let \( A^*_1, \ldots, A^*_m \) be the connected components of \( H'[\alpha^*(t)] \). For each \( i \in [m] \), let \( H'_i := H'[V(A_i) \cup N^H(A'_i)] \) and \( S'_i := \partial^H(H'_i) \). Then \( S'_i = N^H(V(H'_i) \setminus S'_i) \), and \( H'_i \setminus S'_i \) is connected. Replace \( t \) by fresh tips \( t_1, \ldots, t_m \) and define the separator of \( t'_i \) to be \( S'_i \) and the component to be \( V(H'_i) \setminus S'_i \). If we do this for all tips \( t \), we obtain a tight star decomposition where the
17.2. The Main Theorem

However, we have

Subclaim 3b. $|V(H') \setminus \gamma^*(t_i)| > d'$.

Proof. Consider a pair $(S', A')$, where $S' \subseteq V(H')$ and $A'$ is a connected component of $H' \setminus S'$ such that $|S'| \leq d$ and $S' = N^{H'}(A') = \partial^{H'}(V(A') \cup S')$ and

$$|(V(A') \cup S') \setminus \gamma^*(t_i)| \leq d'. \tag{17.2.1}$$

We have to prove that there is a $\pi \in V(\Phi_{U, \pi})$ with $A_{\pi} = A' \setminus X$.

Let $A'' := A' \setminus X$ and $S'' := N^G(A'')$ and $H'' := G[V(A'') \cup S'']$. We have

$$S'' \subseteq S' \cup (X \cap V(A')) \cup Z. \tag{17.2.2}$$

To bound the size of this set, we note that

$$X \cap V(A') \subseteq (V(A') \setminus \gamma^*(t_i)) \cup \sigma^*(t_i) \cup (X \cap \alpha^*(t_i)). \tag{17.2.3}$$

The separation $(C', B')$ with $B' = H \setminus \alpha^*(t_i)$ and $C' = (\gamma^*(t_i) \cup Z, E(H) \setminus E(B'))$ is in $\mathfrak{S}_X$, because the star decomposition $(S^*, \beta^*)$ is $\mathfrak{S}_X \setminus Z$-central. By the definition of the tangle $\mathfrak{S}_X$, this implies $|(X \cap \alpha^*(t_i)| \leq k - 1$. Thus since $|S'| \leq d$ and by (17.2.1)–(17.2.3),

$$|S''| \leq d + d' + p' + k - 1 + \ell' \leq 3k - 2.$$

Moreover, we have $|H''| < |H|$. To see this, note first that $H'' \subseteq H$. As $H' \subseteq H$ as well, it thus suffices to prove that $H' \setminus H'' \neq \emptyset$. Suppose for contradiction that $V(H') \subseteq V(H'')$. Then $V(H') = V(H') \cap V(H'') \subseteq V(A') \cup S'$. However, $|(V(A') \cup S') \setminus \gamma^*(t_i)| \leq d'$ by (17.2.1), whereas $|V(H') \setminus \gamma^*(t_i)| > d'$ by Subclaim 3a. This is a contradiction.

Hence we can apply the induction hypothesis (of the inductive proof of Claim 3) to $A'', S''$ and indeed find a $\pi \in V(\Phi_{U, \pi})$ with $A_{\pi} = A''$. This completes the proof of Subclaim 3b.
Now we apply Claim 2 with $V := U^\infty$ and see that the $\omega$-decomposition $\Delta_{U^\infty, \pi}$ is an ordered completion of $\Phi_{U^\infty, \pi}$. As $U^\infty = U^i$ for $i := |G|^\pi$, this implies that $\pi \in U^{i+1} = U^\infty$.

We are now ready to define the decomposition $\Delta^1 = (D^1, \sigma^1, \alpha^1)$. We let $V(D^1) := U^\infty$. We let $E(D^1)$ be the set of all pairs $(\pi, \overline{\pi}) \in V(D^1)^2$ such that

- either $\pi \in U_r$ and $A_\pi$ is a connected component of $G \setminus X_\pi$,
- or $\pi \in U_c$ and $A_\pi$ is a connected component of $G \setminus (X_\pi \cup Y_\pi)$,
- or $\pi \in U_{uc}$ with $\text{rk}(\pi) = i + 1$ and there is a ground leaf $t$ of the ordered completion $\Delta_{U^1, \pi}$ of $\Phi_{U^1, \pi}$ such that $\sigma_{U^1, \pi}(t) = S_\pi$ and $\alpha_{U^1, \pi}(t) = V(A_\pi)$.

Note that b-nodes have no children. For all $\pi \in V(D^1)$, we let $\sigma^1(\pi) := S_\pi$ and $\alpha^1(\pi) := V(A_\pi)$.

Claim 4. There is d-scheme $\Lambda^1$ such that $\Delta^1 = \Lambda^1[G]$.

Proof. Straightforward.

Claim 5. $\Delta^1$ satisfies axioms [TL.1]–[TL.3] and [TL.5].

Proof. It is straightforward to verify [TL.1]–[TL.3], and [TL.5] follows from Claim 3.

However, the decomposition $\Delta^1$ may violate [TL.4] because the children of a uc-node are the ground leaves of a completion, and they may intersect in arbitrary ways if they are not children of the same node of this completion.

Claim 6. Let $\pi \in V(D^1)$. Then $X_\pi \subseteq \beta^1(\pi)$. Furthermore, if $\pi \in U_r \cup U_b \cup U_c$, then $\beta^1(\pi) \subseteq \overline{\pi}$.

Proof. To verify $X_\pi \subseteq \beta^1(\pi)$, we just need to check that for all $\pi \in N^{D^1}(\pi)$ we have $X_\pi \cap \alpha^1(\pi) = \emptyset$; this is is straightforward.

To prove the second assertion, we make a case distinction. If $\pi \in U_r$, then $\beta^1(\pi) = X_\pi$, because for every connected component $A$ of $G \setminus X_\pi$ there is a $\pi \in N^{D^1}(\pi)$ with $\alpha^1(\pi) = V(A)$. If $\pi \in U_b$, then $\beta^1(\pi) = \gamma^1(\pi) = X_\pi$. If $\pi \in U_c$, then $\beta^1(\pi) = X_\pi \cup Y_\pi$, because for every connected component $A$ of $G \setminus (X_\pi \cup Y_\pi)$ there is a $\pi \in N^{D^1}(\pi)$ with $\alpha^1(\pi) = V(A)$.

Part 2. Lifting the ordered decompositions.

To define an (ordered) treelike decomposition of $G$ we use a product construction similar to the proofs of various lifting lemmas earlier in this book.

We can easily turn $\Lambda(X, \pi)$ into a new od-scheme $\Lambda^2(\pi)$ such that for all $\pi \in U_{uc} \cap U^\infty$ of rank $i := \text{rk}(\pi)$ we have $\Lambda^2[G, \pi] = \Lambda[G, U^{i-1}, \pi]$. In the following, we let $\Delta^2_{\pi} := (D^2_{\pi}, \sigma^2_{\pi}, \alpha^2_{\pi}) := \Lambda^2[G, \pi]$. Recall (from the definition of $U^i$) that $\Delta^2_{\pi}$ is an ordered completion of $\Phi_{U^{i-1}, \pi}$.

To simplify the notation, in the following we denote nodes of $D^1$ by $t, t', t_1$ et cetera and nodes of $\Delta^2_{\pi}$ by $u, u', u_1$ et cetera, and sometimes $x$. Nodes of our new decomposition will be pairs $tu$, which we write without parenthesis and commas. As $\Delta^2_{\pi}$ is only defined for $t \in U_{uc}$, for all $t \in U_r \cup U_b \cup U_c$ we create a fresh node $u'$. Recall (from Claim 2) that for all $t \in U_{uc}$ the decomposition $\Delta^2_{\pi}$ has a unique minimal node, which is a completion node. We denote this minimal node by $u'$.

We shall define an ordered treelike decomposition $\Delta = (D, \sigma, \alpha, \leq)$. 

M. Grohe, Definable Graph Structure Theory
17.2. The Main Theorem

(H) We let \( V(D) \) be the set of all pairs \( tu \) such that \( t \in V(D^1) \) and
- either \( t \in U_r \cup U_b \cup U_c \) and \( u = u^t \),
- or \( t \in U_{uc} \) and \( u \) is a completion node of \( \Delta^2 \).

(I) We let \( E(D) \) be the set of all pairs \( (tu, t'u') \in V(D)^2 \) such that
- either \( t = t' \) and \( t \in U_{uc} \) and \( uu' \in E(D^2_1) \),
- or \( tt' \in E(D^1) \) and \( t, t' \in U_r \cup U_b \cup U_c \),
- or \( tt' \in E(D^1) \) and \( t \in U_r \cup U_b \cup U_c \) and \( t' \in U_{uc} \) and \( u' = u^t \),
- or \( tt' \in E(D^1) \) and \( t \in U_{uc} \) and \( u' = u^{t'} \) and \( u \in V(D^3_1) \) such that \( u \) has a child \( x \in N^{D^3_1}(u) \) that is a ground node of \( \Delta^2 \) with \( \sigma^2(x) = \sigma^1(t') \) and \( \alpha^2(x) = \alpha^1(t') \).

(J) For all \( tu \in V(D) \),
- if \( t \in U_r \cup U_b \cup U_c \) we let \( \sigma(tu) := \sigma^1(t) \) and \( \alpha(tu) := \alpha^1(t) \),
- if \( t \in U_{uc} \) and \( u = u^t \) we let \( \sigma(tu) := \sigma^1(t) \) and \( \alpha(tu) := \alpha^1(t) \),
- if \( t \in U_{uc} \) and \( u \neq u^t \) we let \( \sigma(tu) := \sigma^2(u) \) and \( \alpha(tu) := \alpha^2(u) \).

Claim 7. \((D, \sigma, \alpha)\) is a treelike decomposition of \( G \).

Proof. \([\text{TL.1}],[\text{TL.3}]\) and \([\text{TL.5}]\) follow easily from the definitions and the fact that \( \Delta^1 \) and \( \Delta^2 \) satisfy the corresponding axioms. To see this, note that for all \( t \in U_{uc} \cap V(D^1) \) we have \( \gamma(tu) = \gamma^1(t) = V(G_t) = \gamma^2(tu) = \alpha^2(tu) \) and \( \sigma(tu) = \sigma^1(t) \subseteq X_t \subseteq \beta^2(tu) \) by Claim 6.

It remains to verify \([\text{TL.4}]\). Let \( tu \in V(D) \) and \( t^1u^1, t^2u^2 \in N^D(tu) \). Then \( t \not\in U_b \).

Case 1: \( t \in U_r \).

Then for \( i = 1, 2 \), the graph \( A_{ti} = G[\alpha^1(t^i)] = G[\alpha(t^i u^i)] \) is a connected component of \( G \setminus X_t \), and \( \sigma(t^i u^i) = \sigma^1(t^i) = N^G(A_{ti}) \). This implies \( t^1u^1 \parallel t^2u^2 \) or \( t^1u^1 \perp t^2u^2 \).

Case 2: \( t \in U_c \).

Then for \( i = 1, 2 \), the graph \( A_{ti} = G[\alpha^1(t^i)] = G[\alpha(t^i u^i)] \) is a connected component of \( G \setminus X_t \cup Y_t \), and \( \sigma(t^i u^i) = \sigma^1(t^i) = N^G(A_{ti}) \). Again, this implies \( t^1u^1 \parallel t^2u^2 \) or \( t^1u^1 \perp t^2u^2 \).

Case 3: \( t \in U_{uc} \).

Then it follows from \([\text{TL.4}]\) for \( \Delta^2 \) that either \( t^1u^1 \parallel t^2u^2 \) or \( t^1u^1 \perp t^2u^2 \).

Claim 8. Let \( tu \in V(D) \).

1. If \( t \in U_r \cup U_b \cup U_c \), then \( \beta(tu) = \beta^1(t) \).
2. If \( t \in U_{uc} \), then \( \beta(tu) = \beta^2(u) \).

Proof. Assertion (1) follows immediately from the definitions. To prove (2), suppose that \( t \in U_{uc} \). Then \( \gamma(tu) = \gamma^2(u) \). (We have already noted in the proof of Claim 7 that this also holds if \( u = u^t \).) Moreover, for all \( t' u' \in N^D(tu) \) there is an \( x \in N^{D^2_1}(u) \) such that \( \alpha(t' u') = \alpha^2(x) \). Conversely, we need to prove that for all \( x \in N^{D^2_1}(u) \) there is a \( t' u' \in N^D(tu) \) such that \( \alpha(t' u') = \alpha^2(x) \). So let \( x \in N^{D^2_1}(u) \). If \( x \) is a completion node of \( \Delta^2 \), then
$tx \in N^D(tu)$ with $\alpha(tx) = \alpha_t^2(x)$. So suppose that $x$ is a ground node, and let $i := \text{rk}(t)$. As $\Delta_t^2$ is an ordered completion of $\Phi_{U^{i-1},t}$, there is a $t' \in \Phi_{U^{i-1},t}$ such that $x \parallel^{\Delta_t^2,\Phi_{U^{i-1},t}} t'$, that is, $\sigma_t^2(x) = \sigma^{\Phi_{U^{i-1},t}}(t') = S_{t'} = \sigma^1(t')$ and $\alpha_t^2(x) = \alpha^{\Phi_{U^{i-1},t}}(t') = V(A_{t'}) = \alpha^1(t')$. As $t' \in U^{i-1} \subseteq V(D^1)$, we have $t'u \in N^D(tu)$ and $\alpha(t'u') = \alpha(t') = \alpha_t^2(x)$.

It remains to define the linear orders $\leq_{tu}$. Let $tu \in V(D)$. If $t \in U_{uc}$, we simply let $\leq_{tu} := (\leq_t^2)_u$, which is a linear order of $\beta(tu) = \beta_t^2(u)$, because $u$ is a completion node of the ordered completion $\Delta_t^2$. Suppose that $t \in U_r \cup U_b \cup U_c$ and let $\pi \in V(G)^n$ such that $t = \pi$. By Claims 8 and 6 we have $\beta(tu) \subseteq \widetilde{u}$, and we can order the vertices of $\beta(tu)$ by the indices of their first occurrence in the tuple $\pi$.

Then $\Delta = (D, \sigma, \alpha, t)$ is the desired ordered treelike decomposition of $G$. It is easy to formalise the definition of $\Delta$ in IFP. This completes the proof of our main theorem. 

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_Mein Glücksgefühl wird daher ein jeder verstehen, der, wie ich, am liebsten solchen Beschäftigungen nachgeht, deren Konzeption bereits von vornherein verrät, daß sie zu nichts führen können, deren Ausübung daher reiner, seliger Selbstzweck ist._ [65]
In this final chapter, we collect various small results, related mainly to definable canonisation and the Canonisation Theorem 7.4.1. These results may be interesting in other contexts (those presented in Section 18.3 have already proved to be, see [54]). But as the results were not needed to prove our main theorem, we have delayed them to this last chapter. For the same reason, we only sketch some of the proofs.

We close the chapter with a section where we discuss a few open problems and directions for future research.

18.1 From Graphs to Relational Structures

Recall (from Section 2.2.6) that a relational structure $A$ consists of a finite vertex set $V(A)$ and a relation $R(A)$ on $V(A)$ for each relation symbol $R$ in the vocabulary of $A$. In this section, we show how our main results can be generalised from graphs to arbitrary relational structures. To do this, we associate a graph, the so-called graph Gaifman graph, with each structure.

The Gaifman graph of a $\tau$-structure $A$ is the graph $G_A$ with vertex set $V(G_A) := V(A)$ and edge relation $E(G_A) := \{vw \mid v \neq w, 3R \in \tau, \exists \bar{v} \in R(A) : v, w \in \bar{v}\}$. For every class $C$ of graphs, $S(C)$ denotes the class of all structures whose Gaifman graph is in $C$.

Structures inherit graph theoretic notions and properties from their Gaifman graphs. In particular, a structure $A$ is connected if $G_A$ is connected. The connected components of a structure $A$ are the induced substructures of $A$ whose Gaifman graphs are the connected components of $G_A$. An (ordered) treelike decomposition of a structure $A$ is an (ordered) treelike decomposition of its Gaifman graph. However, when defining a decomposition in a structure we may use the relations of the structure and not just the edge relation of the Gaifman graph. Formally, an (o)d-scheme of vocabulary $\tau$ is defined like an (o)d-scheme, except that its formulas are in $\text{IFP}[\tau]$. The following example shows that this may make a difference.

Example 18.1.1. Consider the class $\mathcal{O}[[\{\leq\}]$ of “linear orders”, that is, $\{\leq\}$ structures $(V(A), \leq^A)$ where $\leq^A$ is a linear order of $V(A)$. Trivially, there is an od-scheme of vocabulary $\{\leq\}$ that defines ordered treelike decompositions on all structures in $\mathcal{O}[[\{\leq\}]$.

However, for each structure $A \in \mathcal{O}[[\{\leq\}]$ its Gaifman graph $G_A$ is the complete graph $K[V(A)]$. As the class of complete graphs does not admit IFP-definable ordered treelike
decompositions, there is no od-scheme of vocabulary \(\{E\}\) that defines ordered treelike decompositions on all Gaifman graphs of structures in \(\mathcal{O}[\{\leq\}]\).

We say that a class \(\mathcal{C}\) of structures (possibly of different vocabularies) \emph{admits \textsc{ifp}-definable ordered treelike decompositions} if for every vocabulary \(\tau\) there is an od-scheme \(\Lambda\) of vocabulary \(\tau\) such that \(\mathcal{C}[\tau] \subseteq \mathcal{O}T_{\Lambda}\).

**Lemma 18.1.2.** Let \(\mathcal{C}\) be a class of graphs. Then \(\mathcal{C}\) that admits \textsc{ifp}-definable ordered treelike decompositions if and only if \(S(\mathcal{C})\) admits \textsc{ifp}-definable ordered treelike decompositions.

**Proof.** To prove the forward direction, let 

\[
\Lambda = (\lambda_V(\bar{x}), \lambda_E(\bar{x}, \bar{x}'), \lambda_\sigma(\bar{x}, y), \lambda_\alpha(\bar{x}, y), \lambda_\leq(\bar{x}, y_1, y_2))
\]

be an od-scheme of vocabulary \(\{E\}\) such that \(\mathcal{C} \subseteq \mathcal{O}T_{\Lambda}\). Observe that there is an \textsc{ifp}[\tau, \{E\}]-transduction \(\Theta\) such that \(\Theta[A] = G_A\) for each \(\tau\)-structure \(A\). Then it follows from the Transduction Lemma (Fact 2.4.6) that

\[
\Lambda^{-\Theta} := (\lambda_V(\bar{x}), \lambda_E^\Theta(\bar{x}, \bar{x}'), \lambda_\sigma^\Theta(\bar{x}, y), \lambda_\alpha^\Theta(\bar{x}, y), \lambda_\leq^\Theta(\bar{x}, y_1, y_2))
\]

is an od-scheme of vocabulary \(\tau\) such that \(S(\mathcal{C})[\tau] \subseteq \mathcal{O}T_{\Lambda^{-\Theta}}\).

The backward direction is trivial, because \(\mathcal{C} \subseteq S(\mathcal{C})\).

**Corollary 18.1.3.** Let \(\mathcal{C}\) be a class of graphs with excluded minors. Then \(S(\mathcal{C})\) admits \textsc{ifp}-definable ordered treelike decompositions.

We can easily generalise the Canonisation Theorem [7.4.1] from graphs to arbitrary structures.

**Theorem 18.1.4 (Canonisation Theorem for Relational Structures).** Let \(\mathcal{C}\) be a class of structures that admits \textsc{ifp}-definable ordered treelike decompositions. Then \(\mathcal{C}\) admits \textsc{ifp+c}-definable canonisation.

**Proof.** The proof of the Canonisation Theorem [7.4.1] goes through with only minor adaptions.

**Corollary 18.1.5.** Let \(\mathcal{C}\) be a class of graphs with excluded minors. Then \(S(\mathcal{C})\) admits \textsc{ifp+c}-definable canonisation.

**Corollary 18.1.6.** Let \(\mathcal{C}\) be a class of graphs with excluded minors. Then \(\textsc{ifp+c}\) captures \textsc{ptime} on \(S(\mathcal{C})\).

It is interesting to note that the analogue of Lemma 18.1.2 for \textsc{ifp+c}-definable canonisation fails, as the following example shows.

**Example 18.1.7.** We have seen in Example 3.3.4 that the class \(\mathcal{K}\) of complete graphs admits \textsc{ifp+c}-definable canonisation. We shall prove that the class \(S(\mathcal{K})\) does not admit \textsc{ifp+c}-definable canonisation.

Let \(\bar{E}\) be a binary relation symbol and \(\tau = \{E, \bar{E}\}\). For every graph \(G\), let \(A_G\) be the \(\tau\)-expansion of \(G\) with \(\bar{E}(A_G) := V(G)^2 \setminus E(G)\). Then the Gaifman graph \(G_{A_G}\) of \(A_G\) is a complete graph, and therefore \(A_G \in S(\mathcal{K})[\tau]\). Let \(\mathcal{H} := \{A_G \mid G \in \mathcal{G}\}\).

Cai, Fürer, and Immerman [16] proved that \textsc{ifp+c} does not capture polynomial time on the class \(\mathcal{G}\) of all graphs. Hence \(\mathcal{G}\) is not \textsc{ifp+c}-canonisable. This implies that \(\mathcal{H}\) is not \textsc{ifp+c}-canonisable, because it is easy to turn an \textsc{ifp+c}-canonisation of \(\mathcal{H}\) into one of \(\mathcal{G}\). But as \(\mathcal{H} \subseteq S(\mathcal{K})\), it follows that \(S(\mathcal{K})\) is not \textsc{ifp+c}-canonisable.
Remark 18.1.8. Another graph associated with a structure is its incidence graph, defined as follows. Let \( A \) be a \( \tau \)-structure. The incidence graph of a \( \tau \)-structure \( A \) is the (bipartite) graph \( I_A \) with vertex set \( V(I_A) := V \cup W \), where \( V := V(A) \) and \( W := \{ S \subseteq V(A) \mid \exists R \in \tau \}, \bar{v} \in R(A) : S = \bar{v}\} \), and edge set

\[
E(I_A) := \{ vS \mid v \in V, S \in W \text{ such that } v \in S \}.
\]

It is an interesting open question whether the analogue of Lemma 18.1.2 holds for incidence graphs as well. It is not obvious how to obtain an ordered treelike decomposition of a structure from an ordered treelike decomposition of its incidence graph.

It is not difficult to prove that if a class \( \mathcal{C} \) of graphs admits \( \text{IFP} \)-definable ordered treelike decompositions, then the class of all structures with incidence graph in \( \mathcal{C} \) admits \( \text{IFP}+\text{C} \)-definable canonisation.

18.2 Lifting Canonisations

It may seem that our approach to canonising graphs through ordered treelike decompositions is overly complicated. Instead, we could just define treelike decompositions whose torsos admit definable canonisation and then inductively lift the canonisation from the torsos to the whole graph. Such a direct approach to canonisation would also be more flexible, because there are structures where the torsos do not admit ordered treelike decompositions, but are still canonisable. For example, the torsos may be complete graphs. This happens when we decompose classes of chordal graphs such as interval graphs \([83]\) or chordal line graphs \([46]\).

However, such a direct approach to canonisation turns out to be far more complicated than the approach via ordered treelike decompositions. One important reason is that we cannot easily detect if a given transduction canonises a torso of a decomposition, whereas we can easily recognise if a given formula defines a linear order on a bag. Nevertheless, in this section we prove a “Canonisation Lifting Lemma”, albeit a fairly weak one that imposes several extra conditions on the torsos (they must admit strong canonisation, to be defined below) and the decomposition (it must have bounded adhesion). The lemma first appeared (without proof) in the conference paper \([45]\) under the name “Second Lifting Theorem”. A closely related result is the “Lifting Lemma” of \([54]\), which we will state later as Lemma 18.3.3.

In our inductive canonisation procedure we have to work with weighted rather than just plain relational structures, even if we just want to canonise graphs. Let \( \tau \) be a vocabulary. A weighted \( \tau \)-structure consists of a vertex set \( V(A) \) and a mapping \( R(A) : V(A)^k \rightarrow \mathbb{N} \) for every \( k \)-ary relation symbol \( R \in \tau \). For technical reasons, we impose the condition \( R(A)(\overline{\tau}) < 2^{|A|+1} \) for all \( \overline{\tau} \in V(A)^k \). To improve readability, we usually write \( R^A \) instead of \( R(A) \). We may view unweighted, or plain, relational structures as weighted structures by identifying each relation with its characteristic function.

Restrictions and expansions can be generalised from plain to weighted structures in a straightforward way. It is not obvious how to define substructures, but there is an obvious definition of induced substructures of weighted structures: if \( A \) is a weighted \( \tau \)-structure and \( W \subseteq V(A) \), then the induced substructure \( A[W] \) is the weighted \( \tau \)-structure with vertex set \( V(A[W]) := W \) and relations \( R^A[W] := R^A|_{W^k} \) for every \( k \)-ary \( R \in \tau \). Note, however, that the induced substructure \( A[W] \) (and similarly the structure \( A \setminus W \)) may violate the technical condition that \( R^A[W](\overline{\tau}) < 2^{|A[W]|+1} \) for all \( k \)-ary \( R \in \tau \) and \( \overline{\tau} \in W^k \) and thus formally may not be a weighted structure. We can safely ignore this technical detail here. We let \( A \setminus W := A[V(A) \setminus W] \). The definition of unions and intersections of weighted
structures is again problematic. We only define the union of two structures that coincide on their intersection. Let $A, B$ be weighted $\tau$-structures such that for all $k$-ary $R \in \tau$ and all $\bar{v} \in V(A)^k \cap V(B)^k$ we have $R^A(\bar{v}) = R^B(\bar{v})$. We define the union $A \cup B$ to be the weighted $\tau$-structure with $V(A \cup B) := V(A) \cup V(B)$ and

$$R^{A \cup B}(\bar{v}) := \begin{cases} R^A(\bar{v}) & \text{if } \bar{v} \in V(A)^k, \\ R^B(\bar{v}) & \text{if } \bar{v} \in V(B)^k, \\ 0 & \text{otherwise.} \end{cases}$$

for all $k$-ary relation symbols $R \in \tau$ and $\bar{v} \in V(A \cup B)^k$.

The underlying plain structure of a weighted $\tau$-structure $A$ is the plain $\tau$-structure $A'$ with $V(A') := V(A)$ and $R(A') := \{ \bar{v} \in V(A)^k \mid R^A(\bar{v}) > 0 \}$ for all $k$-ary $R \in \tau$. The Gaifman graph $G_A$ of $A$ is defined to be the Gaifman graph of $A'$, that is, the graph with vertex set $V(G_A) := V(A)$ and edge set $E(G_A) := \{ vw \mid v \neq w, \exists R \in \tau \text{ and } \bar{v} \text{ such that } R^A(\bar{v}) > 0 \text{ and } v, w \in \bar{v} \}$. For a class $\mathcal{C}$ of graphs, we let $\mathcal{S}_w(\mathcal{C})$ be the class of all weighted structures whose Gaifman graph is in $\mathcal{C}$.

An isomorphism from a weighted $\tau$-structure $A$ to a weighted $\tau$-structure $B$ is a bijective mapping $f : V(A) \to V(B)$ such that for all $k$-ary $R \in \tau$ and all $\bar{v} \in V(A)^k$ we have $R^A(\bar{v}) = R^B(f(\bar{v}))$. Now canonisation mappings and canonisation algorithms for weighted structures are defined in the obvious way.

To speak about weighted structures in our logics, with each weighted $\tau$-structure $A$ we associate a two-sorted structure $\hat{A}$ as follows. The universe of the first sort of $\hat{A}$ is the vertex set $V(\hat{A}) := V(A)$ of $A$ and the universe of the second sort is the set $\text{Num}(\hat{A}) := \text{Num}(A) = [0, |A|] \subseteq \mathbb{N}$. Consequently, elements of the first sort are called vertices and elements of the second sort are called numbers. The vocabulary $\hat{\tau}$ of $\hat{A}$ consist of a $(k+1)$-ary relation symbol $\hat{R}$ for each $k$-ary $R \in \tau$. Just like the relation variables in IFP+C, the relation symbols $\hat{R}$ are typed: $\hat{R}(\hat{A})$ is required to be a subset of $V(A)^k \times \text{Num}(A)$. In addition, $\hat{\tau}$ contains a binary relation symbol $\preceq$ ranging over pairs of numbers. For each $k$-ary $R \in \tau$, we define $\hat{R}(\hat{A})$ in such a way that for each $\bar{v} \in V(A)^k$ the string

$$b_{|A|} b_{|A|-1} \ldots b_0 \in \{0, 1\}^{|A|+1},$$

where $b_i = 1 \iff \forall i \in \hat{R}(\hat{A})$, is the binary representation of $R^A(\bar{v})$. Furthermore, $\preceq^{\hat{A}}$ is the natural linear order on $\text{Num}(A)$.

**Example 18.2.1.** Let $A$ be the weighted directed 5-star with vertex set $V(A) = \{a, b, c, d, e, f\}$ and weighted edge relation $E^A : V(A)^2 \to \mathbb{N}$ defined by

$$E^A(a, b) := 11, \quad E^A(a, c) := 2, \quad E^A(a, d) := 6, \quad E^A(a, e) := 2, \quad E^A(a, f) := 5$$

and $E^A(v, w) := 0$ for all $(v, w) \in V(A)^2 \setminus \{(a, b), (a, c), (a, d), (a, e), (a, f)\}$ (see Figure 18.1).

Then $\hat{A}$ is the structure with $V(\hat{A}) = \{a, b, c, d, e, f\}$, $\text{Num}(\hat{A}) = [0, 6]$, $\hat{E}(\hat{A}) = \{(a, b, 0), (a, b, 1), (a, b, 3), (a, c, 1), (a, d, 1), (a, d, 2), (a, e, 1), (a, f, 0), (a, f, 2)\}$, and $\preceq^{\hat{A}} = \{(i, j) \in [0, 6]^2 \mid i \leq j\}$.

We can easily extend our logics FO, IFP, IFP+C to the two-sorted framework. We use typed individual and relation variables. The logic IFP+C lives in the same two-sorted framework as the structures $\hat{A}$, so they work together smoothly.

M. Grohe, *Definable Graph Structure Theory*
The following formula defines the pairs of minimum positive weight.

For example, if \( w_{\text{minpos}} \) is the weighted star of Example 18.2.1, we have

\[
\{0,5\} \subseteq \{0,5,1,0,5,3,0,1,1,0,4,1,0,4,2,0,2,1,0,3,0,0,3,2\}
\]
and

\[
\{i, j\} \subseteq \{i, j\} \subseteq \{0, 6\}^2 \quad \text{and} \quad \leq_{A'} = \leq_{A'}|_{[0,5]} = \{(i, j) \in [0, 5]^2 \mid i \leq j\}.
\]
Example 18.2.4. The class of weighted complete graphs does not admit IFP+C-definable canonisation. If it would, then the class of all graphs would admit IFP+C-definable canonisation as well, because we can describe each graph by a weighted complete graph where all edges have weight 2 and all non-edges have weight 1, and from a canonical copy of this weighted complete graph we can easily retrieve a canonical copy of the original graph. \[\square\]

The following definition is crucial.

Definition 18.2.5. A class $C$ of graphs admits IFP+C-definable strong canonisation if the class $S_w(C)$ of all weighted structures with Gaifman graph in $C$ admits IFP+C-definable canonisation.

Let us spell this out in more detail: a class $C$ of graphs admits IFP+C-definable strong canonisation if for every vocabulary $\tau$ there is an IFP+C-$\{\dot{\tau}, \dot{\tau} \cup \{\leq\}\}$-transduction that canonises all weighted $\tau$-structures $A$ with $G_A \in C$.

Example 18.2.6. A fairly straightforward generalisation of Example 18.2.3 shows that the class of stars admits IFP+C-definable strong canonisation. On the other hand, Example 18.2.4 shows that the class of complete graphs does not admit IFP+C-definable strong canonisation. \[\square\]

The following theorem is another generalisation of the Canonisation Theorem 7.4.1.

Theorem 18.2.7. Let $C$ be a class of graphs that admits IFP-definable ordered treelike decompositions. Then $C$ admits IFP+C-definable strong canonisation.

Proof. It is easy to adapt the proof of the Canonisation Theorem. \qed

Corollary 18.2.8. Let $C$ be a class of graphs with excluded minors. Then $C$ admits IFP+C-definable strong canonisation.

We are now ready to state the main result of this section. Let $A$ be a class of graphs. We say that a class $C$ of graphs admits IFP-definable treelike decompositions over $A$ of bounded adhesion if there is a $d$-scheme $\Lambda$ and a $k \in \mathbb{N}$ such that for all graphs $G \in C$ the decomposition $\Lambda[G]$ is a treelike decomposition of $G$ over $A$ of adhesion at most $k$.

Lemma 18.2.9 (Canonisation Lifting Lemma). Let $A$ be a class of graphs that admits IFP+C-definable strong canonisation, and let $C$ of graphs admits IFP-definable treelike decompositions over $A$ of bounded adhesion. Then $C$ admits IFP+C-definable strong canonisation.

Proof. The proof is very similar to the proof of the Canonisation Theorem 7.4.1. For simplicity, we restrict our attention to weighted $\{R\}$-structures, where $R$ is a ternary relation symbol. (This way, we also avoid confusion resulting from the symbol $\tau$ being used for the vocabulary and the torsos. In this proof, the vocabulary is $\{R\}$, and $\tau$ is only used for the torsos.) We further assume that all graphs in $C$ and hence all structures in $S_w(C)$ are connected. It is easy to generalise the proof to arbitrary vocabularies and to disconnected structures. Our goal is to define an IFP+C-$\{\dot{R}, \{\leq\}\}$-transduction $\Theta$ that canonises the class $S_w(C)[\{R\}]$.

Let $k \in \mathbb{N}$ and $\Lambda$ be an $\ell$-dimensional $d$-scheme of vocabulary $\{\dot{R}\}$ such that for all structures $C \in S_w(C)[\{R\}]$, the decomposition $\Lambda[C]$ is a strict and normal treelike decomposition of $C$ over $A$ of adhesion at most $k$. Without loss of generality we may assume that $\ell \geq k$ and that for all $C \in S_w(C)[\{R\}]$ and all $t = (v_1, \ldots, v_\ell) \in V(D^{\Lambda[C]})$, if
18.2. Lifting Canonisations

\[ \sigma^{A[C]}(t) \neq \emptyset \] then \[ \sigma^{A[C]}(t) = \{v_1, \ldots, v_k\} \]. Let \( P, Q \) be fresh \( k \)-ary relation variables. Let \( \Theta^1 \) be an \( \text{IFP}+C(\{\hat{R}, \hat{Q}, \hat{P}\}, \{\hat{R}, \hat{Q}, \hat{P}, \leq\}) \)-transduction that canonises the class of all weighted \( \{R, Q, P\} \)-structures \( A \) with Gaifman graph \( G_A \in \mathcal{A} \). By Lemma 3.3.18, we may assume that \( \Theta^1 \) is normal (see Definition 3.3.15).

To explain our construction, we fix a weighted \( \{R\} \)-structure \( C \). Let \( G := G_C \) be the Gaifman graph of \( C \). Let \( \Delta := (D, \sigma, \alpha) := \Lambda[C] \). For all nodes \( t = (v_1, \ldots, v_t) \in V(D) \) let \( \bar{s}_t := (v_1, \ldots, v_k) \). Then if \( \sigma(t) \neq \emptyset \) we have \( \sigma(t) = \bar{s}_t \). We define a weighted \( \{R, P\} \)-structure \( C_t \) as follows. We let \( V(C_t) := \gamma(t) \) and \( R^{C_t} := R^{C_t}[\gamma(t)] \). For all \( \bar{v} \in \gamma(t)^k \setminus \{\bar{s}_t\} \) we let \( P^{C_t}(\bar{v}) := 0 \). If \( \sigma(t) = \emptyset \) we let \( P^{C_t}(\bar{s}_t) := 0 \) and otherwise let \( P^{C_t}(\bar{s}_t) := 1 \). Observe that there is an \( \text{IFP}+C([\hat{R}], \{\hat{R}, \hat{P}\}) \)-transduction \( \Theta^2(\bar{v}) \), where \( \bar{v} \) is an \( \ell \)-tuple of vertex variables, such that \( \Theta^2[C, t] = C_t \). We let \( n_t := |C_t| \).

Let \( \bar{v} \) be an \( \ell \)-tuple of vertex variables, and let \( y_1, \ldots, y_{\max\{3, k\}}, z \) be number variables. Furthermore, let \( X, Y, Z \) be relation variables whose types match the types of \( \bar{v} \) and \( \bar{y}_1 y_2 y_3 z \) and \( y_1 \ldots y_k z \), respectively. That is, \( X \) ranges over subsets of \( V(C)^\ell \) and \( Y \) ranges over subsets of \( V(C)^\ell \times \text{Num}(C)^4 \) and \( Z \) ranges over subsets of \( V(C)^\ell \times \text{Num}(C)^{k+1} \). We shall define a simultaneous fixed-point formula

\[
\text{ifp} \left( \begin{array}{c}
X \bar{v} \\
Y \bar{v} y_1 y_2 y_3 z \\
Z \bar{v} y_1 \ldots y_k z
\end{array} \right) \leftarrow \begin{array}{c}
\varphi(\bar{v}, X, Y, Z) \\
\psi(\bar{v}, y_1, y_2, y_3, z, X, Y, Z) \\
\chi(\bar{v}, y_1, \ldots, y_k, z, X, Y, Z)
\end{array} \right) \bar{v}.
\]

Let \( X^h, Y^h, Z^h \), for \( h \in \mathbb{N} \), be the stages of the fixed-point process in \( C \). We let \( X_\infty := \bigcup_{h \in \mathbb{N}} X^h \) and similarly \( Y_\infty, Z_\infty \) be the fixed points. For all \( \ell \)-tuples \( \bar{v} \in V(G)^\ell \) and all \( h \in \mathbb{N} \cup \{\infty\} \), we let

\[ Y^h(\bar{v}) := \{(i_1, i_2, i_3, j) \in \text{Num}(G)^4 \mid \bar{v}_{i_1 i_2 i_3} j \in Y^h\}. \]

We define \( Z^h(\bar{v}) \subseteq \text{Num}(C)^{k+1} \) similarly. In the following, we denote tuples in \( V(G)^\ell \) by \( t, u \) if they are intended to be nodes of \( D \). Our goal is to define the formulae \( \varphi, \psi, \chi \) such that the following conditions are satisfied.

(A) \( X^h \) is the set of all nodes of \( D \) of depth at most \( h - 1 \).

(B) For all nodes \( t \in X^h \) it holds that \( Y^h(t) \subseteq [0, n_t - 1]^3 \times \text{Num}(C) \).

Furthermore, for all nodes \( t \in X^h \) there is a bijective mapping \( f_t \) from \( \gamma(t) \) to \([0, n_t - 1]\), which only depends on \( t \) and not on \( h \), such that the following conditions are satisfied.

(C) For all \( v_1, v_2, v_3 \in \gamma(t) \) and \( j \in \text{Num}(C) \) it holds that

\[ (v_1, v_2, v_3, j) \in \hat{R}(C) \iff (f_t(v_1), f_t(v_2), f_t(v_3), j) \in Y^h(t). \]

(D) If \( \sigma(t) = \emptyset \) then \( Z^h(t) = \emptyset \).

If \( \sigma(t) \neq \emptyset \) and \( \bar{s}_t = (s_1, \ldots, s_k) \) then \( Z^h(t) = \{(f_t(s_1), \ldots, f_t(s_k), 0)\}. \)

Observe that it follows from (B) \( \text{(D)} \) that the two-sorted structure \(([0, n_t - 1], \text{Num}(C), Y^h(t), Z^h(t))\) is isomorphic to \( C_t \).

Suppose we have defined \( \varphi, \psi, \chi \) such that (A) \( \text{(D)} \) are satisfied. Then we can complete the proof of the lemma as follows. We define an \( \text{IFP}+C(\{\hat{R}\}, \{\hat{R}, \leq\}) \)-transduction \( \Theta^3(\bar{v}) \) such that \( (C, t) \in D_{\Theta^3(\bar{v})} \) if and only if \( t \in V(D) \), and \( \Theta^3[C, t] = ([0, n_t - 1], \text{Num}(C), Y^\infty(t), \leq [0, n_t - 1]) \).
for all \( t \in V(D) \). Now we let \( \Theta^4(\bar{\pi}) \) be the transduction obtained from \( \Theta^4(\pi) \) by restricting the domain to all \((\bar{C}, t)\) such that \( \sigma(t) = \emptyset \) and \( \alpha(t) = V(G) \). Such nodes \( t \) exist by \((\text{TL.5})\) because \( C \) is connected. Then for all \((C, t) \in D_{\Theta^4(\pi)}\) we have \( \Theta^4[C, t] \cong \bar{C} \). Hence \( \Theta^4(\pi) \) canonises \( C \).

It remains to define \( \varphi, \psi, \chi \) such that \([A] [D]\) are satisfied. We let

\[
\varphi(\bar{x}, Y, Z) := \lambda_V(\bar{x}) \land \forall \bar{x}' \left( (\lambda_V(\bar{x}') \land \lambda_E(\bar{x}, \bar{x}')) \rightarrow X \bar{x}' \right).
\]

Here \( \lambda_V(\bar{x}) \) and \( \lambda_E(\bar{x}, \bar{x}') \) are the formulas appearing in the d-scheme \( \Lambda \) that we used to define the decomposition \( \Delta \). Then \( \bar{\pi} \) enters \( X^{h+1} \) if and only if all its children in \( D \) are in \( X^h \). By induction, this implies \([A]\). The difficult part is to define \( \psi \) and \( \chi \). Let \( h \in \mathbb{N} \) and suppose that \( X^h, Y^h \) are defined and satisfy \([A] [D]\). For every \( u \in X^h \), let \( f_u : \gamma(u) \rightarrow [0, n_u - 1] \) be a bijective mapping that satisfies \([C]\) and \([D]\). Let \( \tilde{D}_u \) be the two-sorted ordered \( \{R, \bar{P}, \leq\}\)-structure \( ([0, n_u - 1], \text{Num}(C), Y^h(u), Z^h(u), \leq) \). By the induction hypothesis, \( \tilde{D}_u \) is an ordered copy of \( \hat{C}_u \), and the mapping \( f_u \) is an isomorphism from \( \hat{C}_u \) to \( \tilde{D}_u \). Also note that \( \sigma(u) \neq \emptyset \) by Lemma \( 4.3.8 \) because \( \Lambda[\hat{C}] \) is strict and normal. Thus by the definition of \( P_{C^u} \) and \([D]\), \( \tilde{D} \) and \( Z^h(t) \) both contain exactly one tuple.

Let \( t \in X^{h+1} \). It is our goal to define a mapping \( f_t \) and the relations \( Y^{h+1}(t), Z^{h+1}(t) \) such that \([B][D]\) are satisfied. Furthermore, we have to make sure that the definition of \( Y^{h+1}, Z^{h+1} \) from \( X^h, Y^h, Z^h \) can be formalised in the logic \( \text{IFP} + C \). The mapping \( f_t \) will not be definable in \( \text{IFP} + C \), and we are not allowed to use the mappings \( f_u \) for \( u \in X^h \) in the \( \text{IFP} + C\)-definition of \( Y^{h+1}, Z^{h+1} \).

As in the proof of the Canonisation Theorem \( 7.4.1 \) we proceed in two phases. In the first phase, we merge the structures \( \tilde{D}_u \) for those \( u \) that have the same tuple \( s_u \). In the second phase we use the merged structures together with a canonical copy of the torso at \( t \) to define \( D_t \). In the following, we give a high-level outline of the two phases and leave it to the reader to fill in the details.

**Phase 1.**

For every \( k \)-tuple \( \bar{v} = (v_1, \ldots, v_k) \in \beta(t)^k \), we let \( U^\pi(\bar{v}) \) be the set of all \( u \in N^D_+(t) \) with \( s_u = \bar{v} \). We define the weighted \( \{R, \bar{P}\}\)-structure \( C^\pi \) to be the union of the weighted structures \( C_u \) for \( u \in U^\pi \). Note that this union is well-defined. Observe that the structures \( C_u \) for \( u \in U^\pi \) form a sunflower, because their pairwise intersection is \( \sigma(u) = \bar{s}_u = \bar{v} \). Using an argument similar to the Sunflower Canonisation Lemma \( 7.4.4 \), we combine the structures \( \tilde{D}_u \) for \( u \in U^\pi \) into an ordered copy \( \bar{D}^\pi \) of \( \hat{C}^\pi \).

**Phase 2.**

Let \( \tilde{D}_1, \ldots, \tilde{D}_m \) be an enumeration of all structures \( \tilde{D}^\pi \) for \( \bar{v} \in \beta(t)^k \) with \( U^\pi \neq \emptyset \) in lexicographical order without repetitions. It may happen that \( \tilde{D}^i = \tilde{D}^\pi \) for several tuples \( \bar{v} \). We define a weighted \( \{R, \bar{P}, Q\}\)-structure \( B_t \) as follows:

- \( V(B_t) := \beta(t) \)
- \( R^{B_t} := R^C|_{\beta(t)^2} \)
- \( P^{B_t} := P^C \)
- \( Q^{B_t}(\bar{v}) := \begin{cases} 0 & \text{for all } \bar{v} \in \beta(t)^k \text{ with } U^\pi = \emptyset, \\ i & \text{for all } \bar{v} \in \beta(t)^k \text{ with } U^\pi \neq \emptyset \text{ and } \tilde{D}^\pi = \tilde{D}^i \end{cases} \)

M. Grohe, *Definable Graph Structure Theory*
Let \( \hat{A}_t := \Theta^1[\hat{B}_t] \) be the ordered copy of \( \hat{B}_t \) obtained by applying the canonisation \( \Theta^1 \). We can combine the ordered structures \( \hat{A}_t \) and \( \hat{D}_t \) to the desired structure \( \hat{D}_t \). □

18.3 Invariant Treelike Decompositions and Polynomial Time Canonisation

So far, we have mostly studied logically definable treelike decompositions and used them to prove results about definable canonisation. In this section, we want to study decompositions that are efficiently computable and use them to prove results about efficient canonisation. The definitions and results of this section are from [54].

Definition 18.3.1. A decomposition mapping (for short: d-mapping) for a class \( C \) of graphs is a mapping \( \Lambda \) that associates a decomposition \( \Lambda(G) \) of \( G \) with each graph \( G \in C \).

We generalise basic properties of decompositions to decomposition mappings. Let \( \Lambda \) be a d-mapping for a class \( C \) of graphs. We say that \( \Lambda \) is treelike if \( \Lambda(G) \) is a treelike decomposition, for all \( G \in C \). We say that \( \Lambda \) is over a class \( A \) of graphs if \( \Lambda(G) \) is decomposition over \( A \), for all \( G \in C \). We say that \( \Lambda \) is of bounded adhesion if there is a \( k \in \mathbb{N} \) such that the adhesion of \( \Lambda(G) \) is at most \( k \), for all \( G \in C \). Note that every d-scheme “defines” a d-mapping. We say that a d-mapping \( \Lambda \) over \( C \) is IFP-definable if there is a d-scheme \( \Lambda' \) such that \( \Lambda(G) = \Lambda'[G] \) for all \( G \in C \). We can also speak about polynomial time computable d-mappings. Observe that if \( \Lambda \) is IFP-definable then it is polynomial time computable. This follows from Lemma 3.1.6.

The following definition is crucial if we want to use decompositions to canonise graphs. It formalises the notion of “isomorphism invariant decompositions” that appeared informally in various places earlier in this book.

Definition 18.3.2. Let \( \Lambda \) be a d-mapping for a class \( C \) of graphs. \( \Lambda \) is invariant if for all isomorphic \( G, G' \in C \) with \( \Lambda(G) := (D, \sigma, \alpha) \) and \( \Lambda(G') := (D', \sigma', \alpha') \) and all isomorphisms \( f : G \cong G' \) there is an isomorphism \( g : D \to D' \) such that \( f(\sigma(t)) = \sigma'(g(t)) \) and \( f(\alpha(t)) = \alpha'(g(t)) \) for all \( t \in V(D) \).

Observe that IFP-definable decomposition mappings are always invariant. (Actually, this holds for decomposition mappings definable in any reasonable logic.) We can use polynomial time computable invariant decomposition mappings to obtain polynomial time canonisation algorithms in the same way as we used IFP+C-definable decompositions to obtain IFP+C-definable canonisation. The following lemma is a variant of the Canonisation Lifting Lemma [18.2.9]. We say that a class \( C \) of graphs admits polynomial time strong canonisation if for every vocabulary \( \tau \) there is a polynomial time canonisation algorithm for the class \( S_w(C)[\tau] \).

Lemma 18.3.3 (Grohe and Marx [54]). Let \( \mathcal{A} \) be a class of graphs that admits polynomial time strong canonisation. Let \( \mathcal{C} \) be a class of graphs such that there is a polynomial time computable treelike d-mapping over \( \mathcal{A} \) of bounded adhesion for \( \mathcal{C} \). Then \( \mathcal{C} \) admits polynomial time strong canonisation.

Proof. The proof is very similar to the proof of Lemma [18.2.9]. □

Lemma [18.3.3] was used in [54] to prove that all classes of graphs with excluded topological subgraphs admit polynomial time (strong) canonisation. For this it was shown that for all such classes there is a polynomial time computable d-mapping into torsos that either exclude
some fixed graph as a minor or have bounded degree up to a bounded number of vertices. As classes with excluded minors and classes of bounded degree up to a bounded number of vertices admit polynomial time strong canonisation (the former follows from Corollary 18.2.8 and the latter from the fact due to Babai and Luks [9] that all classes of bounded degree admit polynomial time canonisation), we can apply the Lifting Lemma 18.3.3 to obtain the desired canonisation result.

We can also define o-decomposition mappings (for short: od-mappings). It is then obvious how to define invariant od-mappings. We obtain the following variant of the Canonisation Theorem 7.4.1.

**Theorem 18.3.4.** Let \( C \) be a class of graphs for which there is a polynomial time computable invariant od-mapping \( \Lambda \) such that for all graphs \( G \in C \), the o-decomposition \( \Lambda(G) \) is an ordered treelike decomposition of \( G \). Then \( C \) admits polynomial time strong canonisation.

**Proof.** It is easy to adapt the proof of the Canonisation Theorem 7.4.1. \( \square \)

### 18.4 Directions for Further Research

In this last section, we briefly discuss some open questions and promising research topics. Some were already mentioned earlier in this book.

#### 18.4.1 Definability

Let us start our discussion of open problems with a very specific conjecture. Recall that a minor ideal is a class of graphs that is closed under taking minors.

**Conjecture 18.4.1.** Every minor ideal is definable in IFP.

Robertson and Seymour [109] proved that every minor ideal is decidable in cubic time. As we have seen earlier, together with our result that IFP+C captures PTIME on every class of graphs excluding a minor, this implies that every minor ideal is definable in IFP+C.

Robertson and Seymour’s theorem may suggest an even stronger conjecture stating that every minor ideal is definable in a bounded variable fragment (maybe even the 3-variable fragment of) IFP or rather LFP (cf. [80]), or at least in a bounded arity fragment, where the relation variables appearing in the fixed-point operators have bounded arity. However, it follows from the results of [41] that this is not the case: for each \( k \) there is a minor ideal that is not definable by a \( k \)-ary IFP-formula.

Towards a positive resolution of the conjecture, we have proved that the class of planar graphs, the class of \( K_5 \)-minor free graphs, and for each \( k \) the class of all graphs of tree width at most \( k \) are IFP-definable. Even for these relatively simple classes, the IFP-definitions are fairly complicated. In particular, for planar graphs we take a detour through descriptive complexity. It would be nice to have a more direct IFP-definition for the class of planar graphs.

#### 18.4.2 Other Logics

We exclusively developed our definable structure theory for the logic IFP. It is clear that many of our constructions do not need the full expressive power of IFP and can be carried out in weaker logics. It would be worthwhile investigate this systematically.

M. Grohe, *Definable Graph Structure Theory*
Towards capturing results for logarithmic space on graph classes with excluded minors, it would be particularly interesting to see which of our constructions can be carried out in the logic \( LREC_\leq \) introduced in [52], which captures \( \text{LOGSPACE} \) on the class of all trees.

Of course we may also consider logics stronger than \( \text{IFP} \) to define treelike decomposition, possibly to capture polynomial time on more general graph classes.

The monadic second-order logic \( \text{MSO} \) of graphs has been studied to great depth (see, for example, [20]). Some of our constructions may contribute new insights here, in particular those for graphs of bounded tree width, and the \( \text{MSO} \)-definability of treelike decompositions deserves some attention as well.

### 18.4.3 Other Graph Classes

We proved that \( \text{IFP}+\text{C} \) captures \( \text{PTIME} \) on all classes of graphs with excluded minors. Short of a proof that there is a logic capturing \( \text{PTIME} \) (on all structures), we may ask if there are further interesting classes of graphs for which we can find a logic capturing \( \text{PTIME} \). It follows from Babai and Luks’s canonisation theorem for graphs of bounded degree [9] that there is a logic capturing polynomial time on the class of graphs of bounded degree. By Grohe and Marx’s [54] canonisation theorem for classes with excluded topological subgraphs, including all classes of bounded degree, there even is a logic capturing \( \text{PTIME} \) on all such classes. However, the logics obtained through these canonisation results are not very natural.

Cai, Fürer, and Immerman’s Theorem [16] (Fact 3.1.11) shows that \( \text{IFP}+\text{C} \) does not capture \( \text{PTIME} \) on these classes. It is an interesting open problem to find a natural logic that captures polynomial time on graph classes of bounded degree or even classes with excluded topological subgraphs.

Recently, graph classes closed under taking induced subgraphs have received some attention [46, 56, 83, 84, 116]. Most notably, Laubner [83] proved that \( \text{IFP}+\text{C} \) captures \( \text{PTIME} \) on the class of interval graphs. Interval graphs are intersection graphs of intervals on the real line. Neuen [93] gave a polynomial time isomorphism test for the class of unit disk intersection graphs. It can probably be extended to a polynomial time canonisation algorithm, which would imply that there is a logic capturing \( \text{PTIME} \) on this class. It is open whether \( \text{IFP}+\text{C} \) captures polynomial on this class. On the other hand, it can be shown that if there is a logic capturing \( \text{PTIME} \) on the class of (axis-parallel) rectangle intersection graphs then there is a logic that captures \( \text{PTIME} \) on all graphs [84, 33]; this follows easily from [124].

Notable open questions in this context are whether there are logics capturing polynomial time on the class of unit disk intersection graphs and the class of circular arc graphs (intersection graphs of arcs on a cycle), or if there are polynomial time isomorphism tests for these classes.

Grohe and Schweitzer [56] gave polynomial time isomorphism tests for all graph classes of bounded rank width [96], or equivalently bounded clique width [21]. Again, this can probably be extended to polynomial time canonisation algorithms and thus a logic capturing \( \text{PTIME} \) on these classes, and again it is an interesting open question whether \( \text{IFP}+\text{C} \) captures polynomial time on these classes.

### 18.4.4 The Weisfeiler-Leman Dimension

Via Lemma 3.5.8 we obtain another interesting corollary of our Definable Structure Theorem 17.2.1.

**Corollary 18.4.2.** For every class \( C \) of graphs with excluded minors there is a \( k \) such that
the \( k \)-dimensional Weisfeiler-Leman algorithm identifies all graphs in \( \mathcal{C} \).

By Theorem 3.5.7, we may equivalently say that the \((k+1)\)-variable logic \( \mathcal{C}^{k+1} \) identifies all graphs in \( \mathcal{C} \).

**Definition 18.4.3.** The WL-dimension of a class \( \mathcal{C} \) of graphs is the least \( k \) such that the \( k \)-dimensional Weisfeiler-Leman algorithm identifies all graphs in \( \mathcal{C} \), or \( \infty \) if no such \( k \) exists.

By Corollary 18.4.2 all graph classes with excluded minors have a finite WL-dimension. But our proofs, depending on the structure theory of graphs with excluded minors, give no reasonable bounds on the dimension. Good bounds on the WL-dimension of graph classes with excluded minors would probably require an entirely different proof of Corollary 18.4.2 that avoids heavy graph minor theory. It is conceivable that such a proof exists; after all, Ponomarenko’s [101] polynomial time isomorphism test for graph classes with excluded minors only uses elementary graph theory. Also note that we developed our definable structure theory not only to prove Corollary 18.4.2 but mainly to prove that IFP+C captures PTIME on classes with excluded minors (Corollary 17.2.3); this is a stronger result.

Nevertheless, bounds on the WL-dimension of arbitrary graph classes with excluded minors seem to be out of reach at the moment. However, for graph classes of bounded Euler genus, it seems possible to get exact or at least asymptotically tight bounds. The first question to ask is: what is the WL-dimension of the class of planar graphs? In his Master’s Thesis [102], Redies gave an upper bound of 14 by analysing the proof given in this book. But this bound seems to be far from tight.

Further interesting questions are whether the WL-dimension of classes of bounded Euler genus depends linearly on the WL-dimension; it seems conceivable to obtain a linear upper bound just by analysing the proof given here. Similarly, it may be possible to obtain exact bounds for classes of bounded tree width: I conjecture that the WL-dimension of the class of graphs of tree width \( k \) is \( k \). Note that for all these classes it is not only interesting to prove upper bounds on the WL-dimension, but also lower bounds.

18.4.5 Automorphism Groups and a Generalisation of Ordered Decompositions

In this section, we assume (very) basic knowledge of permutation groups. We denote the symmetric group on a set \( V \) by Sym(\( V \)) and the automorphism group of a graph \( G \) by Aut(\( G \)).

Our Definable Structure Theorem 17.2.1 seems to have strong implications for the automorphism groups of graphs with excluded minors. Observe that if a formula \( \varphi(x_1, \ldots, x_\ell, y_1, y_2) \) of IFP or any other logic defines a linear order on a graph \( G \), then the automorphism group of \( G \) has a base of length \( \ell \), where a base of a permutation group is a sequence of elements whose pointwise stabiliser is the identity. The existence of a base of length \( \ell \) implies that the automorphism group of \( G \) has order at most \( |G|^\ell \). Now it seems that if a graph \( G \) has an ordered treelike decomposition defined by an od-scheme of dimension \( \ell \), then its automorphism group can be written as some form of generalised wreath product of groups with a base of length \( \ell \). It remains a challenging task to turn these intuitive remarks into a precise (and useful) theorem.

The fact that definable ordered treelike decompositions severely restrict the automorphism groups of the graphs in question is also a clear indication that the scope of a definable structure theory based on ordered treelike decompositions is limited. To extend the theory, for example
to graph classes of bounded rank width, we need a less restricted form of decomposition that still carries enough information to canonise graphs.

One idea would be to interpret graphs in other graphs that admit definable ordered treelike decompositions via some definable transformation (that is, a transduction). Such an approach may be suitable for classes like the class of interval graphs, where we can define all cliques with a bounded number of vertices.

A different idea would be to define the automorphism groups of graphs instead of linear orders. Let me briefly sketch one way of doing this. We can specify a permutation \( \pi \) of the vertex set of a graph \( G \) by a formula \( \varphi(y, z) \) that defines the graph of the permutation. That is, \( G \models \varphi[v, w] \iff v^\pi = w \) for all \( v, w \in V(G) \). Here and in the following we denote the action of permutations as “exponentiation”, that is, we denote the image of \( v \) under \( \pi \) by \( v^\pi \).

Then we can specify a set of at most \(|G|^k\) permutations by a formula \( \varphi(x_1, \ldots, x_k, y, z) \): we say that in \( G \) the formula \( \varphi(x_1, \ldots, x_k, y, z) \) defines the set

\[
\Pi_{\varphi} := \{ \pi \in \text{Sym}(V(G)) \mid \exists u_1, \ldots, u_k \in V(G) \forall v, w \in V(G) : G \models \varphi[u_1, \ldots, u_k, v, w] \iff v^\pi = w^\pi \}
\]

of permutations. Observe that the set \( \Pi_{\varphi} \) is invariant under automorphisms of \( G \), that is, if \( \pi \in \Pi_{\varphi} \) then for each \( \alpha \in \text{Aut}(G) \) the permutation \( \pi^\alpha := \alpha^{-1} \pi \alpha \), which satisfies \((v^\alpha)^\pi^\alpha = (v^\pi)^\alpha\), is in \( \Pi_{\varphi} \) as well. A concise way of writing this invariance condition is \( \alpha^{-1} \Pi_{\varphi} \alpha \subseteq \Pi_{\varphi} \) for all \( \alpha \in \text{Aut}(G) \). It is not hard to see that from a formula \( \varphi(\overline{a}, \overline{y}_1, \overline{y}_2) \) that defines a linear order (with parameters) on a graph we can obtain a formula \( \varphi'(\overline{a}, \overline{y}) \) such that \( \Pi_{\varphi'} = \text{Aut}(G) \).

But clearly, in general it is not possible to define automorphism groups this way with a bounded number of parameters, simply because in general the automorphism groups are exponentially large. Instead, we can try to define a generating set for the automorphism group.

Example 18.4.4. The following formula defines the set of all transpositions of the vertex set:

\[
\varphi(x_1, x_2, y, z) := x_1 \neq x_2 \land (y = x_1 \land z = x_2) \lor (y = x_2 \land z = x_1) \lor (y \neq x_1 \land y \neq x_2 \land z = y).
\]

Hence for every complete graph \( K \), the formula \( \varphi \) defines a generating set of the automorphism group \( \text{Aut}(K) \), because \( \text{Aut}(K) = \text{Sym}(V(K)) \), and the symmetric group \( \text{Sym}(V(K)) \) is generated by the set of all transpositions.

We say that a formula \( \varphi(\overline{a}, \overline{y}) \) defines the automorphism group of a graph \( G \) if \( \Pi_{\varphi} \) is a generating set for \( \text{Aut}(G) \), and for some logic \( L \) and class \( C \) of graphs we say that \( C \) admits \( L \)-definable automorphism groups if there is an \( L \)-formula \( \varphi(\overline{a}, \overline{y}) \) that defines the automorphism group of all graphs \( G \) in \( C \).

Now many interesting questions arise. Which classes of structures do admit \( L \)-definable automorphism groups, for various logics \( L \) such as \( \text{FO}, \text{IFP}, \text{IFP}+\text{C}, \text{MSO} \)? There is a related purely group theoretic question: for which graphs \( G \) does \( \text{Aut}(G) \) have a generating set of size \(|G|^k\) that is normal, that is, a generating set \( \Pi \) with \( \alpha^{-1} \Pi \alpha \subseteq \Pi \) for all \( \alpha \in \text{Aut}(G) \). Then the question arises whether a class of graphs that admits, say, \( \text{IFP}+\text{C} \)-definable automorphism groups admits \( \text{IFP}+\text{C} \)-definable canonisation, or if at least \( \text{IFP}+\text{C} \)-captures \( \text{PTIME} \) on such a class. Conversely, we may ask whether a class on which \( \text{IFP}+\text{C} \)-captures \( \text{PTIME} \) admits \( \text{IFP}+\text{C} \)-definable automorphism groups.

Once these basic questions are understood, we may proceed to studying treelike decomposition where, instead of linear orders, we can define generating sets for the automorphism group.
groups of all torsos, or other graphs associated with the bags of the decomposition (it may be useful to consider richer structures instead of the torsos, similar as in the proof of the Canonisation Lifting Lemma 18.2.9). Then similar questions as those above arise regarding classes that admit such decompositions. These ideas may allow us to extend a definable structure theory far beyond classes of graphs with excluded minors.
Appendix A

Robertson and Seymour’s Version of the Local Structure Theorem

In this appendix, we explain how our Local Structure Theorem 17.1.3 relates to Robertson and Seymour’s version of the theorem (Theorem 3.1 of [111]), which reads as follows.

**Fact A.0.1.** For every graph \( L \), there are integers \( q, p, \ell \geq 0 \) and \( k \geq 1 \) with the following property. Let \( \mathfrak{T} \) be a tangle of order at least \( k \) in a graph \( G \), controlling no \( L \)-minor of \( G \). Then there exists a set \( \zeta \subseteq V(G) \) with \( |\zeta| \leq \ell \), and a \( \mathfrak{T}\setminus \zeta \)-central segregation of \( G\setminus \zeta \) of type \((p,q)\) which has a proper arrangement in some surface in which \( L \) cannot be drawn.

In the following we will explain the terminology used in this statement, and it will become gradually clear how the Local Structure Theorem follows. For the statement to make sense, we must assume that \( L \) is nonplanar and that \( k > \ell \), the latter to make sure that \( \mathfrak{T}\setminus \zeta \) is defined.

First of all, let us get rid of the graph \( L \) and the restriction on the tangle to control no \( L \)-minor. Let \( \mathcal{C} \) be a class of graph with excluded minors, and let \( L \) be a graph that is not a minor of any \( G \in \mathcal{C} \). Without explaining in detail what it means that a tangle controls an \( L \)-minor (intuitively it means that there is an image of \( L \) in \( G \) that is located in the region described by the tangle), let us note that no tangle of any graph in \( \mathcal{C} \) controls an \( L \)-minor, simply because graphs in \( \mathcal{C} \) exclude \( L \) as a minor. We may assume without loss of generality that \( L \) is not planar. We can do this because every supergraph of \( L \) is also an excluded minor of all graphs in \( \mathcal{C} \). Let \( r \) be the maximum Euler genus of a surface in which \( L \) cannot be embedded. Then Fact A.0.1 yields the following.

**Corollary A.0.2.** Let \( \mathcal{C} \) be a class of graph with excluded minors. Then there are \( k, \ell, p, q, r \in \mathbb{N} \) with \( k > \ell \) such that the following holds. Let \( \mathfrak{T} \) be a tangle of order at least \( k \) in a graph \( G \in \mathcal{C} \). Then there exists a set \( \zeta \subseteq V(G) \) with \( |\zeta| \leq \ell \) and a \( \mathfrak{T}\setminus \zeta \)-central segregation of \( G\setminus \zeta \) of type \((p,q)\) which has a proper arrangement in some surface of Euler genus at most \( r \).

A *society* is a pair \((G, \Omega)\), where \( G \) is a graph and \( \Omega \) is a cyclic permutation of a subset of \( V(G) \) denoted by \( \tilde{\Omega} \).

Let \( p \in \mathbb{N} \). A *society* \((R, \Omega)\) is a \( p\)-RS-vortex\(^1\) if for all distinct \( v, w \in \tilde{\Omega} \) there do not exist \((p+1)\) mutually distinct paths of \( R \) between \( I \cup \{v\} \) and \( J \cup \{w\} \), where \( I \) denotes the set

---

\(^1\)We use “RS-vortex” to distinguish it from our version of “vortex”. In [111], an RS-vortex is just called “vortex.”
of all vertices of \( \tilde{\Omega} \) between \( v \) and \( w \) in the cyclic order determined by \( \Omega \) and \( J \) denotes the set of all vertices of \( \Omega \) between \( w \) and \( v \) in the cyclic order. The following claim shows that RS-vortices and our vortices are essentially the same. To relate them, we need one additional piece terminology. Let \((R, \Omega)\) be a society and \( \pi = (r_1, \ldots, r_n) \in V(R)^n \), for some \( n \in \mathbb{N} \). Then we say that \( \pi \) is compatible with \( \Omega \) if \( \tilde{\pi} = \tilde{\Omega} \) and \( \Omega(r_i) = r_{i+1} \) for all \( i \in [n-1] \) and \( \Omega(r_n) = r_1 \). It is an immediate consequence of Menger’s Theorem (Fact [2.2.1]) that for every tuple \( \pi \) of vertices of \( R \) that is compatible with \( \Omega \), the pair \((R, \Omega)\) is a \( p \)-RS-vortex if and only if \((R, \pi)\) is a \( p \)-vortex.

A graph \( R \) is a \( p \)-RS-ring with perimeter \( r_1, \ldots, r_n \) if \( R \) has a path decomposition \((P, \beta)\) of width at most \( p - 1 \), where \( P \) is the natural path on \([n]\) and for all \( i \in [n] \) we have \( r_i \in \beta(i) \). Observe that if \( R \) is a \( p \)-RS-ring with perimeter \( r_1, \ldots, r_n \) then \((R, (r_1, \ldots, r_n))\) is a \( p \)-ring. The converse does not necessarily hold.

A design is a pair \((H, \delta)\), where \( G \) is a graph and \( \delta \subseteq 2^V(H) \). A torso of a design \((H, \delta)\) is a graph \( H' \) such that

\[
H \subseteq H' \subseteq H \cup \bigcup_{X \in \delta} K[X].
\]

A location in graph \( G \) is a set \( \mathcal{L} \) of separations of \( G \) such that for all distinct \((A, B), (A', B') \in \mathcal{L} \) it holds that \( A \subseteq B' \). The design of a location \( \mathcal{L} \) in \( G \) is

\[
\left( \bigcap_{(A, B) \in \mathcal{L}} B, \{V(A \cap B) \mid (A, B) \in \mathcal{L}\} \right).
\]

We observe that a location is more or less the same as a star decomposition. Indeed, let \( \mathcal{L} = \{(A_1, B_1), \ldots, (A_n, B_n)\} \) be a location in a graph \( G \). Let \( S_\mathcal{L} \) be a star with centre \( s \) and tips \( t_1, \ldots, t_n \). We define \( \beta_\mathcal{L} : V(S_\mathcal{L}) \to 2^V(G) \) by letting \( \beta_\mathcal{L}(s) := \bigcap_{i=1}^n V(B_i) \) and \( \beta_\mathcal{L}(t_i) := V(A_i) \) for all \( i \in [n] \). It is easy to see that \((S_\mathcal{L}, \beta_\mathcal{L})\) is indeed a star decomposition of \( G \). Moreover, the torso \( \tau_\mathcal{L}(s) \) of the centre \( s \) is a torso of the design of the location \( \mathcal{L} \). Conversely, it is easy to see that every star decomposition \((S, \beta)\) of \( G \) yields a location \( \mathcal{L} \) such that \((S, \beta) \cong (S_\mathcal{L}, \beta_\mathcal{L})\).

The following fact is claim (3.2) of [III].

**Fact A.0.3.** Let \( p \in \mathbb{N} \). Let \((G, \Omega)\) be a \( p \)-RS-vortex, and let \((r_1, \ldots, r_n) \in V(G)^n \) be compatible with \( \Omega \). Then there are separations \((A_1, B_1), \ldots, (A_n, B_n)\) of \( G \) such that the following conditions are satisfied.

1. \( r_1, \ldots, r_n \in \bigcap_{i=1}^n V(B_i) \).
2. For every \( i \in [n] \) we have \( r_i \in V(A_i) \).
3. \( \mathcal{L} := \{(A_1, B_1), \ldots, (A_n, B_n)\} \) is a location in \( G \).
4. For every \( i \in [n] \) the separation \((A_i, B_i)\) has order at most \( 2p + 1 \).
5. Every torso of the design of \( \mathcal{L} \) is a \((2p + 1)\)-RS-ring with perimeter \( r_1, \ldots, r_n \).

Translated into our terminology, Fact A.0.3 immediately yields the following.

**Corollary A.0.4.** Let \( p \in \mathbb{N} \), and let \((G, \pi)\) be a \( p \)-vortex, and suppose that \( \pi = (r_1, \ldots, r_n) \). Then there exists a star decomposition \((S, \beta)\) such that the following conditions are satisfied.
(i) \( \bar{r} \subseteq \beta(s) \) for the centre \( s \) of \( S \).

(ii) \( S \) has exactly \( n \) tips \( t_1, \ldots, t_n \), and for every \( i \in [n] \) it holds that \( r_i \in \sigma(t_i) \).

(iii) The adhesion of \((S, \beta)\) is at most \( 2p + 1 \).

(iv) \((\tau(s), \bar{r}) \) is a \((2p + 1)\)-ring.

A segregation of a graph \( G \) is a set \( \mathcal{S} \) of societies such that

\[
G = \bigcup_{(A, \Omega) \in \mathcal{S}} A,
\]

and for all distinct \((A, \Omega), (A', \Omega') \in \mathcal{S}\) it holds that

\[
V(A \cap A') \subseteq \overline{\Omega} \cap \overline{\Omega}' \quad \text{and} \quad E(A \cap A') = \emptyset.
\]

We let \( V(\mathcal{S}) := \bigcup_{(A, \Omega) \in \mathcal{S}} \overline{\Omega} \). A segregation \( \mathcal{S} \) is of type \((p, q)\) if there are at most \( q \) societies \((A, \Omega) \in \mathcal{S}\) with \( |\Omega| > 3 \), and if all societies \((A, \Omega) \in \mathcal{S}\) with \( |\Omega| > 3 \) are \( p \)-RS-vortices.

Let \( \mathcal{T} \) be a tangle in a graph \( G \). A segregation \( \mathcal{S} \) of \( G \) is \( \mathcal{T} \)-central if for all \((A, B) \in \mathcal{T}\) and all \((A', \Omega) \in \mathcal{S}\) it holds that \( B \nsubseteq A' \).

Let \( \mathcal{S} \) be a segregation of a graph \( G \), and let \( S \) be a surface without boundary. An arrangement of \( \mathcal{S} \) in \( S \) is a pair \((D, \Pi)\) of mappings, where \( D : \mathcal{S} \to 2^S \) and \( \Pi : V(\mathcal{S}) \to S \), such that the following conditions are satisfied for all distinct \((A, \Omega), (A', \Omega') \in \mathcal{S}\):

- \( D(A, \Omega) \) is a closed disk in \( S \).
- \( D(A, \Omega) \cap D(A', \Omega') \subseteq \Pi(V(\mathcal{S})) \).
- \( \Pi \) is injective.
- \( \Pi(\overline{\Omega}) \subseteq \text{bd}(D(A, \Omega)) \), and the vertices \( \Pi(v) \) for \( v \in \Omega \) appear on \( \text{bd}(D(A, \Omega)) \) in the cyclic order determined by the cyclic permutation \( \Omega \).

An arrangement \((D, \Pi)\) of \( \mathcal{S} \) in \( S \) is proper if \( D(A, \Omega) \cap D(A', \Omega') = \emptyset \) for all distinct \((A, \Omega), (A', \Omega') \in \mathcal{S}\) with \( |\Omega|, |\Omega'| > 3 \).

To relate arrangements of a segregation with our arrangements of a graph, let \( \mathcal{S} \) be a segregation of a graph \( G \) of type \((p, q)\), and let \((D, \Pi)\) be a proper arrangement of \( \mathcal{S} \) in a surface \( \overline{S} \) without boundary. Let \( \mathcal{S} = \{(A_1, \Omega_1), \ldots, (A_m, \Omega_m)\} \), and suppose that \( |\Omega_i| > 3 \) for all \( i \in [q] \) and \( |\overline{\Omega}_i| \leq 3 \) for all \( i \in [q+1, m] \).

Let

\[
H' := \left( G \setminus \bigcup_{i=q+1}^{m} (V(A_i) \setminus \overline{\Omega}_i) \right) \cup \bigcup_{i=q+1}^{m} K[\overline{\Omega}_i].
\]

Note that \( G \) has a star decomposition \((S', \beta')\), where \( S' \) is a star with centre \( s \) and tips \( t_{q+1}, \ldots, t_m \) and \( \beta' \) is defined by \( \beta'(s) := V(H) \) and \( \beta'(t_i) := V(A_i) \). Then \( H' = \tau'(s) \).

For all \( i \in [q] \), let \( D_i := D(A_i, \Omega_i) \). Let \( S \) be the surface obtained from \( \overline{S} \) by deleting the interiors of the disks \( D_1, \ldots, D_q \). Let \( H_0 \) be the graph with vertex set

\[
V(H_0) := H' \setminus \bigcup_{i=1}^{q} (V(A_i) \setminus \overline{\Omega}_i).
\]
Chapter A. Robertson and Seymour’s Version of the Local Structure Theorem

and edge set

\[ E(H_0) := E(H') \setminus \bigcup_{i=1}^{q} E(A_i). \]

The arrangement \((D, \Pi)\) of \(S\) in \(\mathcal{S}\) yields an embedding of \(H_0\) in \(S\). To simplify the notation, let us view \(H_0\) as being embedded in \(S\) in the following and let \(\pi\) be the identity mapping on \(V(H_0)\). For every \(i \in [q]\), let \(\pi_i\) be a tuple of vertices of \(A_i\) that is compatible with \(\Omega_i\). Then \((A_i, \pi_i)\) is a \(p\)-vortex and thus a \(2 \left\lceil p/2 \right\rceil\)-vortex, and \((H_0, \pi, A_1, \pi_1, \ldots, A_q, \pi^q)\) is a local \([p/2]\)-arrangement of \(H\) in \(S\). However, this is not yet the \(\mathcal{AE}\)-star decomposition we need for the Local Structure Theorem.

For every \(i \in [q]\) we apply Corollary A.0.4 to the \(p\)-vortex \((A_i, \pi^i)\). Let \((S^i, \beta^i)\) be the resulting star decomposition. Let \(s^i\) be the centre of \(S^i\) and \(R^i := \tau(s^i)\). Then \(R^i\) is a \((2p+1)\)-ring. Let \(t^i_1, \ldots, t^i_{\ell_i}\) be the tips of \(S_i\).

Let \(H := H_0 \cup \bigcup_{i=1}^{q} R^i\). Then \((H_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q)\) is a \((2p+1)\)-arrangement of \(H\) in \(S\). Let \(S\) be the star with centre \(s\) and tips \(t^i_j\), for \(i \in [q]\) and \(j \in [\ell_i]\), and \(t_i\), for \(i \in [q+1, m]\).

We define \(\beta : V(S) \to 2^{V(G)}\) by \(\beta(s) := V(H)\) and \(\beta(t^i_j) := \beta^i(t^i_j)\) and \(\beta(t_i) := \beta(t_i)\). Then

\[ (S, \beta, s, S, H_0, \pi, R^1, \pi^1, \ldots, R^q, \pi^q) \]

is a \(\mathcal{AE}_{(2p+1), q, r}\)-star decomposition of \(G\), where \(r := \text{eg}(S)\).

To summarise: we have shown how to construct an \(\mathcal{AE}_{(2p+1), q, r}\)-star decomposition of a graph \(G\) from a segregation of \(G\) of type \((p, q)\) that has a proper arrangement in some surface of Euler genus at most \(r\). To prove the Local Structure Theorem from Corollary A.0.2 we need to show that if we start with a segregation that is \(\mathcal{I}\)-central for some tangle \(\mathcal{I}\) in \(G\), then the resulting star decomposition \((S, \beta)\) will also be \(\mathcal{I}\)-central. So let \(\mathcal{I}\) be a tangle of \(G\) of order \(k > \max\{2p+1, 3\}\) such that \(\mathcal{I}\) is \(\mathcal{I}\)-central. Let \((A, B) \in \mathcal{I}\). Let \(t\) be a tip of \(S\), and suppose for contradiction that \(V(B) \subseteq \beta(t)\).

**Case 1:** \(t = t^i_j\) for some \(i \in [q]\) and \(j \in [\ell_i]\).

Then \(V(B) \subseteq V(A_i)\). As the segregation \(S\) is \(\mathcal{I}\)-central, we have \(B \not\subseteq A_i\) and thus \(E(B) \setminus E(A_i) \neq \emptyset\). It follows from the definition of a segregation that both edges in \(E(B) \setminus E(A_i)\) have both endvertices in \(\Omega_i\). Also note that \(V(B) \cap \Omega_i \subseteq \sigma(t)\) and thus \(|V(B) \cap \Omega_i| \leq 2p + 1\).

Let \((A', B')\) be the separation of \(G\) with \(A' := A_i[V(B)]\) and \(B' := (V(A), E(G) \setminus E(A'))\). Then \((A', B') \in \mathcal{I}\), because \(A' \subseteq A_i\). Let \((A'', B'')\) be the separation of \(G\) with \(A'' := G[V(B) \cap \Omega_i]\) and \(B'' := (V(G), E(G) \setminus E(A''))\). Then \((A', B') \in \mathcal{I}\) by (TA.3) Note that \(B \subseteq A' \cup A''\) and thus \(A \cup A' \cup A'' = G\). This contradicts (TA.2).

**Case 2:** \(t = t_i\) for some \(i \in [q+1, m]\).

We can argue similarly to the first case, using that in this case we have \(|V(B) \cap \Omega_i| \leq 3\).

**Proof of the Local Structure Theorem** (17.1.3). Let \(C\) be a class of graphs with excluded minors, and let \(\ell', p', q', r', k'\) be the parameters we obtain from Corollary A.0.2. We let \(\ell := \max\{\ell', 1\}, p := 2p' + 1, q := q', r := r'\) and \(k := \max\{k', \ell' + 2p' + 1, \ell' + 3\}\). Let \(G \in C\), and let \(\mathcal{I}\) be a tangle of \(G\) of order at least \(k\). From Corollary A.0.2 we obtain a set \(\zeta \subseteq V(G)\) with \(|\zeta| \leq \ell\) and a \(\mathcal{I} \setminus \zeta\)-central segregation \(S\) of \(G \setminus \zeta\) of type \((p, q)\) which has a proper arrangement in some surface \(S\) of Euler genus at most \(r\). By the construction described above, from \(S\) we obtain a \(\mathcal{AE}_{p, q, r}\)-star decomposition of \(G \setminus \zeta\) that is \(\mathcal{I} \setminus \zeta\)-central. □
Bibliography


Preliminary Version


Symbol Index

\[2^S\] powerset of set \(S\), page 19

\([m, n]\) \([\{m, m + 1, \ldots, n\}\), page 19

\([n]\) \([\{1, \ldots, n\}\), page 19

\[#x \varphi = y\] counting formula in \(\text{IFP} + C\), page 30

\(\alpha(t)\) component at node \(t\) in a decomposition, page 92

\(\angle(f)\) set of angles of face \(f\) of an embedded graph, page 190

\(\angle(C)\) set of angles of facial cycle \(C\) of a 3-connected planar graph, page 190

\(\angle(G)\) set of angles of embedded or 3-connected planar graph \(G\), page 190

\(\langle A, \leq^A \rangle\) representation of ordered structure \((A, \leq^A)\) by binary string, page 43

\(\text{art}(Q)\) set of articulation vertices of shortest path system \(Q\), page 372

\(\beta(t)\) bag at node \(t\) in a decomposition, page 92

\(\text{Bd}(f)\) boundary subgraph of face \(f\), page 182

\(\bigcup S\) union of all sets in \(S\), page 19

\(\binom{S}{k}\) set of all \(k\)-element subsets of set \(S\), page 19

\(\mathcal{A}\mathcal{E}_{p,S}\) class of graphs \(p\)-almost embeddable in \(S\), page 360

\(\mathcal{A}\mathcal{E}_{p,q,r}\) class of graphs \(p\)-almost embeddable in a surface of Euler genus \(r\) with \(q\) cuffs, page 360

\(\mathcal{A}\mathcal{P}_p\) class of \(p\)-almost planar graphs, page 282
\( \mathcal{C}[\tau] \) class of all \( \tau \)-structures in class \( \mathcal{C} \) of arbitrary structures, page 25
\( \mathcal{E}_g \) class of all graphs of Euler genus at most \( g \), page 186
\( \mathcal{E}_X \) class of all graphs embeddable in topological space \( X \), page 182
\( \text{cf}(S) \) number of cuffs of surface \( S \), page 178
\( \mathcal{P}_{\text{CFI}} \) Cai-Fürer-Immerman property, page 50
\( \mathcal{G} \) class of all graphs, page 20
\( \mathcal{G}_k \) class of all graphs of order at most \( k \), page 20
\( \mathcal{K} \) class of all complete graphs, page 55
\( \mathcal{K}_k \) class of complete graphs of order at most \( k \), page 94
\( \mathcal{L} \) class of all graphs isomorphic to \( L \), page 253
\( \text{Clins}(C) \) graph in \( \text{clins}(C) \), page 286
\( \mathcal{M}(C) \) class of all minors of graphs in \( C \), page 23
\( \mathcal{M}(G) \) class of all minors of \( G \), page 23
\( \mathcal{MAE}_{p,q,r} \) class of all graphs that have a \( p \)-m-arrangement in a surface of Euler genus \( r \) with \( q \) cuffs, page 428
\( \mathcal{MAE}_{\text{red},p,q,r} \) class of all graphs that have a reduced \( p \)-m-arrangement in a surface of Euler genus \( r \) with \( q \) cuffs, page 428
\( \mathcal{MAP}_p \) class of all graphs that have a \( p \)-m-arrangement, page 323
\( \mathcal{N}_k(C) \) class of all \( k \)-enlargements of graphs in \( C \), page 20
\( \mathcal{O} \) class of all ordered structures, page 25
\( \text{Com}(\Gamma) \) compass of grid \( \Gamma \), page 334
\( \gamma(t) \) cone at node \( t \) in a decomposition, page 92
\( \simeq \) isomorphism between graphs and structures, page 20
\( \mathcal{OT}_\Lambda \) class of all graphs \( G \) such that \( \Lambda[G] \) is an ordered treelike decomposition of \( G \), page 148
\( \mathcal{P} \) class of all planar graphs, page 189
\( \mathcal{Q}(u,u') \) canonical shortest path system, page 372
\( \mathcal{Q}[v,w] \) induced shortest path system, page 372
\( \mathcal{QZ}_4^* \) class of quasi-4-connected graphs, page 237
\( \mathcal{S} \) class of all structures, page 25
Symbol Index

\( \mathcal{S}(\mathcal{C}) \) class of all structures whose Gaifman graph is in \( \mathcal{C} \), page 455

\( \mathcal{S}_w(\mathcal{C}) \) class of all weighted structures whose Gaifman graph is in \( \mathcal{C} \), page 458

\( \mathcal{SK}_{p,\ell} \) class of all 3-connected graphs whose definable \( p \)-skeleton has tree width at most \( \ell \), page 321

\( \mathcal{ST} \) class of all stars, page 55

\( \mathcal{T} \) class of all trees, page 23

\( \mathcal{T}(\mathcal{C}) \) class of all graphs that have a tree decomposition over \( \mathcal{C} \), page 90

\( \mathcal{T}_\Lambda(\mathcal{A}) \) class of all graphs \( G \) such that \( \Lambda[G] \) is a treelike decomposition of \( G \) over \( \mathcal{A} \), page 116

\( \mathcal{T}_k \) class of all graphs of tree width at most \( k \), page 90

\( \mathcal{T}_\Lambda \) class of all graphs \( G \) such that \( \Lambda[G] \) is a treelike decomposition, page 116

\( \mathcal{U}(\mathcal{C}) \) closure of class \( \mathcal{C} \) under disjoint unions, page 20

\( \text{Cut}(\mathcal{B}) \) graph obtained by cutting through belt \( \mathcal{B} \), page 390

\( \mathcal{X}(\mathcal{C}) \) class of all graphs that do not contain any graph from \( \mathcal{C} \) as a minor, page 23

\( \mathcal{X}(G) \) class of all \( G \)-minor free graphs, page 23

\( \mathcal{Z} \) class of all connected graphs, page 21

\( \mathcal{Z}_k \) class of all \( k \)-connected graphs, page 21

\( \mathcal{Z}_k^* \) union of \( \mathcal{Z}_k \) with all complete graphs of order at most \( k \), page 21

\( \preceq^D, \prec^D \) partial order on directed acyclic graph \( G \), page 24

\( \preceq^Q \) partial order of shortest path system \( Q \), page 371

\( \text{dep}^D(v) \) depth of vertex \( v \) in directed acyclic graph \( G \), page 24

\( \delta(t) \) design at node \( t \) in a decomposition, page 92

\( \text{dfp} \) deflationary fixed-point operator, page 28

\( \text{dist}^G(W_1,W_2) \) minimum distance between \( w_1 \in W_1 \) and \( w_2 \in W_2 \) in \( G \), page 21

\( \text{dist}^G(w_1,w_2) \) distance between \( w_1 \) and \( w_2 \) in \( G \), page 21

\( \hat{\mathcal{A}} \) two-sorted structure associated with weighted structure \( \mathcal{A} \), page 458

\( \text{eg}(\mathcal{S}) \) Euler genus of surface \( \mathcal{S} \), page 178

\( \text{eg}(G) \) Euler genus of graph \( G \), page 186

\( \exists^{\geq i} \) counting quantifiers, page 72
$B^n$  
$n$-ball, page 177

$bd_X(Y), bd(Y)$ boundary of subspace $Y$ in topological space $X$, page 177

$cl_X(Y), cl(Y)$ closure of subspace $Y$ in topological space $X$, page 177

$clins(g)$ closure of $ins(g)$, page 286

$I$ closed unit interval, page 177

$ins(g)$ inside of simple closed curve $g$, page 286

$int_X(Y), int(Y)$ interior of subspace $Y$ in topological space $X$, page 177

$N_h$ nonorientable surface of genus $h$, page 178

$R(B)$ region of belt $B$, page 389

$R^n$ Euclidean $n$-space, page 177

$S^n$ $n$-sphere, page 177

$S_g$ orientable surface of orientable genus $g$ (and Euler genus $2g$), page 178

$\Gamma_k$ elementary grid of radius $k$, page 332

$\hat{A}$ $\hat{A} := \hat{A}^G := G[V(A) \cup N(A)] \cup K[N(A)]$ for a subgraph $A \subseteq G$, page 217

$ht^D(v)$ height of vertex $v$ in directed acyclic graph $G$, page 24

$H_G(W)$ 3-hinges of $G$ that are inseparable from $W$, page 220

ifp inflationary fixed-point operator, page 26

$Ins(C)$ graph in $ins(C)$, page 286

$Int(\Gamma)$ interior of grid $\Gamma$, page 333

$\Lambda[G]$ decomposition defined by d-scheme $\Lambda$ in graph $G$, page 116

$C$ (syntactical) extension of FO by counting quantifiers, page 72

$C^\omega_{\infty\omega}$ finite-variable variable infinitary logic with counting, page 73

$C^k$ $k$-variable fragment of $C$, page 73

$C^k_{\infty\omega}$ $k$-variable infinitary logic with counting, page 73

$C_{\infty\omega}$ infinitary logic with counting, page 73

$\leq$ natural order on the natural numbers (and integers, rationals, reals), page 25

$\leq^*$ simplicity order on surfaces, page 361

$\leq$ order symbol reserved for ordered structures, page 25

$\leq_{lex}$ lexicographical order of sets or tuples of integers, page 19
Symbol Index

\[ L[\tau] \] L-formulae of vocabulary \( \tau \), page 26

\[ L^\omega_{\omega} \] finite-variable variable infinitary logic, page 69

\[ L^k \] \( k \)-variable fragment of FO, page 68

\[ L^k_{\omega} \] \( k \)-variable infinitary logic, page 69

\[ L_{\omega} \] infinitary logic, page 69

\( \models \) satisfaction relation between structures or interpretations and formulas, page 26

\( N^D_-(v) \) in-neighbours (parents) of \( v \) in digraph \( D \), page 24

\( \mathbb{N} \) set of nonnegative integers, page 19

\( N^D_+(v) \) out-neighbours (children) of \( v \) in digraph \( D \), page 24

\( \text{Num}(A) \) number set of structure \( A \) in two-sorted framework, page 30

\( \sqcup_\prec \) ordered union of ordered graphs, page 63

\( \parallel \) parallel nodes between decomposition and pre-decomposition, page 258

\( \partial^G(W), \partial^G(H) \) boundary of \( W \subseteq V(G), H \subseteq G \), respectively, in \( G \), page 20

\( \text{Per}(\Gamma) \) perimeter of grid \( \Gamma \), page 333

\( \Phi = (V(\Phi), \sigma^\Phi, \alpha^\Phi) \) pre-decomposition, page 257

\( \varphi[A, P, X] \) relation defined by formula \( \varphi \), page 33

\( \Phi^{(d,d')} \) \((d,d')\)-derivation of pre-decomposition \( \Phi \), page 264

\( \varphi^{-\Theta} \) formula obtained by applying the Transduction Lemma to \( \varphi \) and \( \Theta \), page 36

\( \Phi_{G,k} \) generic pre-decomposition of \( G \) of adhesion \( k \), page 259

\( \mathbb{N}^+ \) set of positive integers, page 19

\( \preceq \) containment of pre-decompositions, page 259

\( \Psi = (\psi_V(\pi), \psi_\alpha(\pi,y), \psi_\alpha(\pi,y)) \) pd-scheme, page 258

\( \Psi[G] \) pre-decomposition defined by pd-scheme \( \Psi \) on graph \( G \), page 258

\( \rho(G) \) representativity of embedded graph \( G \), page 187

\( S\text{Cen}_{p,q}(G) \) \((p,q)\)-supercentre of \( G \), page 346

\( \sigma(t) \) separator at node \( t \) in a decomposition, page 92

\( S\text{Skel}_{p,q}(G) \) \((p,q)\)-superskeleton of \( G \), page 346

\( \Theta[A,P] \) image of \((A,P)\) under transduction \( \Theta(X) \), page 35

\( \tilde{v} \) set of elements of tuple \( \pi \), page 19
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau(t))</td>
<td>torso at node (t) in a decomposition, page 92</td>
</tr>
<tr>
<td>(v, n)</td>
<td>vertex and number type, page 30</td>
</tr>
<tr>
<td>(\varpi \prec^G \varpi, \varpi \prec \varpi)</td>
<td>angle (\varpi) is aligned with angle (\varpi), page 190</td>
</tr>
<tr>
<td>(Z)</td>
<td>set of integers, page 19</td>
</tr>
<tr>
<td>(A</td>
<td>_{\tau})</td>
</tr>
<tr>
<td>(D^Q)</td>
<td>directed acyclic graph of shortest path system (Q), page 372</td>
</tr>
<tr>
<td>(E^G(v))</td>
<td>set of edges incident with (v) in (G), page 20</td>
</tr>
<tr>
<td>(E_x(G,W))</td>
<td>crossedges of pairs of crossing 3-hinges in (H(G,W)), page 223</td>
</tr>
<tr>
<td>(F(G))</td>
<td>set of faces of embedded graph (G), page 182</td>
</tr>
<tr>
<td>(f(G))</td>
<td>image of (G) under mapping (f), page 20</td>
</tr>
<tr>
<td>(G + F)</td>
<td>graph obtained from (G) by adding edges in (F), page 20</td>
</tr>
<tr>
<td>(G + k)</td>
<td>normal ordered graph shifted by (k), page 63</td>
</tr>
<tr>
<td>(G - F)</td>
<td>graph obtained from (G) by deleting edges in (F), page 20</td>
</tr>
<tr>
<td>(G/A, G/F)</td>
<td>graph obtained from (G) by contracting subgraph (A) or edge set (F), page 22</td>
</tr>
<tr>
<td>(G[W])</td>
<td>induced subgraph of (G) with vertex set (W), page 20</td>
</tr>
<tr>
<td>(G \preceq H)</td>
<td>(G) is a minor of (H), page 22</td>
</tr>
<tr>
<td>(G \setminus S)</td>
<td>induced subgraph of (G) with vertex set (V(G) \setminus S), page 20</td>
</tr>
<tr>
<td>(G \setminus V(H))</td>
<td>induced subgraph of (G) with vertex set (V(G) \setminus V(H)), page 20</td>
</tr>
<tr>
<td>(G \approx H)</td>
<td>embedded graphs (G) and (H) are homeomorphic, page 182</td>
</tr>
<tr>
<td>(G \subseteq Y)</td>
<td>subgraph of embedded graph (G) in (Y), page 182</td>
</tr>
<tr>
<td>(G \times k)</td>
<td>ordered union of (k) copies of normal ordered graph, page 63</td>
</tr>
<tr>
<td>(G \cup H)</td>
<td>disjoint union of graphs (G) and (H), page 20</td>
</tr>
<tr>
<td>(G_A)</td>
<td>Gaifman graph of structure (A), page 455</td>
</tr>
<tr>
<td>(I(\mathcal{B}))</td>
<td>internal graph of belt (\mathcal{B}), page 389</td>
</tr>
<tr>
<td>(I(\mathcal{Q}))</td>
<td>internal graph of non-simplifying patch (\mathcal{Q}), page 382</td>
</tr>
<tr>
<td>(I(\mathcal{Q}))</td>
<td>internal graph of simplifying patch (\mathcal{Q}), page 384</td>
</tr>
<tr>
<td>(J(G,W))</td>
<td>torso of quasi-4-connected component of (G) at (W), page 232</td>
</tr>
<tr>
<td>(J^*(G,W))</td>
<td>quasi-4-connected component of (G) at (W), page 230</td>
</tr>
<tr>
<td>(K[V])</td>
<td>complete graph with vertex set (V), page 21</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>$K_n$</td>
<td>complete graph with vertex set $[n]$, page 21</td>
</tr>
<tr>
<td>$K_{m,n}$</td>
<td>complete bipartite graph with vertex set $[m+n]$ and edge set ${ij \mid i \in [m], j \in [m+1,n]}$, page 21</td>
</tr>
<tr>
<td>$L$</td>
<td>graph obtained from cycle of length 8 by adding edges between all pairs of opposite vertices, page 253</td>
</tr>
<tr>
<td>$N^G(v), N^G(W), N^G(H)$</td>
<td>set of neighbours of $v \in V(G)$, $W \subseteq V(G)$, $H \subseteq G$, respectively, in $G$, page 20</td>
</tr>
<tr>
<td>$N^G_r(v)$</td>
<td>$r$-neighbourhood of $v$ in $G$, page 317</td>
</tr>
<tr>
<td>$O(B)$</td>
<td>outside of belt $B$, page 390</td>
</tr>
<tr>
<td>$s/\equiv, S/\equiv$</td>
<td>equivalence class and set of equivalence classes, page 19</td>
</tr>
<tr>
<td>$t \approx t'$</td>
<td>nodes $t$ and $t'$ are indistinguishable, page 98</td>
</tr>
<tr>
<td>$t \parallel t'$</td>
<td>nodes $t$ and $t'$ are parallel, page 92</td>
</tr>
<tr>
<td>$t \perp t'$</td>
<td>nodes $t$ and $t'$ are orthogonal, page 92</td>
</tr>
<tr>
<td>$tTu$</td>
<td>unique path from $t$ to $u$ in tree $T$, page 24</td>
</tr>
<tr>
<td>$W_n$</td>
<td>wheel with $n$ spokes, page 281</td>
</tr>
<tr>
<td>FO+C</td>
<td>first-order logic with counting, page 32</td>
</tr>
<tr>
<td>FO</td>
<td>first-order logic, page 25</td>
</tr>
<tr>
<td>IFP</td>
<td>inflationary fixed-point logic, page 26</td>
</tr>
<tr>
<td>LFP</td>
<td>least fixed-point logic, page 30</td>
</tr>
</tbody>
</table>
Index

2-cell, 182
2-connected component, 166
   index, 166
   proper, 166
2CC Decomposition Lemma, 165
3-Hinge Lemma, 218
3-connected component, 171
   index, 171
   proper, 171
3CC Decomposition Lemma, 170
3CC Lifting Lemma, 171
abstract logic, 49
adhesion, see tree decomposition, see decomposition
   see pre-decomposition
\( \mathcal{A} \varepsilon_{p,q,r} \)-star completion, 432
\( \mathcal{A} \varepsilon_{p,q,r} \)-star decomposition, 432
   simple, 432
aligned, 190
almost all, 50
almost embeddable, 360
Almost Embeddable Completion Th., 432
almost planar, 282
Almost Planar Completion for Quasi-4-Connected Graphs, 348
Almost Planar Completion Theorem, 332
ancestor, 24
angle, 190
   aligned, 190
\( \mathcal{A} \varepsilon_{p} \)-star completion, 332
\( \mathcal{A} \varepsilon_{p} \)-star decomposition, 331
   simple, 332
   wide, 340
arity, 25
arrangement, 281, 359 see segregation
   \( p \), 282, 360
   injective, 281, 360
   local \( p \), 282, 360
   trivial, 282
articulation vertex, see shortest path system,
   see belt
assignment, 26, 33
atomic formula, 26
atomic formula, 25, 30
attachment point, 363, 364
automorphism, 20
automorphism-invariant, 96
bag, 92
belt, 388
   articulation vertex, 389
   cut through, 390
   endvertex, 389
   exit vertex, 390
   length, 388
   major articulation vertex, 389
   minor articulation vertex, 389
   outside, 390
   patches, 389
   proper articulation vertex, 389
   reducing, 389
Belt Lemma, 391
β-γ-σ-Lemma, 97
bijective pebble game, 74
bisimilar
decompositions, 109
nodes, 109
bisimulation, 109
block, see shortest path system
boundary, 20, see topological space
boundary subgraph, 182
Bounded Width Completion Lemma, 260
branch set, 23
branch vertex, 21
bridge, 22, 292
C, 72
canonical, 42, see shortest path system
canonisation
L, 54
abstract mapping, 54
algorithm, 53
definable, 54
mapping, 53
polynomial time, 53
Canonisation Lifting Lemma, 460
Canonisation Theorem, 157
Canonisation Theorem for Relational Structures, 456
canonise, 54
capture, 44
PTIME, 49
on a class, 44
cardinality pebble game, 74
central, 285, see star decomposition, see segregation
definably, 319
definable, 319
geometrically, 319
within cycles, 285
within disks, 285
central cycle, 333
central cycle, 333
centrality, 286
centre
definable, 319
definitional, 319
definable, 319
definable, 319
geometric, 317
geometric, 319
centre (of a star), 331
CFI gadget, 80
CFI-Theorem, 75
child, 24
chordless, 21, 187
CωωCωωωω, 73
CkCkCkCk, 73
clique, 21
closed loop, 363
closed disk, 177
coincide, 188
strongly, 188
colour refinement, 76
distinguish, 77
Combination Lemma, 399
quasi-4-connected, 238
Component Lifting Lemma, 156
Component Lifting Lemma, 156
cone, 90, 92
connected, 21
contract
edge, 22
subgraph, 22
contractible, 179
contraction, 22
contraction transduction, 38
counting operator, 30
counting quantifier, 72
crack, 142, 445
crackable, 445
crosscap, 178
crossedge, 223
crossing separators, 168
cuff, 178
cuff separating curve, 365
curve
simple, 177
simple closed, 178
cut vertex, 166
cutting through a belt, 390
cycle, 21
chordless, 21, 187
length, 21
noncontractible, 187
nonseparating, 187
cyclic order, 178
cylinder, 179
d-mapping, 463
d-scheme, see decomposition scheme
decision problem, 42
decomposition, 92
adhesion, 92
automorphism-invariant, 96
disjoint union, 123
ordered treelike, 147
over, 92
tight, 103
tree, 89, 112
treelike, 92
Decomposition Lemma
CC, 117, 165
decomposition mapping, 463
IFP-definable, 463
bounded adhesion, 463
invariant, 463
over a class, 463
polynomial time computable, 463
treelike, 463
decomposition scheme, 115
defined decomposition, 116
dimension, 115
parametrised, 122
Definability Lifting Lemma, 121
definable centre, 319
definable order, 51
definable skeleton, 320
Definable Structure Theorem, 147
Definable Structure Theorem for Almost Embeddable Graphs, 426
Definable Structure Theorem for Almost Planar Graphs, 321
Definable Structure Theorem for Embeddable Graphs, 202
Definable Structure Theorem for Graphs of Bounded Tree Width, 139
definable witness, 346
definably central, 319
define decomposition within, 124
define o-decomposition within, 152
define order, 51
deflationary fixed-point operator, 28
degree, 21
maximum, 21
minimum, 21
depth, 24
derivation, 264, 332
descendant, 24
design, 90, 92, 470
of a location, 470
torso, 470
diameter, 68
digraph
seedirected graph, 24
dimension, see decomposition scheme, see o-decomposition scheme, see pd-scheme
seedtransduction, 35
directed graph, 19, 24
acyclic, 24
disconnected, 21
disk
closed, 177
open, 177
distance, 21 see entrance hinge
distinguish, see colour refinement, see Weisfeiler-Leman algorithm
double torus, 178
Duplicator, 70
embeddable, 182
embedded graph, 182
face, 182
homeomorphic, 182
isomorphic, 182
underlying graph, 182
embedding, 182
2-cell, 182
directed graph, 24
directed graph, 20
homeomorphic, 182
minimal, 186
Index

normal, 186
polyhedral, 187
structure, 25
endvertex, see shortest path system, see belt
edge, 20
enlargement, 20
entrance hinge, 232
entrance hinge
distance, 232
equivalence relation
generated, 34
∃SO, see existential second-order logic
Euler genus, 186
Euler characteristic, 184
Euler genus, 178, 179
Euler’s formula, 184
exclude, 23
Excluded Grid Theorem, 334
existential second-order logic, 45
exit vertex, 390
expansion, 25
extension lemma, 120

face, 182
facial cycle, 189
facial subgraph, 182
Fagin’s Theorem, 45
faithful, 23
Finite Extension Lemma, 153
Finite Extension Lemma for Definable Orders,
52
Finite Extension Lemma for Ordered Completions,
267
First Angle Lemma, 192
first-order logic, 25
finite-variable fragments with counting, 73
first-order logic
finite-variable fragments, 68
first-order logic with counting, 32
fixed point operator
deflationary, 28
simultaneous, 29
fixed point operator
inflationary, 26
fixed-point operator
greatest, 30
least, 30
FO, see first-order logic
FO+C, see first-order logic with counting
forest, 23
fractional isomorphism, 85
free variable, 27

g-reduction, 366
Gaifman graph, 455
generic pre-decomposition, 259
genus
Euler, 178, 179
nonorientable, 178
orientable, 178
geometric centre, 317, 319
geometric skeleton, 317, 319
geometrically central, 319
GI, see graph isomorphism problem
gluing a disk on a cuff, 178
graph, 19
complete, 21
complete bipartite, 21
covnected, 21
disconnected, 21
disjoint union, 20
empty, 20
enlargement, 20
Euler genus, 186
intersection, 20
order, 20
ordered, 25
planar, 189
plane, 189
property, 20
regular, 21
union, 20
graph interpretation, 33
graph isomorphism problem, 76
Graph Minor Theorem, 23
graph transduction, 37
greatest fixed-point operator, 30
grid, 218, 333
j-central vertex, 334
central cycle, 333
central vertex, 333
centre, 334
compass, 334
elementary, 332

Preliminary Version
facial cycle, 333
hexagon, 333
interior, 333
perimeter, 333
plane, 334
ground leaf, 258
handle, 178
head, 24
height, 24
hexagon, 333
hexagonal grid, see grid
elementary, 332
hinge, 218
crossedge, 223
crossing, 223
homeomorphic, 177, see embedded graph, see embedding
homomorphism
decomposition, 109
directed graph, 24
graph, 20
structure, 25
homotopic, 360
horizon, 313
identify, 71, see colour refinement, see Weisfeiler-Leman algorithm
IFP, see inflationary fixed-point logic
IFP+C, see inflationary fixed-point logic with counting
image, 23
branch set, 23
faithful, 23
Immerman-Vardi Theorem, 45
in-degree, 24
incidence graph, 457
incident, 182
increasing tuple, 279
independent set, 21
index, see 2-connected component, see 3-connected component
indistinguishable, 98
infinitary logic, 69
finite-variable fragments, 69
infinitary logic

with counting, 73
inflationary fixed-point logic with counting, 30
inflationary fixed-point logic, 26
inflationary fixed-point operator, 26
injective arrangement, 281, 360
inseparable, 166, 168, 220
inside, 179
instance, 42
internal face, see patch
internal graph, see patch, see patch
internal region, 384
internally disjoint, 21
interpretation, 26, 33
definable, 33
graph, 33
invariant, see decomposition mapping
irrelevant, see separator
isomorphic, 20
isomorphism, see embedded graph
decomposition, 109
directed graph, 24
graph, 20
structure, 25
Jordan Curve Theorem, 178
K₅-minor free graph, 253
edge-maximal, 253
Klein bottle, 178
Kuratowski’s Theorem, 189
L, 7
L-equivalent, 70
Last Extension Lemma, 405
leaf, 24
least fixed-point logic, 30
least fixed-point operator, 30
Lemma on Simultaneous Induction, 29
lifting lemma, 120
L∞ω, 69
link, 364
attachment point, 364
Kₖ, 68
Lₖ, 69
local p-arrangement, 282, 360
local m-arrangement, 323, 428
Local Structure Theorem, 443

M. Grohe, Definable Graph Structure Theory
Index

location, 470
design, 470
logic
abstract, 49
first-order, 25
inflationary fixed-point, 26
$L_{\infty}^k$, 69
loop
attached, 363
attachment point, 363
closed, 363
noncontractible, 364
open, 363
m-arrangement, 322
local, 323
reduced, 428
Möbius strip, 179
$\mathcal{M}\mathcal{A}\mathcal{E}_{p,q,r}$-star completion, 433
$\mathcal{M}\mathcal{A}\mathcal{E}_{p,q,r}$-star decomposition, 433
reduced, 433
simple, 433
$\mathcal{M}\mathcal{A}\mathcal{P}_{p}$-star decomposition, 352
$\mathcal{M}\mathcal{A}\mathcal{P}_{p}$-star decomposition, 352
simple, 352
matching, 21
see quasi-4-connected component
Menger’s Theorem, 22
minimally embedded, 186
minor, 22
excluded, 23
minor free, 23
minor ideal, 23
naive minor transduction, 39
naive vertex classification, 76
natural path, 21
neighbour, 20
neighbourhood
$r$, 317
node, 23
noncontractible, 179
see cycle
noncontractible loop, 364
nonseparating, 22
see simple closed curve normal, see structure, see transduction, 98
186
Normalisation Lemma for Definable Treelike Decompositions, 119
Normalisation Lemma for Treelike Decompositions, 102
normally embedded, 186
null-homotopic, 360
o-decomposition, 147
tight, 151
underlying plain decomposition, 147
o-decomposition scheme, 147
dimension, 148
o-decomposition mapping, 464
o-decomposition scheme
defined o-decomposition, 148
underlying decomposition scheme, 148
od-mapping, 464
od-scheme, see o-decomposition scheme
open loop, 363
open disk, 177
order, see graph, see separation
ordered completion, 260
ordered copy, 54
Ordered Decomposition Lifting Lemma, 149
ordered decomposition scheme
parametrised, 151
Ordered Extension Lemma, 153
ordered treelike decomposition, 147
ordered union, 63
orthogonal, 92
out-degree, 24
outside, see belt
parallel, 92
parameter independent, 58
Parameter Elimination Lemma, 152
Parameterised Sunflower Canonisation Lemma, 160
Parametrised Definability Lifting Lemma, 122
Parametrised Normalisation Lemma, 122
parent, 24
partial isomorphism, 70
patch, 377
bridge, 384
external bridge, 384
external component, 384
internal bridge, 384
internal component, 384
internal face, 384
internal graph, 382, 384
internal region, 384
minimal simplifying, 383
path, 21
  endvertex, 21
  internal vertex, 21
  internally disjoint, 21
  isolated, 21
  length, 21
  natural, 21
  path decomposition, 279
  pc-node, 272
  pd-scheme, 258
    dimension, 258
    parametrised, 259
  pebble game
    bijective, 74
    cardinality, 74
  perimeter, 333
  pg-node, 272
  planar, 189
  plane, 189
  plane (grid), 334
  polyhedrally embedded, 187
  potential completion node, 272
  potential ground node, 272
  pre-decomposition, 257
    adhesion, 257
    completion, 257
    contained, 259
    defined, 258
    defined within, 259
    derivation, 264
    equivalent, 259
    generic, 259
    ordered completion, 260
    proper, 257
    restriction, 259
    subdecomposition, 259
    tight, 257
    width-$k$ completion, 260
  projective plane, 178
  proper, see 2-connected component, see 3-connected component, see pre-decompo-
sition
  proper noncontractible curve, 362
  property, 20, 25
  protect, 370
  Q-bridge, 384
  Q4C Completion Lemma, 268
  Q4C Decomposition Lemma, 237
  Q4C Lifting Lemma, 245
  quasi-4-connected, 237
  quasi-4-connected component, 238
    index, 238
    matching, 238
    torso, 238
  query, 34
    decide, 44
  reduced, see arrangement 428, see $\mathcal{MAE}_{p,q,r}$
  star decomposition
  reducing, see belt
  regular, 21
  representation, 42
  representativity, 187, 366
  restriction, 25, see pre-decomposition
  ring, 280
  root, 24
  rooted tree, 24
  RS-ring, 470
  3
  perimeter, 470
  RS-vortex, 469
  safe
  shortest path system, 377
  subgraph, 375
  Second Angle Lemma, 193
  second-order logic, 45
    existential, 45
  segment, 21
  segregation, 471
  3-central, 471
  arrangement, 471
  proper arrangement, 471
  type, 471
  separate, 22
  separating, see simple closed curve
  separation, 442
  order, 442
  separator, 22, 90, 92, 166
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>((W_1, W_2))</td>
<td>22</td>
</tr>
<tr>
<td>(k)</td>
<td>22</td>
</tr>
<tr>
<td>cross</td>
<td>168</td>
</tr>
<tr>
<td>irrelevant</td>
<td>231</td>
</tr>
<tr>
<td>maximal</td>
<td>90</td>
</tr>
<tr>
<td>order</td>
<td>22</td>
</tr>
<tr>
<td>proper</td>
<td>168</td>
</tr>
<tr>
<td>Sherali-Adams hierarchy</td>
<td>87</td>
</tr>
<tr>
<td>shortest path system</td>
<td>371</td>
</tr>
<tr>
<td>articulation vertex</td>
<td>372</td>
</tr>
<tr>
<td>block</td>
<td>373</td>
</tr>
<tr>
<td>canonical</td>
<td>372</td>
</tr>
<tr>
<td>endvertex</td>
<td>371</td>
</tr>
<tr>
<td>proper articulation vertex</td>
<td>372</td>
</tr>
<tr>
<td>sink</td>
<td>371</td>
</tr>
<tr>
<td>source</td>
<td>371</td>
</tr>
<tr>
<td>trivial</td>
<td>371</td>
</tr>
<tr>
<td>signature</td>
<td>33</td>
</tr>
<tr>
<td>simple, see transduction</td>
<td>371</td>
</tr>
<tr>
<td>(AP_p)-star decomposition, see (\mathcal{MA}P_p)-star decomposition</td>
<td>371</td>
</tr>
<tr>
<td>(\mathcal{AE}_p,q,r)-star decomposition, see (\mathcal{MAE}_p,q,r)-star decomposition</td>
<td>371</td>
</tr>
<tr>
<td>simpler</td>
<td>361</td>
</tr>
<tr>
<td>simple closed-curve</td>
<td></td>
</tr>
<tr>
<td>inside</td>
<td>179</td>
</tr>
<tr>
<td>simple closed curve</td>
<td>178</td>
</tr>
<tr>
<td>contractible</td>
<td>179</td>
</tr>
<tr>
<td>noncontractible</td>
<td>179</td>
</tr>
<tr>
<td>nonseparating</td>
<td>179</td>
</tr>
<tr>
<td>separating</td>
<td>179</td>
</tr>
<tr>
<td>simple curve</td>
<td>177</td>
</tr>
<tr>
<td>endpoints</td>
<td>177</td>
</tr>
<tr>
<td>internal points</td>
<td>178</td>
</tr>
<tr>
<td>simplifying curve</td>
<td>365</td>
</tr>
<tr>
<td>shortest path system</td>
<td>377</td>
</tr>
<tr>
<td>subgraph</td>
<td>375</td>
</tr>
<tr>
<td>simulate</td>
<td>109</td>
</tr>
<tr>
<td>simultaneous fixed-point operator</td>
<td>29</td>
</tr>
<tr>
<td>sink</td>
<td>371</td>
</tr>
<tr>
<td>skeleton</td>
<td></td>
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<tr>
<td>definable</td>
<td>320</td>
</tr>
<tr>
<td>geometric</td>
<td>317</td>
</tr>
<tr>
<td>SO, see second-order logic</td>
<td></td>
</tr>
<tr>
<td>society</td>
<td>469</td>
</tr>
<tr>
<td>source</td>
<td>371</td>
</tr>
<tr>
<td>spanning tree</td>
<td>24</td>
</tr>
<tr>
<td>split extension</td>
<td>169</td>
</tr>
<tr>
<td>Spoiler</td>
<td>70</td>
</tr>
<tr>
<td>sps, see shortest path system</td>
<td>55</td>
</tr>
<tr>
<td>star</td>
<td>331</td>
</tr>
<tr>
<td>star decomposition</td>
<td>331</td>
</tr>
<tr>
<td>(\Sigma)-central</td>
<td>443</td>
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<tr>
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<td>98</td>
</tr>
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</tr>
<tr>
<td>(IFP+C)-definable</td>
<td>460</td>
</tr>
<tr>
<td>polynomial time computable</td>
<td>463</td>
</tr>
<tr>
<td>strong embedding</td>
<td></td>
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<tr>
<td>directed graph</td>
<td>24</td>
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<td>plain</td>
<td>157</td>
</tr>
<tr>
<td>property</td>
<td>25</td>
</tr>
<tr>
<td>union</td>
<td>25</td>
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<td>457</td>
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<td>Structure Theorem</td>
<td>441</td>
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<td>subdecomposition</td>
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</tr>
<tr>
<td>see pre-decomposition</td>
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<td>23</td>
</tr>
<tr>
<td>subgraph</td>
<td>20</td>
</tr>
<tr>
<td>induced</td>
<td>20</td>
</tr>
<tr>
<td>subgraph transduction</td>
<td>38</td>
</tr>
<tr>
<td>subgrid</td>
<td>333</td>
</tr>
<tr>
<td>central</td>
<td>333</td>
</tr>
<tr>
<td>subpatch</td>
<td>383</td>
</tr>
<tr>
<td>proper</td>
<td>383</td>
</tr>
<tr>
<td>substructure</td>
<td>25</td>
</tr>
<tr>
<td>induced</td>
<td>25</td>
</tr>
<tr>
<td>subtree</td>
<td>24</td>
</tr>
<tr>
<td>subtuple</td>
<td>19</td>
</tr>
<tr>
<td>Sunflower Canonisation Lemma</td>
<td>157</td>
</tr>
<tr>
<td>supercentre</td>
<td>346</td>
</tr>
<tr>
<td>superskeleton</td>
<td>346</td>
</tr>
<tr>
<td>surface</td>
<td>178</td>
</tr>
</tbody>
</table>

Preliminary Version
boundary, 178
cuff, 178
simpler, 361
syntactical interpretation, see transduction

$\mathcal{T}$-central, see star decomposition, see segregation
tail, 21
tangle, 442
Third Angle Lemma, 195
tight, see decomposition, see o-decomposition, see pre-decomposition
tip, 331
topological minor, 23
topological subgraph, 23
topological space, 177
  boundary, 177
  closure, 177
  interior, 177
  subspace, 177
torso, 90, 92, see quasi-4-connected component, see design
torus, 178
transduction, 35
  contraction, 38
  dimension, 35
  domain, 35
  domain variable, 35
  graph, 37
  naive minor, 39
  normal, 61
  numerical, 35
  parameter, 35
  parameter independent, 58
  simple, 35
  subgraph, 38
  Transduction Lemma, 36
  Transduction Lemma for Definable Decompositions, 124
  Transduction Lemma for Definable O-Decompositions, 152
  Transitivity Lemma, 125
tree, 23
  directed, 24
  rooted, 24
  spanning, 24
treelike decomposition

normal, 98
strict, 98
tree decomposition, 89, 112
  adhesion, 90
  height, 141
  over, 90
  tree width, 90
  triple torus, 178
  trivial, see shortest path system
  Turing machine, 42
  type, see segregation
uncrackable, 445
  Union Lemma for Definable Decompositions, 121
  Union Lemma for Definable Ordered Decompositions, 150
  Union Lemma for Definable Orders, 52
variable
  individual, 25, 30
  number, 30
  relation, 26
  type, 30
  vertex, 30
  vertex of attachment, 22
  vocabulary, 25
  vortex, 280
  protect, 370
walk, 21
  closed, 21
  endvertex, 21
  length, 21
  simple, 21
  simple closed, 21
wall, 334
weighted structure, 457
  definable canonisation, 459
  expansion, 457
  Gaifman graph, 458
  induced substructure, 457
  isomorphism, 458
  restriction, 457
  underlying plain structure, 458
  union, 458
  Weisfeiler-Leman algorithm, 79
distinguish, 80
Index

identify, 80
wheel, 281
Whitney’s Theorem, 189
wide, 340
width, 90 92
width-k completion, 260
WL, see Weisfeiler-Leman algorithm
WL-dimension, 466